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Willi Freeden · Michael Schreiner

Spherical Functions of Mathematical Geosciences

A Scalar, Vectorial, and Tensorial Setup

Springer

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This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

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Willi Freeden was born in 1948 in Kaldenkirchen/Germany, Studies in Mathematics, Geography, and Philosophy at the RWTH Aachen, 1971 'Diplom' in Mathematics, 1972 'Staatsexamen' in Mathematics and Geography, 1975 PhD in Mathematics, 1979 'Habilitation' in Mathematics, 1981/1982 Visiting Research Professor at the Ohio State University, Columbus (Department of Geodetic Science and Surveying), 1984 Professor of Mathematics at the RWTH Aachen (Institute of Pure and Applied Mathematics), 1989 Professor of Technomathematics (Industrial Mathematics), 1994 Head of the Geomathematics Group, 2002–2006 Vice-President for Research and Technology at the University of Kaiserslautern.

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This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

Contents

Preface

During the last decades, geosciences and -engineering were influenced by two essential scenarios. First, the technological progress has changed completely the observational and measurement techniques. Modern high speed computers and satellite-based techniques are entering more and more all (geo) disciplines. Second, there is a growing public concern about the future of our planet, its climate, its environment, and about an expected shortage of natural resources. Obviously, both aspects, viz. (i) efficient strategies of protection against threats of a changing Earth and (ii) the exceptional situation of getting terrestrial, airborne as well as spaceborne, data of better and better quality explain the strong need for new mathematical structures, tools, and methods. In consequence, mathematics concerned with geoscientific problems, i.e., *geomathematics*, is becoming more and more important. Nowadays, geomathematics may be regarded as the key technology to build the bridge between real Earth processes and their scientific understanding. In fact, it is the intrinsic and indispensable *means* to handle geoscientifically relevant data sets of high quality within high accuracy and to improve significantly modeling capabilities in Earth system research.

From modern satellite-positioning, it is well known that the Earth's surface deviates from a sphere by less than 0.4% of its radius. This is the reason why spherical functions and concepts play an essential part in all geosciences. In particular, spherical polynomials and zonal functions constitute fundamental ingredients of modern (geo-)research – wherever spherical fields are significant, be they electromagnetic, gravitational, hydrodynamical, solid body, etc. Surprisingly enough, it turned out that essential features involving spherical vector and tensor structures were not available in the geosciences, when W. Freeden, first at the RWTH Aachen and later as head of the Geomathematics Group of the TU Kaiserslautern, started with the vector and/or tensor analysis of (Earth's) gravity field data obtained by satellite-to-satellite tracking (SST) and/or satellite gravity gradiometry (SGG). This is the reason why, based on results about Green's function with respect to the scalar Beltrami operator, a series of papers was initiated to establish vector and tensor counterparts of the Legendre polynomials, to verify vector and tensor extensions of the addition theorem, and to introduce vectorial and tensorial generalizations of the famous Funk-Hecke formula. Even more, the concept of zonal (kernel) functions (i.e., radial basis functions in the jargon of approximation theory), the theory of splines and wavelets etc could be generalized to the spherical vector/tensor case. All these new concepts were successfully applied in diverse areas such as climate and weather, deformation analysis, geomagnetics, gravitation, and ocean circulation.

This book collects all material developed by the Geomathematics Group, TU Kaiserslautern, during the last years to set up a theory of spherical functions of mathematical (geo-)physics. The work shows a twofold transition: First, the natural transition from the scalar to the vectorial and tensorial theory of spherical harmonics is given in coordinate-free representation, based on new variants of the addition theorem and the Funk–Hecke formulas. Second, the canonical transition from spherical harmonics via zonal (kernel) functions to the Dirac kernel is presented in close orientation to an uncertainty principle classifying the space/frequency (momentum) behavior of the functions for purposes of constructive approximation and data analysis. In doing so, the whole palette of spherical (trial) functions is provided for modeling and simulating phenomena and processes of the Earth system.

The main purpose of the book is to serve as a self-consistent introductory textbook for (graduate) students of mathematics, (geo-)physics, geodesy, and (geo-)engineering. In addition, the work should also be a valuable reference for scientists and practitioners facing spherical problems in their professional tasks. Essential ingredients of the work are the theses of W. Freeden (1979a), T. Gervens (1989), M. Schreiner (1994), S. Beth (2000), and H. Nutz (2002). Preliminary material can be found in the work by C. Müller (1952, 1966, 1998) and W. Freeden et al. (1998).

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Kaiserslautern and Buchs, May 2008 W. F., M. S.

1 Introduction

Spherical harmonics are the analogues of trigonometric functions for Fourier expansion theory on the sphere. They were introduced in the 1780s to study gravitational theory (cf. P.S. de Laplace (1785), A.M. Legendre (1785)). Early publications on the theory of spherical harmonics in their original physically motivated meaning as multipoles are, e.g., due to R.F.A. Clebsch (1861), T. Sylvester (1876), E. Heine (1878), F. Neumann (1887), and J.C. Maxwell (1891). Today, the use of spherical harmonics in diverse procedures is a well-established technique in all geosciences, particularly for the purpose of representing scalar potentials. A great incentive came from the fact that global geomagnetic data became available in the first half of the 19^{th} century (cf. C.F. Gauß (1838)). Nowadays, reference models for the Earth's gravitational or magnetic field, for example, are widely known by tables of coefficients of the spherical harmonic Fourier expansion of their potentials. It is characteristic for the Fourier approach that each spherical harmonic, as an 'ansatz-function' of polynomial nature, corresponds to exactly one degree, i.e., in the jargon of signal processing to exactly one frequency. Thus, orthogonal (Fourier) expansion in terms of spherical harmonics amounts to the superposition of summands showing an oscillating character determined by the degree (frequency) of the Legendre polynomial (see Table 1.1). The more spherical harmonics of different degrees are involved in the Fourier (orthogonal) expansion of a signal, the more the oscillations grow in number, and the less are the amplitudes in size.

Concerning the mathematical representation of spherical vector and tensor fields in applied sciences, one is usually not interested in their separation into their (scalar) cartesian component functions. Instead, we have to observe inherent physical constraints. For example, the external gravitational field is curl-free, the magnetic field is divergence-free, and the equations for incompressible Navier–Stokes equations in meteorological applications or the geostrophic formulation of ocean circulation include divergence-free vector solutions. In many cases, certain quantities are related to each other in an obvious manner by vector operators like the surface gradient or the surface curl gradient. In this respect, the gravity field, the magnetic field, the wind field, the field of oceanic currents, or electromagnetic waves generated by surface currents should be mentioned as important examples.

In addition, spherical modeling in terms of spherical harmonics arises naturally in the analysis of the elastic-gravitational free oscillations of a spherically symmetric, non-rotating Earth. Altogether, vector/tensor spherical harmonics are used throughout mathematics, theoretical physics, geo- and astrophysics, and engineering – indeed, wherever one deals with physically based fields.

Table 1.1: Fourier expansion of scalar square-integrable functions on the unit sphere Ω.

Weierstraß approximation theorem: use of homogeneous polynomials	↓	(geo)physical constraint of harmonicity
spherical harmonics $Y_{n,j}$ as restrictions of homogeneous harmonic polynomials $H_{n,j}$ to the unit sphere $\Omega \subset \mathbb{R}^3$		
orthonormality and orthogonal invariance	↓	addition theorem
one-dimensional Legendre polynomial P_n satisfying $P_n(\xi \cdot \eta) = \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta), \ \xi, \eta \in \Omega$		
convolution against the Legendre kernel	↓	Funk–Hecke formula
Legendre transform of F: $(P_n * F)(\xi) = \frac{2n+1}{4\pi} \int_\Omega P_n(\xi \cdot \eta) F(\eta) d\omega(\eta), \ \xi \in \Omega$		
superposition over frequencies	↓	orthogonal (Fourier) series expansion
Fourier series of $F \in \mathrm{L}^2(\Omega)$: $F(\xi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_\Omega P_n(\xi \cdot \eta) F(\eta) d\omega(\eta), \ \xi \in \Omega$		

1.1 Motivation

In the second half of the last century, a physically motivated approach for the decomposition of spherical vector and tensor fields was presented based on a spherical variant of the Helmholtz theorem (see, e.g., P.M. Morse, H. Feshbach (1953), G.E. Backus (1966); G.E. Backus (1967, 1986)). Following this concept, e.g., the tangential part of a spherical vector field is split up into a curl-free and a divergence-free field by use of two differential operators, viz. the already mentioned surface gradient and the surface curl gradient. Of course, an analogous splitting is valid in tensor theory.

Table 1.2: Twofold transition.

Scalar Legendre kernels	⟶	Vector Legendre kernels	⟶	Tensor Legendre kernels
↓		↓		↓
scalar zonal kernels	⟶	vector zonal kernels	⟶	tensor zonal kernels
↓		↓		↓
scalar Dirac kernel	⟶	vector Dirac kernel	⟶	tensor Dirac kernel

In subsequent publications during the second half of the last century, however, the vector spherical harmonic theory was usually written in local coordinate expressions that make mathematical formulations lengthy and hard to read. Tensor spherical harmonic settings are even more difficult to understand. In addition, when using local coordinates within a global spherical concept, differential geometry tells us that there is no representation of vector and tensor spherical harmonics which is free of singularities. In consequence, the mathematical arrangement involving vector and tensor spherical harmonics has led to an inadequately complex and less consistent literature, yet. Coordinate free explicit formulas on vector and/or tensor variants of the Legendre polynomial could not be found in the literature. As an immediate result, the orthogonal invariance based on specific vector /tensor extensions of the Legendre polynomials was not worked out suitably in a unifying scalar/vector/tensor framework. Even more, the concept of zonal (kernel) functions was not generalized adequately to the spherical vector/tensor case. All these new structures concerning spherical functions in mathematical (geo-)physics are successfully developed in this work. Basically two transitions are undertaken in our approach, namely the transition from spherical harmonics via zonal kernel functions to the Dirac kernels on the one hand and the transition from scalar to vector and tensor theory on the other hand (see Table 1.2).

To explain the *transition from the theory of scalar spherical harmonics to its vectorial and tensorial extensions* (see Chapters 3, 4, 5, and 6 for details), our work starts from physically motivated dual pairs of operators (the reference space being always the space of signals with finite energy, i.e., the space of square-integrable fields). The pair $o^{(i)}, O^{(i)}, i \in \{1,2,3\}$, is originated in the constituting ingredients of the Helmholtz decomposition of a vector field (see Chapter 5), while $o^{(i,k)}, O^{(i,k)}, i,k \in \{1,2,3\}$, take the analogous role for the Helmholtz decomposition of tensor fields (see Chapter 6). For example, in vector theory, $o^{(1)}F$ is assumed to be the normal field $\xi \mapsto o_\xi^{(1)}F(\xi) = F(\xi)\xi$, $\xi \in \Omega$, $o^{(2)}F$ is the surface gradient field $\xi \mapsto o_\xi^{(2)}F(\xi) = \nabla_\xi^* F(\xi)$, $\xi \in \Omega$, and $o^{(3)}F$ is the surface curl gradient field $\xi \mapsto o_\xi^{(3)}F(\xi) = \mathrm{L}_\xi^* F(\xi), \xi \in \Omega$, with $\mathrm{L}_\xi^* = \xi \wedge \nabla_\xi^*$ applied to a scalar valued function F, while $O^{(1)}f$ is the normal component $\xi \mapsto O_\xi^{(1)}f(\xi) = f(\xi)\cdot\xi$, $\xi \in \Omega$, $O^{(2)}f$ is the negative surface divergence $\xi \mapsto O_\xi^{(2)}f(\xi) = -\nabla_\xi^*\cdot f(\xi)$, $\xi \in \Omega$, and $O^{(3)}f$ is the negative surface curl $\xi \mapsto O_\xi^{(3)}f(\xi) = -\mathrm{L}_\xi^*\cdot f(\xi)$, $\xi \in \Omega$ taken over a vector valued function f. Clearly, the operators $o^{(i,k)}, O^{(i,k)}$ are also definable in orientation to the tensor Helmholtz decomposition theorem (for reasons of simplicity, however, their explicit description is omitted here). It should be noted that, in vector as well as tensor theory, the connecting link from the operators to the Helmholtz decomposition is the Green function with respect to the (scalar) Beltrami operator and its iterations (for more details, the reader is referred to Chapter 4 of this work).

The pairs $o^{(i)}, O^{(i)}$ and $o^{(i,i)}, O^{(i,i)}$ of dual operators lead us to an associated palette of Legendre kernel functions, all of them generated by the classical one-dimensional Legendre polynomial P_n of degree n. To be more concrete, three types of Legendre kernels occur in the vectorial as well as tensorial context (see Table 1.3).

Table 1.3: Legendre kernel functions.

Scalar Legendre polynomial $P_n = \frac{O^{(i)}O^{(i)}\mathbf{p}_n^{(i,i)}}{\mu_n^{(i)}} = \frac{O^{(i,k)}O^{(i,k)}\mathbf{P}_n^{(i,k)}}{\mu_n^{(i,k)}}$	
application of $o^{(i)}$ $\downarrow\uparrow$ application of $O^{(i)}$	application of $o^{(i,k)}$ $\downarrow\uparrow$ application of $O^{(i,k)}$
vector Legendre kernel $p_n^{(i)} = \frac{o^{(i)}P_n}{(\mu_n^{(i)})^{1/2}} = \frac{O^{(i)}\mathbf{p}_n^{(i,i)}}{(\mu_n^{(i)})^{1/2}}$	tensor Legendre kernel (order 2) $\mathbf{p}_n^{(i,k)} = \frac{o^{(i,k)}P_n}{(\mu_n^{(i,k)})^{1/2}} = \frac{O^{(i,k)}\mathbf{P}_n^{(i,k)}}{(\mu_n^{(i,k)})^{1/2}}$
application of $o^{(i)}$ $\downarrow\uparrow$ application of $O^{(i)}$	application of $o^{(i,k)}$ $\downarrow\uparrow$ application of $O^{(i,k)}$
tensor Legendre kernel (order 2) $\mathbf{p}_n^{(i,i)} = \frac{o^{(i)}p_n^{(i)}}{(\mu_n^{(i)})^{1/2}} = \frac{o^{(i)}o^{(i)}P_n}{\mu_n^{(i)}}$	tensor Legendre kernel (order 4) $\mathbf{P}_n^{(i,k,i,k)} = \frac{o^{(i,k)}\mathbf{p}_n^{(i,k)}}{(\mu_n^{(i,k)})^{1/2}} = \frac{o^{(i,k)}o^{(i,k)}P_n}{\mu_n^{(i,k)}}$
vectorial context	*tensorial context*

The Legendre kernels $o^{(i)}P_n$, $o^{(i)}o^{(i)}P_n$ are of concern for the vector approach to spherical harmonics, whereas $o^{(i,i)}P_n$, $o^{(i,i)}o^{(i,i)}P_n, i = 1,2,3$, form the analogues in tensorial theory. Corresponding to each Legendre kernel, we are led to two variants for representing square-integrable fields by orthogonal (Fourier) expansion, where the reconstruction – as in the scalar case – is undertaken by superposition over all frequencies.

The Tables 1.3, 1.4, and 1.5 bring together – into a single unified notation – the formalisms for the vector/tensor spherical harmonic theory based on the following principles:

- The vector/tensor spherical harmonics involving the $o^{(i)}, o^{(i,i)}$-operators, respectively, are obtainable as restrictions of three-dimensional homogeneous harmonic vector/tensor polynomials, respectively, that are computable exactly exclusively by integer operations.
- The vector/tensor Legendre kernels are obtainable as the outcome of sums extended over a maximal orthonormal system of vector/tensor spherical harmonics of degree (frequency) n, respectively.

- The vector/tensor Legendre kernels are zonal kernel functions, i.e., they are orthogonally invariant (in vector/tensor sense, respectively) with respect to orthogonal transformations (leaving one point of the unit sphere Ω fixed).
- Spherical harmonics of degree (frequency) n form an irreducible subspace of the reference space of (square-integrable) fields on Ω.
- Each Legendre kernel implies an associated Funk–Hecke formula that determines the constituting features of the convolution of a square-integrable field against the Legendre kernel.
- The orthogonal Fourier expansion of a square-integrable field is the sum of the convolutions of the field against the Legendre kernels being extended over all frequences.

Unfortunately, the vector spherical harmonics generated by the operators $o^{(i)}, O^{(i)}$, $i = 1, 2, 3$, do not constitute eigenfunctions with respect to the Beltrami operator. But it should be mentioned that certain operators $\tilde{o}^{(i)}$, $i = 1, 2, 3$, can be introduced in terms of the operators $o^{(i)}$, $i = 1, 2, 3$, which define alternative classes of vector spherical harmonics that represent eigensolutions to the Beltrami operator. The price to be paid is that the separation of spherical vector fields into normal and tangential parts is lost. More precisely, the operators $\tilde{o}^{(i)}, i \in \{1, 2\}$, generate so-called spheroidal fields, while $\tilde{o}^{(3)}$ generates poloidal fields. In fact, all statements involving orthogonal (Fourier) expansion of spherical fields remain valid for this new class of operators. Moreover, analogous classes of tensor spherical harmonics can be introduced by operators $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}, i, k = 1, 2, 3$, in close analogy to the vector case. In addition, it should be noted that the spherical harmonics based on the $\tilde{o}^{(i)}, \tilde{O}^{(i)}, \tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$-operators play a particular role whenever the Laplace operator comes into play, i.e., in gravitation for representing any kind of harmonic fields (see Chapter 10).

To summarize, the theory of spherical harmonics as presented in this book (see Chapters 3, 4, 5, and 6) is a unifying attempt of consolidating, reviewing and supplementing the different approaches in real scalar, vector, and tensor theory. The essential tools are the Legendre kernels which are shown to be explicitly available and tremendously significant in rotational invariance and in orthogonal Fourier expansions. The work is self-contained: the reader is told how to derive all equations occuring in due course. Most importantly, our coordinate-free setup yields a number of formulas and theorems that previously were derived only in coordinate representation (such as polar coordinates). In doing so, any kind of singularities is avoided at the poles. Finally, our philosophy opens new promising perspectives of constructing important, i.e., zonal classes of spherical trial functions by summing up Legendre kernel expressions, thereby providing (geo-)physical relevance and

Table 1.4: Fourier expansion of (square-integrable) vector fields f.

Vector spherical harmonics $y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} o^{(i)} Y_{n,j}$	
addition theorem $\downarrow$ vectorial variant	addition theorem $\downarrow$ tensorial variant
${}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta)$	$p_n^{(i)}(\xi,\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) Y_{n,j}(\eta)$
Funk–Hecke formula $\downarrow$ tensorial variant	Funk–Hecke formula $\downarrow$ vectorial variant
Legendre transform $\frac{2n+1}{4\pi} \times \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) f(\eta) d\omega(\eta)$	Legendre transform $\frac{2n+1}{4\pi} (\mu_n^{(i)})^{-1/2} \times \int_\Omega p_n^{(i)}(\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta)$
superposition $\downarrow$ over frequencies	superposition $\downarrow$ over frequencies
$f(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \times \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) f(\eta) d\omega(\eta)$	$f(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (\mu_n^{(i)})^{-1/2} \times \int_\Omega p_n^{(i)}(\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta)$
rank–2 tensorial approach	*vectorial approach*

increasing local applicability.

To understand the *transition from the theory of spherical harmonics to zonal kernel function up to the Dirac kernel* (for details see Chapters 7, 8, and 9), we have to realize the relative advantages of the classical Fourier expansion method by means of spherical harmonics not only in the frequency domain, but also in the space domain. Obviously, it is characteristic for Fourier techniques that the spherical harmonics as polynomial trial functions admit no localization in space domain, while in the frequency domain

Table 1.5: Fourier expansion of a square-integrable tensor fields $\mathbf{f}$.

Tensor spherical harmonics $\mathbf{y}_{n,j}^{(i,k)} = (\mu_n^{(i,k)})^{-1/2}\mathbf{o}^{(i,k)}Y_{n,j}$	
addition theorem $\downarrow$ rank-4 tensorial variant	addition theorem $\downarrow$ rank-2 tensorial variant
$\mathbf{P}_n^{(i,k,i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(i,k)}(\eta)$	${}^t\mathbf{p}_n^{(i,k)}(\xi,\eta) = \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) Y_{n,j}(\eta)$
Funk–Hecke formula $\downarrow$ rank-4 tensorial variant	Funk–Hecke formula $\downarrow$ rank-2 tensorial variant
Legendre transform $\frac{2n+1}{4\pi} \times \int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\xi,\eta)\mathbf{f}(\eta)d\omega(\eta)$	Legendre transform $\frac{2n+1}{4\pi}(\mu_n^{(i,k)})^{-1/2} \times \int_\Omega {}^t\mathbf{p}_n^{(i,k)}(\xi,\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)$
superposition $\downarrow$ over frequencies	superposition $\downarrow$ over frequencies
$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \times \int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\xi,\eta)\mathbf{f}(\eta)d\omega(\eta)$	$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \frac{1}{(\mu_n^{(i,k)})^{1/2}} \times \int_\Omega {}^t\mathbf{p}_n^{(i,k)}(\xi,\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)$
rank-4 tensorial approach	*rank-2 tensorial approach*

(more precisely, momentum domain), they always correspond to exactly one degree, i.e., frequency, and therefore, are said to show ideal frequency localization. Because of the ideal frequency localization and the simultaneous absence of space localization, in fact, local changes of fields (signals) in the space domain affect the whole table of orthogonal (Fourier) coefficients. This, in turn, causes global changes of the corresponding (truncated) Fourier series in the space domain. Nevertheless, the ideal frequency localization usually proves to be helpful for meaningful physical interpretations (e.g., within Meissl schemes in physical geodesy (see, e.g., P.A. Meissl (1971), E.W. Grafarend (2001), H. Nutz (2002) and the references therein) relating – for a frequency being fixed – the different observables of the Earth's gravitational potential to each other.

Taking these aspects on spherical harmonic modeling by Fourier series into account, trial functions which simultaneously show ideal frequency localization as well as ideal space localization would be a desirable choice. In fact, such an ideal system of trial functions would admit models of highest spatial resolution which were expressible in terms of single frequencies. However, the uncertainty principle (see, e.g., F.J. Narcowich, J.D. Ward (1996), W. Freeden (1998), N. Laín Fernández (2003)) – connecting space and frequency localization – tells us that both characteristics are mutually exclusive. Extreme trial functions in the sense of such an uncertainty principle are, on the one hand, the Legendre kernels (no space localization, ideal frequency localization) and, on the other hand, the Dirac kernel (ideal space localization, no frequency localization). In conclusion, Fourier expansion methods are well suited to resolve low and medium frequency phenomena, i.e., the 'trend' of a signal, while their application to obtain high resolution in global or local models is critical. This difficulty is also well known to theoretical physics, e.g., when describing monochromatic electromagnetic waves or considering the quantum-mechanical treatment of free particles. In this case, plane waves with fixed frequencies (ideal frequency localization, no space localization) are the solutions of the corresponding differential equations, but do certainly not reflect the physical reality. As a remedy, plane waves of different frequencies are superposed to so-called wave-packages which gain a certain amount of space localization, while losing their ideal spectral localization. In a similar way, a suitable superposition of polynomial Legendre kernel functions leads to so-called zonal kernel functions, in particular to kernel functions with a reduced frequency, but increased space localization.

Additive clustering of weighted Legendre kernels – the weights are usually said to define the Legendre symbol – generates zonal kernel functions. The uncertainty principle (see Chapter 7) describes a trade-off between two 'spreads' of the zonal kernels, one for the space and the other for the frequency. The main statement is that sharp localization of zonal kernels in space and in frequency is mutually exclusive. The reason for the validity of the uncertainty relation is that the aforementioned operators $o^{(1)}$ and $o^{(3)}$ do not commute. Thus, $o^{(1)}$ and $o^{(3)}$ cannot be sharply defined simultaneously. As already mentioned, extremal members in the space/frequency (momentum) relation are the Legendre kernels and the Dirac kernels (see Table 1.6). More explicitly, the uncertainty principle allows us to give a

Table 1.6: From Legendre kernels via zonal kernels to the Dirac kernel.

Legendre kernels	zonal kernels		Dirac kernel
	general case		
	bandlimited	spacelimited	

quantitative classification in the form of a canonically defined hierarchy of the space/frequency localization properties of zonal kernel functions, be they of scalar, vectorial, or tensorial nature. For simplicity, restricting ourselves to scalar zonal kernels of the form

$$K(\xi \cdot \eta) = \sum_{k=0}^{\infty} \frac{2n+1}{4\pi} K^{\wedge}(n) P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega \tag{1.1}$$

(with $K^{\wedge}(n), n = 0, 1, \ldots,$ being the symbol of the kernel K), we are led to the following conclusion: In view of the amount of space/frequency (momentum) localization, it is remarkable to distinguish bandlimited kernels (i.e., $K^{\wedge}(n) = 0$ for all $n \geq N$) and non-bandlimited ones, for which infinitely many numbers $K^{\wedge}(n)$ do not vanish. Non-bandlimited kernels show a much stronger space localization than their bandlimited counterparts. Empirically, if $K^{\wedge}(n) \approx K^{\wedge}(n+1) \approx 1$ for many successive large integers n, then the support of the series (1.1) in the space domain is small, i.e., the kernel is spacelimited (i.e., in the jargon of approximation theory, locally supported). Assuming the condition $\lim_{n\to\infty} K^{\wedge}(n) = 0$, we are confronted with the situation that the slower the sequence $\{K^{\wedge}(n)\}_{n=0,1,\ldots}$ converges to zero, the lower is the frequency localization, and the higher is the space localization.

Our considerations lead us to the following characterization of trial functions in constructive approximation: Fourier expansion methods with polynomial ansatz functions offer the canonical ‘trend-approximation’ of low-frequent phenomena (for global modeling), while bandlimited kernels can be used for the transition from long-wavelength to short-wavelength phenomena (global to local modeling). Because of their excellent localization properties in the space domain, the non-bandlimited kernels can be used for the modeling of short-wavelength phenomena (local modeling). Using kernels of different scales reflecting the different stages of space/frequency localization (see, e.g., W. Freeden (1998), W. Freeden, V. Michel (1999) and the references therein), the modeling process can be adapted to the localization properties of the physical phenomena (see Table 1.7).

Table 1.7: Multiscale expansion of scalar (square-integrable) spherical functions F.

Sequence of scale-dependent zonal kernels (i.e., scaling functions) Φ_j	↓	convolutions against Φ_j
low-pass filtered versions of F $(\Phi_j * F)(\xi) = \int_\Omega \Phi_j(\xi \cdot \eta) F(\eta)\, d\omega(\eta), \quad \xi \in \Omega$		
continuous 'summation' over positions $\eta \in \Omega$	↓	'zooming in' ($\Phi_j \to \delta$ as $j \to \infty$)
multiscale expansion of F involving a Dirac family of zonal scalar kernels $F(\xi) = \lim_{j\to\infty} \int_\Omega \Phi_j(\xi \cdot \eta) F(\eta)\, d\omega(\eta), \xi \in \Omega$		

In case of so-called scaling functions, the width of the corresponding frequency bands and, consequently, the amount of space localization is controlled (in continuous and/or discrete way) using a so-called scale-parameter, such that the Dirac kernel acts as limit kernel as the scale-parameter takes its limit. Typically, the generating kernels of scaling functions have the characteristics of low-pass filters, i.e., the zonal kernels *involved* in the convolution of the field against the Legendre kernels are significantly based on low frequencies, while the higher frequencies are attenuated or even completely left out in the summation. Conventionally, the difference between successive members in a scaling function is called a wavelet function. Clearly, it is again a zonal kernel. In consequence, wavelet functions have the typical properties of band-pass filters, i.e., the weighted Legendre kernels of low and high frequency within the wavelet kernel are attenuated or even completely left out. According to their particular construction, wavelet-techniques provide a decomposition of the reference space into a sequence of approximating subspaces – the scale spaces – corresponding to the scale parameter. In each scale space, a filtered version of a spherical field under consideration is calculated as a convolution of the field against the respective member of the scaling function and, thus, leading to an approximation of the field at certain resolutions. For increasing scales, the approximation improves and the information obtained on coarse levels is usually contained in foregoing levels. The difference between two successive bandpass filtered version of the signal is called the detail information and is collected in the so-called detail space. The wavelets constitute the basis functions of the detail spaces and, summarizing our excursion to multiscale modeling, every element of the reference space can be represented as a structured linear combination of scaling functions and wavelets corresponding to different scales and at different positions. That is, using scaling functions und wavelets at different scales, the corresponding multiscale technique can be constructed as to be suitable for the specific local field structure. Consequently, although most fields show a correlation in space as well as in frequency, the zonal kernel functions with their simultaneous space and frequency localization allow for the efficient detection and approximation of essential features by only using fractions of the original information (decorrelation).

The Tables 1.7, 1.8, and 1.9 bring together, into a unified nomenclature, the formalisms for zonal kernel function theory based on the following principles:

- Weighted Legendre kernels are the constituting summands of zonal kernel functions.
- The only zonal kernel that is both band- and spacelimited is the trivial kernel; the Legendre kernel is ideal in frequency localization, the Dirac kernel is ideal in space localization.
- The convolution of a field (signal) against a zonal kernel function provides a filtered version of the original.
- Scaling kernels, i.e., certain sequences of (parameter-dependent) zonal kernels tending to the Dirac kernel, provide better and better approximating low-pass filtered versions of the field (signal) under consideration.

To summarize, the theory of zonal kernels as presented in this book (see Chapters 7, 8, and 9) is a unifying attempt of reviewing, clarifying and supplementing the different additive clusters of weighted Legendre kernels. The kernels exist as bandlimited and non-bandlimited, spacelimited, and non-spacelimited variants. The uncertainty principle determines the frequency/ space window for approximation. A fixed space window is used for the windowed Fourier transform of fields (signals), where the approximation is still taken over the frequencies. The power of the scaling function

Table 1.8: Interrelations between space and frequency localization, kernel type, correlation, integral transform and resolution.

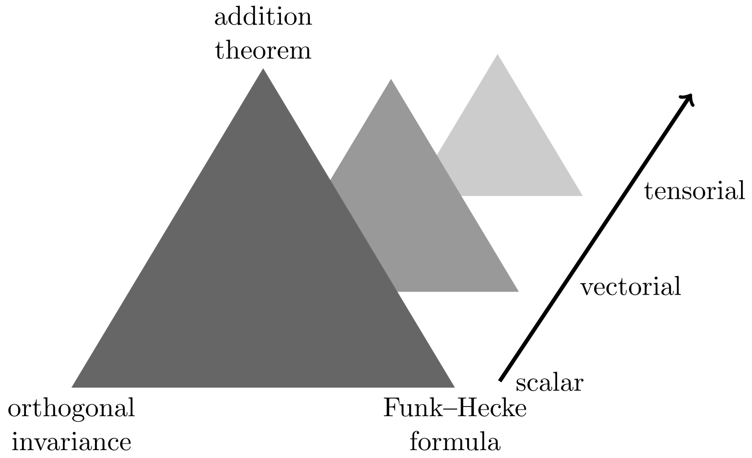
Space localization			
no space localization			ideal space localization
frequency localization			
ideal frequency localization			no frequency localization
kernel type			
Legendre kernel	bandlimited	locally supported	Dirac kernel
correlation			
ideal correlation			no correlation
integral transform			
Fourier	windowed Fourier		wavelet
resolution			
low			high

lies in the fact that zonal kernels with a variable (space localizing) support come into use. The multiscale transform using scaling (kernel) functions is a space-reflected replacement of the Fourier transform, however, giving the dynamical space-varying frequency distribution of a field. Due to the possibility that variable kernel functions (i.e., scaling functions as sequential space localizing reductions) are being applied, a substantial better modeling of the high-frequency 'short wavelength' part of a field (signal) is possible. This finally amounts to the transition from global to (scale-dependent) local approximation (including multiresolution by spherical wavelets).

1.2 Layout

Chapter 2 gives an introduction into spherical nomenclature and settings. Fundamental results of spherical vector analysis are recapitulated. Orthogonal invariance is explained within the scalar, vectorial, and tensorial concept (see Table 1.9).

Table 1.9: The fundamentals of the book.



In Chapter 3, the scalar surface theory of spherical harmonics is formulated based on the work of C. Müller (1952, 1966) and W. Freeden (1979a); W. Freeden (1980b). Important ingredients are the addition theorem of spherical harmonics and the formula of Funk and Hecke. The closure and completeness of scalar spherical harmonics in the space of square-integrable functions is shown by Bernstein or Abel-Poisson summability. Exact generation of linearly independent systems of homogeneous harmonic polynomials only by integer operations is investigated briefly. Fourier (orthogonal) expansions are discussed, (the energy of) a square-integrable function (signal) is split into degree variances in terms of spherical harmonics. The scalar spherical harmonics are recognized to be eigenfunctions of the scalar Beltrami operator on the (unit) sphere. The Legendre polynomial is identified as the only scalar spherical harmonic invariant under orthogonal transformations. Zonal, tesseral, and sectorial spherical harmonics, i.e., associated Legendre harmonics, are introduced by use of associated Legendre functions. Scalar angular derivatives are seen to produce anisotropic operators within the scalar framework.

Chapter 4 presents the theory of Green functions with respect to the scalar Beltrami operator (as proposed by W. Freeden (1979a); W. Freeden (1980b, 1981a)). Its definition is given by formulating four constituting properties, i.e., the Beltrami differential equation relating the Green function to the Dirac function(al), the characteristic logarithmic singularity, the rotational symmetry, and a certain normalization condition to assure uniqueness. Integral formulas are formulated that enable us to estimate the error between a (sufficiently smooth) function and its truncated orthogonal expansion in terms of scalar spherical harmonics. Integral expressions are deduced which act as solutions of the equations involving surface gradient, surface curl gradient, and (iterated) Beltrami differential operators. The results on Green functions are meant to be the preparatory material for decomposition theorems of spherical vector and tensor fields, respectively, in accordance with the Helmholtz approach. Iterated Beltrami equations are solved by integral expressions involving Green functions.

In Chapter 5, the vector theory of spherical harmonics is developed in consistency with its scalar counterpart (based on the work T. Gervens (1989), W. Freeden, T. Gervens (1989, 1991), W. Freeden et al. (1998)). A particular role is played by the Helmholtz decomposition theorem which separates a spherical vector field into three field components, namely a radial part, a tangential divergence-free, and a tangential curl-free part. As already pointed out, an essential tool for representing a spherical vector field is the Green function with respect to the Beltrami operator. The physical background for the Helmholtz decomposition is based on well-known facts of surface vector analysis, viz. the existence of surface potentials and stream functions, and the characterization of tangential vector fields such as surface (curl) gradient fields. To be more concrete, the surface gradient field on the sphere is seen to be generated by a potential function, while the surface curl gradient field is canonically related to a stream function. Vectorial analogues of the Legendre polynomials are introduced, their properties are analyzed in detail. Outstanding keystones in the vectorial framework of vector spherical harmonics are the addition theorem and the formulas of Funk and Hecke. The closure and completeness of vector spherical harmonics for the space of square-integrable vector fields is shown via Bernstein summability. Two different ways of expanding square-integrable fields in terms of (an orthonormal system of) vector spherical harmonics are described alternatively based on a (one-step) tensor-vector multiplication or on a consecutive (two-step) vector-scalar and scalar-vector multiplication.

Chapter 6 deals with the theory of tensor spherical harmonics (in close orientation to M. Schreiner (1994), W. Freeden et al. (1994, 1998)). All essential results known from the scalar and vectorial approach are extended to the tensor case. Orthonormal tensor spherical harmonics are introduced in the space of square-integrable tensor fields on the unit sphere. In particular, the addition theorem for tensor spherical harmonics is formulated and the decomposition theorem for spherical tensor fields is verified by use of the Green function with respect to iterations of the Beltrami operator. The tensor spherical harmonics are characterized as eigenfunctions of a tensorial analogue of the Beltrami operator. Alternative approaches to tensor spherical harmonics are studied. Tensorial versions of the Funk–Hecke formula are described in more detail.

Chapter 7 presents the mathematical classification of zonal kernel functions. The verification and interpretation of an uncertainty principle for fields (with second distributional derivatives) on the the unit sphere is the essential tool for the classification. Frequency as well as space localization are formulated by means of the expectation value and the variance of the surface curl gradient and the radial projection operator, respectively. The results obtained by certain tools of spherical vector analysis are used for a large class of band/spacelimited and non-band/spacelimited zonal kernel functions. The particular role of the Legendre kernel and the Dirac kernel is pointed out. The series expansions of vector/tensor zonal kernel functions in terms of (zonal) Legendre kernels are indicated by the specification of their symbols. All representations are coordinate-free.

Chapter 8 considers two different ways of generating vectorial and tensorial zonal kernel functions (cf. H. Nutz (2002)). In particular, scale-dependent bandlimited and non-bandlimited zonal kernel functions are listed such that the scale parameter acts as regulation for the amount of space/frequency localization. The Funk-Hecke formulas enable us to establish filtered versions of spherical fields by forming convolutions. The sequences of zonal kernel functions tending to the Dirac kernel, i.e., the so-called scaling functions, provide a 'zooming in' approximation of square-integrable fields from global to local features under (geophysically) constraints.

Chapter 9 presents the concept of tensorial zonal kernel functions. Their description is given in parallel to the vectorial case. Particular emphasis is laid on tensor scaling functions.

Finally, Chapter 10 is an application of our spherically oriented approach to geoscientifically relevant gravitation. The essential goal is to present the mathematical concepts, structures, and tools for the understanding of mass balance and mass transport seen in the closely interrelated Earth's gravity field. The key observables in gravitational field determination such as gravity anomalies, gravity disturbances, geoidal undulations, deflections of the vertical, dynamic ocean topography etc are mathematically characterized, both in terms of spherical harmonics and zonal kernel functions. The problems of determining the (geostrophic) ocean circulation, the elastic field from ground displacements, and the density distribution inside the Earth are studied in more detail. Finally, vector and tensor outer harmonic zonal kernels are shown to be the adequate means for ‘downward continuation’ of vectorial and tensorial gravitational data from satellite orbits to the Earth’s surface.

A brief view over the contents of the chapters of this book is given in Table 1.10.

Table 1.10: Contents (in brief).

	Scalar framework	**Vector framework**	**Tensor framework**
Basic settings (differential operators, orthogonal invariance)	Chapter 2	Chapter 2	Chapter 2
Green's functions, integral theorems	Chapter 4		
Spherical harmonics (definition, Legendre functions, addition theorems, Funk–Hecke formulas)	Chapter 3	Chapter 5	Chapter 6
Zonal kernel Functions (definition, classification, scaling functions, Dirac kernel)	Chapter 7	Chapter 8	Chapter 9
Applications (mass distribution interrelated to gravity field quantities)	Chapter 10	Chapter 10	Chapter 10

2 Basic Settings and Spherical Nomenclature

In this chapter, we start with some notation in the three-dimensional Euclidean space $\mathbb{R}^3$. The most important differential operators in $\mathbb{R}^3$ are listed. We give the representation of the gradient and the Laplace operator and split them into their radial and angular parts.

Certain differential operators on the unit sphere Ω in $\mathbb{R}^3$ are introduced, including the surface gradient, the surface curl gradient, the surface divergence, the surface curl, and the Beltrami operator. Although we rely on coordinate-free representations throughout this book (to avoid coordinate-implied singularities on the (global) sphere Ω), these operators will be discussed, for the convenience of the reader, in the particular system of spherical coordinates. Function spaces of scalar- and vector-valued functions on the unit sphere are characterized. Basic theorems on vector analysis are recapitulated in spherical language. Finally, we are concerned with basic results on spherical symmetry and orthogonal invariance in the scalar, vector, and tensor context, respectively.

2.1 Scalars, Vectors, and Tensors

The letters $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the set of positive, non-negative integers, integers, real numbers, and complex numbers, respectively. Let us use $x, y, \ldots$ to represent the elements of the Euclidean space $\mathbb{R}^3$. For all $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, different from the origin, we have

$$x = r\xi, \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The *unit sphere* in $\mathbb{R}^3$ is denoted by Ω:

$$\Omega = \left\{ \xi \in \mathbb{R}^3 \middle| |\xi| = 1 \right\}$$

If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$

$$\varepsilon^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.2}$$

we may represent the points $x \in \mathbb{R}^3$ in cartesian coordinates $x_i = x \cdot \varepsilon^i$, $i = 1, 2, 3$, by

$$x = \sum_{i=1}^{3} (x \cdot \varepsilon^i)\varepsilon^i = \sum_{i=1}^{3} x_i \varepsilon^i \ . \tag{2.3}$$

The *inner (scalar), vector,* and *dyadic (tensor) product* of two elements $x, y \in \mathbb{R}^3$, are defined by

$$x \cdot y = x^T y = \sum_{i=1}^{3} x_i y_i, \tag{2.4}$$

$$x \wedge y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T , \tag{2.5}$$

$$x \otimes y = x y^T = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}, \tag{2.6}$$

respectively. Clearly, $x^2 = |x|^2 = x \cdot x = x^T x$, $x \in \mathbb{R}^3$. Moreover, for $x, y \in \mathbb{R}^3$, we have the *Cauchy-Schwarz inequality*

$$|x \cdot y| \leq |x| \cdot |y| \tag{2.7}$$

and the *triangle inequality*

$$||x| - |y|| \leq |x \pm y| \leq |x| + |y|. \tag{2.8}$$

With the *alternator* (Levi–Cività alternating symbol)

$$\varepsilon^{ijk} = \begin{cases} +1 & \text{if} \quad (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if} \quad (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if} \quad (i,j,k) \text{ is not a permutation of } (1,2,3) \end{cases} \tag{2.9}$$

we obtain

$$(x \wedge y) \cdot \varepsilon^i = (x \wedge y)_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^{ijk} x_j y_k \ . \tag{2.10}$$

Moreover, we have

$$\sum_{i=1}^{3} \varepsilon^{ijk} \varepsilon^{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}, \tag{2.11}$$

where δ_{ij} is the *Kronecker delta*

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} . \tag{2.12}$$

As usual, a *tensor* $\mathbf{x} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ *of second rank (of rank 2 or second order)* is understood to be a linear mapping that assigns to each $x \in \mathbb{R}^3$ a vector $y \in \mathbb{R}^3$: $y = \mathbf{x}x$. The (cartesian) components $\mathbf{x}_{ij}$ of $\mathbf{x}$ are defined by

$$\mathbf{x}_{ij} = \varepsilon^i \cdot (\mathbf{x}\varepsilon^j) = (\varepsilon^i)^T(\mathbf{x}\varepsilon^j), \tag{2.13}$$

so that $y = \mathbf{x}x$ is equivalent to

$$y_i = y \cdot \varepsilon^i = \sum_{j=1}^{3} \mathbf{x}_{ij}(x \cdot \varepsilon^j) = \sum_{j=1}^{3} \mathbf{x}_{ij} x_j \quad . \tag{2.14}$$

The *inner product* $\mathbf{x} \cdot \mathbf{y}$ of two rank–2 tensors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ (also known as double dot product $\mathbf{x} : \mathbf{y}$) is defined by

$$\mathbf{x} \cdot \mathbf{y} = \mathrm{tr}(\mathbf{x}^T\mathbf{y}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{x}_{ij}\mathbf{y}_{ij}, \tag{2.15}$$

while

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} \tag{2.16}$$

is called the *norm* of $\mathbf{x} \in \mathbb{R}^3 \otimes \mathbb{R}^3$.

Given any tensor $\mathbf{x}$ and any pair x, $y \in \mathbb{R}^3$, we have

$$x \cdot (\mathbf{x}y) = \mathbf{x} \cdot (x \otimes y) \quad . \tag{2.17}$$

In connection with (2.17) it is easy to see that

$$(\varepsilon^i \otimes \varepsilon^j) \cdot (\varepsilon^k \otimes \varepsilon^l) = \delta_{ik}\ \delta_{jl}, \tag{2.18}$$

so that the nine tensors $\varepsilon^i \otimes \varepsilon^j$ are orthonormal. Moreover, it follows that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \left(\mathbf{x}_{ij}\varepsilon^i \otimes \varepsilon^j\right) x = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{x}_{ij}(x \cdot \varepsilon^j)\varepsilon^i = \mathbf{x}x. \tag{2.19}$$

Thus, $\mathbf{x} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ can be written in the form

$$\mathbf{x} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{x}_{ij}\varepsilon^i \otimes \varepsilon^j \quad . \tag{2.20}$$

The identity tensor $\mathbf{i}$ is given by

$$\mathbf{i} = \sum_{i=1}^{3} \varepsilon^i \otimes \varepsilon^i \ . \tag{2.21}$$

Moreover, we write $\mathrm{tr}(\mathbf{x})$ for the *trace* of $\mathbf{x}$ and $\det(\mathbf{x})$ for the *determinant* of $\mathbf{x}$. It is not hard to see that

$$\mathrm{tr}(x \otimes y) = x \cdot y, \quad x, y \in \mathbb{R}^3. \tag{2.22}$$

Furthermore,

$$\mathbf{x} \cdot (\mathbf{y}\mathbf{z}) = \left(\mathbf{y}^T \mathbf{x}\right) \cdot \mathbf{z} = \left(\mathbf{x}\mathbf{z}^T\right) \cdot \mathbf{y}, \qquad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3 \otimes \mathbb{R}^3 \ . \tag{2.23}$$

We write $\mathbf{x}^T$ for the *transpose* of $\mathbf{x}$; it is the unique tensor satisfying

$$(\mathbf{x}y) \cdot x = y \cdot \left(\mathbf{x}^T x\right), \tag{2.24}$$

for all $x, y \in \mathbb{R}^3$. We call $\mathbf{x}$ *symmetric* if $\mathbf{x} = \mathbf{x}^T$, and *skew* if $\mathbf{x} = -\mathbf{x}^T$. Every tensor $\mathbf{x}$ admits the unique decomposition

$$\mathbf{x} = \mathrm{sym}\ \mathbf{x} + \mathrm{skw}\ \mathbf{x}, \tag{2.25}$$

into the *symmetric part* sym $\mathbf{x}$ and the *skew part* skw $\mathbf{x}$. More explicitly,

$$\mathrm{sym}\ \mathbf{x} = \frac{1}{2}\left(\mathbf{x} + \mathbf{x}^T\right), \quad \mathrm{skw}\ \mathbf{x} = \frac{1}{2}\left(\mathbf{x} - \mathbf{x}^T\right) \ . \tag{2.26}$$

It should be noted that there is a one-to-one correspondence between vectors and skew tensors: Given any skew tensor $\mathbf{w}$, there exists a unique vector w such that $\mathbf{w}x = w \wedge x$ for every $x \in \mathbb{R}^3$; indeed,

$$w_i = -\frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^{ijk} \mathbf{w}_{jk} \ . \tag{2.27}$$

We call w the *axial vector* corresponding to $\mathbf{w}$. Conversely, given a vector w, there exists a unique skew tensor $\mathbf{w}$ such that the above relation holds; in fact,

$$\mathbf{w}_{ij} = -\sum_{k=1}^{3} \varepsilon^{ijk} w_k \ . \tag{2.28}$$

The dyadic (tensor) product $x \otimes y$ of two elements $x, y \in \mathbb{R}^3$ (see (2.6)) is the tensor that assigns to each $u \in \mathbb{R}^3$ the vector $(y \cdot u)x$. More explicitly,

$$(x \otimes y)u = (y \cdot u)x \tag{2.29}$$

for every $u \in \mathbb{R}^3$.

By use of the canonical orthonormal basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ of $\mathbb{R}^3$, a tensor $\mathbf{F}$ of rank k is written in the form

$$\mathbf{F} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1,\ldots,i_k} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k}, \quad F_{i_1,\ldots,i_k} \in \mathbb{R}, \tag{2.30}$$

and the set $\{\varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k}\}_{i_1,\ldots,i_k \in \{1,2,3\}}$ is an orthonormal basis of the linear space of all tensors of rank k.

The *scalar product* $\mathbf{F} \cdot \mathbf{G}$ of two tensors of rank k is defined by

$$\mathbf{F} \cdot \mathbf{G} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1,\ldots,i_k} G_{i_1,\ldots,i_k}, \tag{2.31}$$

and the *Euclidean norm* is

$$|\mathbf{F}| = (\mathbf{F} \cdot \mathbf{F})^{1/2}. \tag{2.32}$$

If $\mathbf{F} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1,\ldots,i_k} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k}$ and $\mathbf{G} = \sum_{i_1,\ldots,i_l}^{3} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_l}$ are tensors of rank k and l, respectively, then $\mathbf{F} \otimes \mathbf{G}$ is the tensor of rank $k+l$ defined by

$$\mathbf{F} \otimes \mathbf{G} = \sum_{i_1,\ldots,i_k=1}^{3} \sum_{j_1,\ldots,j_l=1}^{3} \mathbf{F}_{i_1,\ldots,i_k} \mathbf{G}_{j_1,\ldots,j_l} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_l}. \tag{2.33}$$

A tensor of rank two,

$$\mathbf{f} = \sum_{i,k=1}^{3} F_{i,k} \varepsilon^i \otimes \varepsilon^k, \tag{2.34}$$

can be viewed as a linear operator on vectors (tensors of rank one) $g = \sum_{i=1}^{3} G_i \varepsilon^i$ in the sense of

$$\mathbf{f} g = \sum_{i,k=1}^{3} F_{i,k} G_k \varepsilon^i. \tag{2.35}$$

Interpreting a tensor of rank four as a linear operator on tensors of rank two, we define

$$\mathbf{F}\mathbf{g} = \sum_{i,j,k,l=1}^{3} F_{i,j,k,l} G_{k,l} \varepsilon^i \otimes \varepsilon^j, \tag{2.36}$$

where

$$\mathbf{F} = \sum_{i,j,k,l=1}^{3} F_{i,j,k,l} \varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k \otimes \varepsilon^l \tag{2.37}$$

is a *tensor of rank four* (also called *rank-4 tensor*) and

$$\mathbf{g} = \sum_{k,l=1}^{3} G_{k,l}\varepsilon^k \otimes \varepsilon^l \tag{2.38}$$

is a *tensor of rank two (i.e., rank-2 tensor).*

As usual, we define the *product of two tensors of rank two*, $\mathbf{f} = \sum_{i,k=1}^{3} F_{i,k}\varepsilon^i \otimes \varepsilon^k$ and $\mathbf{g} = \sum_{k,j=1}^{3} G_{k,j}\varepsilon^k \otimes \varepsilon^j$, by

$$\mathbf{fg} = \sum_{i,j,k=1}^{3} F_{i,k}G_{k,j}\varepsilon^i \otimes \varepsilon^j. \tag{2.39}$$

Furthermore, the *product* of two rank-4 tensors

$$\mathbf{F} = \sum_{i,j,m,n=1}^{3} F_{i,j,k,l}\varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^m \otimes \varepsilon^n \tag{2.40}$$

and

$$\mathbf{G} = \sum_{m,n,k,l=1}^{3} G_{m,n,k,l}\varepsilon^m \otimes \varepsilon^n \otimes \varepsilon^k \otimes \varepsilon^l \tag{2.41}$$

is analogously defined by

$$\mathbf{FG} = \sum_{i,j,k,l,m,n=1}^{3} F_{i,j,m,n}G_{m,n,k,l}\varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k \otimes \varepsilon^l. \tag{2.42}$$

2.2 Differential Operators

If Γ is a set of points in $\mathbb{R}^3$, $\partial\Gamma$ will denote its *boundary.* The set $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ will be called the *closure* of Γ. A set $\Gamma \subset \mathbb{R}^3$ is called a *region* if and only if it is open and connected.

By a scalar, vector, or tensor function (field) on a region $\Gamma \subset \mathbb{R}^3$, we mean a function that assigns to each point of Γ, a scalar, vectorial, or tensorial function value, respectively. Unless otherwise specified, all fields are assumed to be real valued throughout this book. It will be of advantage to use the following general scheme of notations:

capital letters F, G	:	scalar functions,
lower-case letters f, g	:	vector fields,
boldface lower-case letters $\mathbf{f}, \mathbf{g}$	:	tensor fields of second rank,
boldface capital letters $\mathbf{F}, \mathbf{G}$	:	tensor fields of fourth rank.

The restriction of a scalar-valued function F, a vector-valued function f, or a tensor-valued function $\mathbf{f}$ to a subset M of its domain is denoted by $F|M$, $f|M$, or $\mathbf{f}|M$, respectively. For a set S of functions, we set $S|M = \{F|M \;\big|F \in S\}$.

Let $\Gamma \subset \mathbb{R}^3$ be a region. Suppose that $F : \Gamma \to \mathbb{R}$ is differentiable. $\nabla F : x \mapsto (\nabla F)(x)$, $x \in \Gamma$, denotes the *gradient* of F on Γ. The *partial derivatives* of F at $x \in \Gamma$, briefly written $F_{|i}$, $i \in \{1,2,3\}$, are given by

$$F_{|i}(x) = \frac{\partial F}{\partial x_i}(x) = (\nabla F)(x) \cdot \varepsilon^i = ((\nabla F)(x))_i \quad . \tag{2.43}$$

$\mathrm{L}F : x \mapsto \mathrm{L}F(x) = x \wedge (\nabla F)(x)$, $x \in \Gamma$, is called the *curl gradient* of F on Γ. We say that the scalar function $F : \Gamma \to \mathbb{R}$, the vector function $f : \Gamma \to \mathbb{R}^3$, and the tensor function $\mathbf{f} : \Gamma \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, is of class $\mathrm{C}^{(1)}$on Γ, $\mathrm{c}^{(1)}$ on Γ, and $\mathbf{c}^{(1)}$ on Γ, if $F, f, \mathbf{f}$, respectively, is differentiable at every point of Γ and $\nabla F, \nabla f, \nabla \mathbf{f}$, respectively, is continuous on Γ. The gradient of ∇F, ∇f, $\nabla \mathbf{f}$ is denoted by $\nabla^{(2)} F$, $\nabla^{(2)} f$, $\nabla^{(2)} \mathbf{f}$. Continuing in this manner, we say that $F, f, \mathbf{f}$, respectively, is of class $\mathrm{C}^{(n)}, \mathrm{c}^{(n)}, \mathbf{c}^{(n)}$ on $\Gamma, n \geq 1$ (briefly, $F \in \mathrm{C}^{(n)}(\Gamma)$, $f \in \mathrm{c}^{(n)}(\Gamma)$, $\mathbf{f} \in \mathbf{c}^{(n)}(\Gamma)$) if it is of class $\mathrm{C}^{(n-1)}, \mathrm{c}^{(n-1)}, \mathbf{c}^{(n-1)}$ and its $(n-1)$st gradient $\nabla^{(n-1)} F$, $\nabla^{(n-1)} f$,$\nabla^{(n-1)} \mathbf{f}$, respectively, is continuously differentiable (note that we usually write $\mathrm{C}, \mathrm{c}, \mathbf{c}$ instead of $\mathrm{C}^{(0)}, \mathrm{c}^{(0)}, \mathbf{c}^{(0)}$, respectively).

Obviously, the gradient of a differentiable scalar field is a vector field, while the gradient of a differentiable vector field is a tensor field, etc. We say that F is of class $\mathrm{C}^{(n)}$ on $\overline{\Gamma}$, $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ (briefly, $F \in \mathrm{C}^{(n)}(\overline{\Gamma})$), if F is of class $\mathrm{C}^{(n)}$ on Γ and, for each $k \in \{0, \ldots, n\}$, $\nabla^{(k)} F$ has a continuous extension to $\overline{\Gamma}$ (in this case, we also write $\nabla^{(n)} F$ for the extended function). Analogous definitions can be given for the vectorial and tensorial cases.

Let $u : \Gamma \to \mathbb{R}^3$ be a vector field, and suppose that u is differentiable at a point $x \in \Gamma$. The partial derivatives of u at $x \in \Gamma$ are given by

$$u_{i|j}(x) = \frac{\partial u_i}{\partial x_j}(x) = \varepsilon^i \cdot (\nabla u)(x) \varepsilon^j \quad . \tag{2.44}$$

Then, the *divergence* of u at $x \in \Gamma$ is the scalar value

$$\nabla_x \cdot u(x) = \mathrm{div}_x u(x) = \mathrm{tr}\ (\nabla u)(x) \quad . \tag{2.45}$$

Thus we have the identity

$$\nabla_x \cdot u(x) = \mathrm{div}_x u(x) = \sum_{i=1}^{3} u_{i|i}(x) \quad . \tag{2.46}$$

The *curl* of u at $x \in \Gamma$, denoted by

$$\mathrm{L}_x \cdot u(x) = \mathrm{curl}_x u(x),$$

is the unique vector with the property

$$\left((\nabla u)(x) - (\nabla u)(x)^T \right) a = (\mathrm{curl}_x u(x)) \wedge a = (\mathrm{L}_x \cdot u(x)) \wedge a \tag{2.47}$$

for every $a \in \mathbb{R}^3$. In components, we have

$$(\mathrm{L}_x \cdot u(x)) \cdot \varepsilon^i = \mathrm{curl}_x u(x) \cdot \varepsilon^i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^{ijk} u_{k|j}(x) \ . \tag{2.48}$$

We write $(\widehat{\nabla}_x u)(x)$ for the *symmetric gradient* of u given by

$$(\widehat{\nabla} u)(x) = \mathrm{sym}\ (\nabla u)(x) = \frac{1}{2} \left((\nabla u)(x) + (\nabla u)(x)^T \right) \ . \tag{2.49}$$

Let $\mathbf{f} : \Gamma \to \mathbb{R}^3 \otimes \mathbb{R}^3$ be a tensor field of second order, and suppose that $\mathbf{f}$ is differentiable at $x \in \Gamma$. The *partial derivatives* of $\mathbf{f}$ at $x \in \Gamma$ are given by

$$\mathbf{f}_{ij|k}(x) = \frac{\partial \mathbf{f}_{ij}}{\partial x_k}(x) = \varepsilon^i \cdot \left((\nabla \mathbf{f})(x) \varepsilon^k \right) \varepsilon^j, \tag{2.50}$$

Then the tensor field $\mathbf{f}^T : x \mapsto (\mathbf{f}(x))^T$, $x \in \Gamma$, is also differentiable at $x \in \Gamma$. The *divergence* of $\mathbf{f}$ at x, written by $\nabla_x \cdot \mathbf{f}(x) = \mathrm{div}_x \mathbf{f}(x)$, is the unique vector with the property

$$\begin{aligned} (\nabla_x \cdot \mathbf{f}(x)) \cdot a &= \mathrm{div}_x \mathbf{f}(x) \cdot a \\ &= \mathrm{div}_x \left(\mathbf{f}^T(x) a \right) = \nabla_x \cdot \left(\mathbf{f}^T(x) a \right) \end{aligned} \tag{2.51}$$

for every (fixed) vector $a \in \mathbb{R}^3$. In the same manner, we define the *curl* of $\mathbf{f}$ at x, written by $\mathrm{L}_x \cdot \mathbf{f}(x) = \mathrm{curl}_x \mathbf{f}(x)$, to be the unique tensor with the property

$$(\mathrm{L}_x \cdot \mathbf{f}(x))\, a = \mathrm{curl}_x \mathbf{f}(x) a = \mathrm{curl}_x \left(\mathbf{f}^T(x) a \right) = \mathrm{L}_x \cdot \left(\mathbf{f}^T(x) a \right) \tag{2.52}$$

for every (fixed) vector $a \in \mathbb{R}^3$. Clearly,

$$(\nabla_x \cdot \mathbf{f}(x))_i = \mathrm{div}_x \mathbf{f}(x) \cdot \varepsilon^i = \sum_{j=1}^{3} \mathbf{f}_{ij|j}(x), \tag{2.53}$$

$$\varepsilon^i \cdot \left(\mathrm{L}_x \cdot \mathbf{f}(x) \varepsilon^j \right) = \varepsilon^i \cdot \mathrm{curl}_x \mathbf{f}(x) \varepsilon^j = \sum_{p=1}^{3} \sum_{q=1}^{3} \varepsilon^{ipq} \mathbf{f}_{jq|p}(x) \ . \tag{2.54}$$

Let $F : \Gamma \to \mathbb{R}$ be a differentiable scalar field, and suppose that ∇F is differentiable at $x \in \Gamma$. Then we introduce the *Laplace operator (Laplacian)* of F at $x \in \Gamma$ by

$$\Delta_x F(x) = \mathrm{div}_x\left((\nabla F)(x)\right) = \nabla_x \cdot \left((\nabla F)(x)\right) \quad . \tag{2.55}$$

Analogously, we define the *Laplacian* of a vector field $f : \Gamma \to \mathbb{R}^3$ (with ∇f being differentiable at $x \in \Gamma$) by

$$\Delta_x f(x) = \mathrm{div}_x\left((\nabla f)(x)\right) = \nabla_x \cdot \left((\nabla f)(x)\right) \quad . \tag{2.56}$$

Clearly, for sufficiently often differentiable F, f,

$$\Delta_x F(x) = \sum_{i=1}^{3} F_{|i|i}(x), \tag{2.57}$$

$$\Delta_x f(x) \cdot \varepsilon^i = \sum_{j=1}^{3} f_{i|j|j}(x) \quad . \tag{2.58}$$

Finally, the Laplacian $\Delta_x \mathbf{f}(x)$ of a sufficiently smooth tensor field $\mathbf{f}$ is the unique tensor (of second order) with the property

$$(\Delta \mathbf{f})(x)a = \Delta_x(\mathbf{f}(x)a) \tag{2.59}$$

for every fixed $a \in \mathbb{R}^3$. In components,

$$\varepsilon^i \cdot (\Delta \mathbf{f})(x)\varepsilon^j = \sum_{q=1}^{3} \mathbf{f}_{ij|q|q}(x) \quad . \tag{2.60}$$

Of future interest are the following identities

$$\mathrm{L} \cdot \nabla F = \mathrm{curl}\ \nabla F = 0, \tag{2.61}$$

$$\nabla \cdot (\mathrm{L} \cdot u) = \mathrm{div\ curl}\ u = 0, \tag{2.62}$$

$$\mathrm{L} \cdot (\mathrm{L} \cdot u) = \mathrm{curl\ curl}\ u = \nabla \mathrm{div}\ u - \Delta u = \nabla(\nabla \cdot u) - \Delta u, \tag{2.63}$$

$$\mathrm{L} \cdot (\nabla u) = \mathrm{curl}\ \nabla u = 0, \tag{2.64}$$

$$\mathrm{L} \cdot \left(\nabla u^T\right) = \mathrm{curl}\ \left(\nabla u^T\right) = \nabla \mathrm{curl}\ u = \nabla(\mathrm{L} \cdot u), \tag{2.65}$$

$$\nabla u = -\nabla u^T \Rightarrow \nabla \nabla u = 0, \tag{2.66}$$

$$\nabla \cdot (\mathrm{L} \cdot \mathbf{f}) = \mathrm{div\ curl}\ \mathbf{f} = \mathrm{curl\ div}\ \mathbf{f}^T = \mathrm{L} \cdot \left(\nabla \cdot \mathbf{f}^T\right), \tag{2.67}$$

$$(\mathrm{L} \cdot (\mathrm{L} \cdot \mathbf{f}))^T = (\mathrm{curl\ curl}\ \mathbf{f})^T = \mathrm{curl\ curl}\ \mathbf{f}^T = \mathrm{L} \cdot \left(\mathrm{L} \cdot \mathbf{f}^T\right), \tag{2.68}$$

$$\nabla \cdot \left(\mathbf{f}^T u\right) = \mathrm{div}\ \left(\mathbf{f}^T u\right) = u \cdot \mathrm{div}\ \mathbf{f} + \mathbf{f} \cdot \nabla u = u \cdot (\nabla \cdot \mathbf{f}) + \mathbf{f} \cdot \nabla u, \tag{2.69}$$

provided that F is a scalar field, u is a vector field, and $\mathbf{f}$ is a tensor field, sufficiently often differentiable on Γ.

If Γ is a bounded region in three-dimensional Euclidean space with (smooth) boundary $\partial\Gamma$ and unit outward normal ν, then the *Gauss theorem* tells us that

$$\int_\Gamma \nabla_x F(x)\ dV(x) = \int_{\partial\Gamma} F(x)\nu(x)\ d\omega(x), \tag{2.70}$$

$$\int_\Gamma \nabla_x f(x)\ dV(x) = \int_{\partial\Gamma} f(x)\otimes\nu(x)\ d\omega(x), \tag{2.71}$$

$$\int_\Gamma \nabla_x \cdot f(x)\ dV(x) = \int_{\partial\Gamma} f(x)\cdot\nu(x)\ d\omega(x), \tag{2.72}$$

$$\int_\Gamma \mathrm{L}_x \cdot f(x)\ dV(x) = \int_{\partial\Gamma} \nu(x)\wedge f(x)\ d\omega(x), \tag{2.73}$$

$$\int_\Gamma \nabla_x \cdot \mathbf{f}(x)\ dV(x) = \int_{\partial\Gamma} \mathbf{f}(x)\nu(x)\ d\omega(x), \tag{2.74}$$

whenever the integrand on the left is continuously differentiable on $\overline{\Gamma} = \Gamma\cup\partial\Gamma$ (dV is the volume element, $d\omega$ is the surface element).

By letting $f = \nabla F$, $F \in \mathrm{C}^{(1)}(\overline{\Gamma})$, we obtain from (2.72)

$$\int_\Gamma \Delta_x F(x)\ dV(x) = \int_{\partial\Gamma} \frac{\partial F}{\partial\nu}(x)\ d\omega(x). \tag{2.75}$$

Consequently, for all functions $F \in \mathrm{C}^{(1)}(\overline{\Gamma}) \cup \mathrm{C}^{(2)}(\Gamma)$ satisfying the Laplace equation $\Delta F = 0$ in Γ, we have

$$\int_{\partial\Gamma} \frac{\partial F}{\partial\nu}(x)\ d\omega(x) = 0. \tag{2.76}$$

Furthermore, for all $f = F\nabla G$, $F \in \mathrm{C}^{(1)}(\overline{\Gamma})$, $G \in \mathrm{C}^{(2)}(\overline{\Gamma})$, we get

Theorem 2.1. *(First Green Theorem) For* $F \in \mathrm{C}^{(1)}(\overline{\Gamma}), G \in \mathrm{C}^{(2)}(\overline{\Gamma})$

$$\int_\Gamma (F(x)\Delta_x G(x) + \nabla_x F(x)\cdot\nabla_x G(x))\, dV(x) = \int_{\partial\Gamma} F(x)\frac{\partial G}{\partial\nu}(x)\ d\omega(x).$$

Taking $f = F\nabla G - G\nabla F$, $F, G \in \mathrm{C}^{(2)}(\overline{\Gamma})$, we obtain

Theorem 2.2. *(Second Green Theorem) For* $F, G \in \mathrm{C}^{(2)}(\overline{\Gamma})$

$$\int_\Gamma (F(x)\Delta_x G(x) - G(x)\Delta_x F(x))\ dV(x)$$
$$= \int_{\partial\Gamma} \left(F(x)\frac{\partial G}{\partial\nu}(x) - G(x)\frac{\partial F}{\partial\nu}(x)\right)\ d\omega(x).$$

For all $x \in \mathbb{R}^3 \backslash \{y\}, x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3)^T$, an easy calculation shows us that

$$\frac{1}{|x-y|} = \frac{1}{\left(\sum_{k=1}^{3} (x_k - y_k)^2\right)^{1/2}}, \tag{2.77}$$

$$\frac{\partial}{\partial x_k} \frac{1}{|x-y|} = -\frac{x_k - y_k}{|x-y|^3}, \ k = 1, 2, 3, \tag{2.78}$$

and

$$\frac{\partial}{\partial x_k \partial x_j} \frac{1}{|x-y|} = \frac{3(x_k - y_k)(x_j - y_j) - \delta_{kj}|x-y|^2}{|x-y|^5}, \tag{2.79}$$

$j, k = 1, 2, 3$. In other words, for all $x \in \mathbb{R}^3$, $x \neq y$, we have

$$\Delta_x \frac{1}{|x-y|} = 0, \tag{2.80}$$

i.e., $x \mapsto |x-y|^{-1}$, $x \neq y$, is a radial-symmetric solution of the Laplace equation in $\mathbb{R}^3 \backslash \{y\}$. In potential theory, it is called the *fundamental solution of* Δ.

Suppose that y is an element of Γ. Then, for all sufficiently small $\varepsilon > 0$, the Second Green Theorem (Theorem 2.2) gives us

$$\begin{aligned} \int_{\substack{x \in \Gamma \\ |x-y| \geq \varepsilon}} & \left(F(x) \underbrace{\Delta_x \frac{1}{|x-y|}}_{=0} - \frac{1}{|x-y|} \Delta_x F(x) \right) dV(x) \\ = & \int_{x \in \partial\Gamma} \left(F(x) \frac{\partial}{\partial \nu_x} \frac{1}{|x-y|} - \frac{1}{|x-y|} \frac{\partial F}{\partial \nu}(x) \right) d\omega(x) \\ + & \int_{\substack{|x-y| = \varepsilon \\ x \in \overline{\Gamma}}} \left(F(x) \frac{\partial}{\partial \nu_x} \frac{1}{|x-y|} - \frac{1}{|x-y|} \frac{\partial F}{\partial \nu}(x) \right) d\omega(x) \end{aligned} \tag{2.81}$$

provided that F is of class $\mathrm{C}^{(2)}(\overline{\Gamma})$. Now, because of the continuity of $\frac{\partial F}{\partial \nu}$, we find

$$\begin{aligned} \left| \int_{\substack{|x-y| = \varepsilon \\ x \in \overline{\Gamma}}} \frac{1}{|x-y|} \frac{\partial F}{\partial \nu}(x) \, d\omega(x) \right| & \leq \frac{1}{\varepsilon} \int_{\substack{|x-y| = \varepsilon \\ x \in \overline{\Gamma}}} \left| \frac{\partial F}{\partial \nu}(x) \right| d\omega(x) \\ & \leq \frac{C}{\varepsilon} 4\pi\varepsilon^2 \stackrel{\varepsilon \to 0}{\longrightarrow} 0. \end{aligned} \tag{2.82}$$

Moreover, we have

$$\begin{aligned}\int_{\substack{|x-y|=\varepsilon\\ x\in\overline{\Gamma}}} & F(x)\frac{\partial}{\partial\nu_x}\frac{1}{|x-y|}\, d\omega(x) \\ &= -\int_{\substack{|x-y|=\varepsilon\\ x\in\overline{\Gamma}}} F(x)\ \nu(x)\cdot\frac{x-y}{|x-y|^3}\, d\omega(x) \\ &= \frac{1}{\varepsilon^2}\int_{\substack{|x-y|=\varepsilon\\ x\in\overline{\Gamma}}} F(x)\ d\omega(x).\end{aligned} \tag{2.83}$$

The Mean Value Theorem allows us to write

$$\int_{\substack{|x-y|=\varepsilon\\ x\in\overline{\Gamma}}} F(x)\ d\omega(x) = 4\pi F(\overline{x}_\varepsilon)\varepsilon^2, \tag{2.84}$$

where $\overline{x}_\varepsilon$ is a point on the sphere in $\mathbb{R}^3$ with center y and radius ε. Observing the continuity of F, we are able to deduce that $F(\overline{x}_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} F(y)$ as $\overline{x}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} y$. Thus we see that

$$\lim_{\varepsilon\to 0}\int_{\substack{|x-y|=\varepsilon\\ x\in\overline{\Gamma}}} F(x)\frac{\partial}{\partial\nu_x}\frac{1}{|x-y|}d\omega(x) = 4\pi F(y), \quad y\in\Gamma. \tag{2.85}$$

Remark 2.3. Similar arguments apply to the cases $y \in \partial\Gamma$ and $y \notin \overline{\Gamma}$.

Summarizing our results we finally obtain the following theorem.

Theorem 2.4. *(Third Green Theorem) Let F be of class $\mathrm{C}^{(2)}(\overline{\Gamma})$. Then*

$$\begin{aligned}\int_\Gamma & \frac{1}{|x-y|}\Delta_x F(x)\ dV(x) \\ &- \int_{\partial\Gamma}\left(\frac{1}{|x-y|}\frac{\partial F}{\partial\nu}(x) - F(x)\frac{\partial}{\partial\nu_x}\frac{1}{|x-y|}\right) d\omega(x) \\ &= \begin{cases} -4\pi F(y) & , \quad y\in\Gamma, \\ -2\pi F(y) & , \quad y\in\partial\Gamma, \\ 0 & , \quad y\notin\overline{\Gamma}. \end{cases}\end{aligned} \tag{2.86}$$

2.3 Spherical Notation

As already mentioned, the *unit sphere* in $\mathbb{R}^3$ is denoted by Ω:

$$\Omega = \left\{\xi\in\mathbb{R}^3 \middle| |\xi| = 1\right\}. \tag{2.87}$$

We set Ω^{int} for the 'inner space' of Ω, while Ω^{ext} denotes the 'outer space' of Ω. More explicitly,

$$\Omega^{\text{int}} = \left\{x \in \mathbb{R}^3 \middle| \; |x| \; < 1\right\}, \tag{2.88}$$
$$\Omega^{\text{ext}} = \{x \in \mathbb{R}^3 | \; |x| \; > 1\}. \tag{2.89}$$

The sphere in $\mathbb{R}^3$ with radius R around the origin will be denoted by Ω_R:

$$\Omega_R = \left\{x \in \mathbb{R}^3 \middle| |x| = R\right\}. \tag{2.90}$$

We set Ω_R^{int} for the 'inner space' of Ω_R, while Ω_R^{ext} denotes the 'outer space' of Ω_R:

$$\Omega_R^{\text{int}} = \left\{x \in \mathbb{R}^3 \middle| \; |x| \; < R\right\}, \tag{2.91}$$
$$\Omega_R^{\text{ext}} = \{x \in \mathbb{R}^3 | \; |x| \; > R\}. \tag{2.92}$$

It is well known that the total surface $\|\Omega_R\|$ of Ω_R is equal to $4\pi R^2$:

$$\|\Omega_R\| = \int_{\Omega_R} d\omega(\xi) = 4\pi R^2. \tag{2.93}$$

We may represent the points $x \in \mathbb{R}^3$, $x = r\xi$, $\xi \in \Omega$ in *polar coordinates* as follows (see Fig. 2.1):

$$\begin{aligned} x &= r\xi, \; r = |x|, \\ \xi &= t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2), \\ & \quad -1 \le t \le 1 \; , \; 0 \le \varphi < 2\pi \; , \; t = \cos\vartheta \; , \end{aligned} \tag{2.94}$$

($\vartheta \in [0, \pi]$: (co-)latitude, φ: longitude, t: polar distance), i.e.,

$$\xi = (\sin\vartheta \; \cos\varphi, \; \sin\vartheta \; \sin\varphi, \; \cos\vartheta)^T \; . \tag{2.95}$$

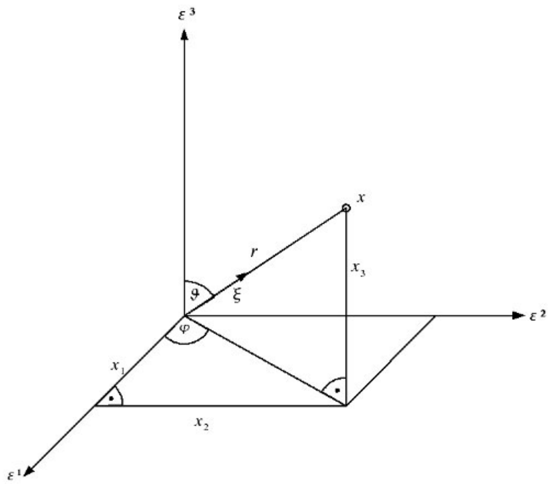


Fig. 2.1: Polar coordinates in three-dimensional Euclidean space $\mathbb{R}^3$.

More explicitly,

$$\xi = (\sin\vartheta_\xi \cos\varphi_\xi, \sin\vartheta_\xi \sin\varphi_\xi, \cos\vartheta_\xi)^T . \tag{2.96}$$

The scalar product between two unit vectors ξ and η reads as follows:

$$\begin{aligned} \eta \cdot \xi &= \sin\vartheta_\eta \cos\varphi_\eta \sin\vartheta_\xi \cos\varphi_\xi \\ &\quad + \sin\vartheta_\eta \sin\varphi_\eta \sin\vartheta_\xi \sin\varphi_\xi \\ &\quad + \cos\vartheta_\eta \cos\vartheta_\xi \\ &= (\cos\varphi_\eta \cos\varphi_\xi + \sin\varphi_\eta \sin\varphi_\xi) \sin\vartheta_\eta \sin\vartheta_\xi + \cos\vartheta_\eta \cos\vartheta_\xi \\ &= \cos(\varphi_\eta - \varphi_\xi) \sin\vartheta_\eta \sin\vartheta_\xi + \cos\vartheta_\eta \cos\vartheta_\xi \\ &= \cos(\varphi_\eta - \varphi_\xi)\sqrt{1-t_\eta^2}\,\sqrt{1-t_\xi^2} + t_\eta t_\xi . \end{aligned} \tag{2.97}$$

2.4 Function Spaces

The set of scalar functions $F : \Omega \to \mathbb{R}$ which are measurable and for which

$$\|F\|_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |F(\xi)|^p \ d\omega(\xi) \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, \tag{2.98}$$

is known as $\mathrm{L}^p(\Omega)$. Clearly, $\mathrm{L}^p(\Omega) \subset \mathrm{L}^q(\Omega)$ for $1 \leq q < p$. A function $F : \Omega \to \mathbb{R}$ possessing k continuous derivatives on the unit sphere Ω is said to be of class $\mathrm{C}^{(k)}(\Omega)$, $(0 \leq k \leq \infty)$. $\mathrm{C}(\Omega)$ $(= \mathrm{C}^{(0)}(\Omega))$ is the class of continuous scalar-valued functions on Ω. $\mathrm{C}(\Omega)$ is a complete normed space endowed with

$$\|F\|_{\mathrm{C}(\Omega)} = \sup_{\xi \in \Omega} |F(\xi)|. \tag{2.99}$$

By $\mu(F;\delta)$, we denote the *modulus of continuity* of the function $F \in \mathrm{C}(\Omega)$

$$\mu(F;\delta) = \max_{\xi,\zeta \in \Omega; 1-\xi\cdot\zeta \leq \delta} |F(\xi) - F(\zeta)|, \quad 0 < \delta < 2. \tag{2.100}$$

A function $F : \Omega \to \mathbb{R}$ is said to be *Lipschitz-continuous* if there exists a (Lipschitz) constant $C_F > 0$ such that the inequality

$$|F(\xi) - F(\eta)| \leq C_F |\xi - \eta| = \sqrt{2} C_F \sqrt{1 - \xi \cdot \eta} \tag{2.101}$$

holds for all $\xi, \eta \in \Omega$. The class of all Lipschitz-continuous functions on Ω is denoted by $\mathrm{Lip}(\Omega)$. Clearly, $\mathrm{C}^{(1)}(\Omega) \subset \mathrm{Lip}(\Omega)$.

$\mathrm{L}^2(\Omega)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$ defined by

$$(F,G)_{\mathrm{L}^2(\Omega)} = \int_\Omega F(\xi) G(\xi) \, d\omega(\xi), \quad F, G \in \mathrm{L}^2(\Omega). \tag{2.102}$$

In connection with $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$, $\mathrm{C}(\Omega)$ is a pre-Hilbert space. For each $F \in \mathrm{C}(\Omega)$ we have the *norm estimate*

$$\|F\|_{\mathrm{L}^2(\Omega)} \leq \sqrt{4\pi} \, \|F\|_{\mathrm{C}(\Omega)}. \tag{2.103}$$

$\mathrm{L}^2(\Omega)$ is the completion of $\mathrm{C}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Omega)}$, i.e.,

$$\mathrm{L}^2(\Omega) = \overline{\mathrm{C}(\Omega)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}}. \tag{2.104}$$

Any function of the form

$$G_\xi : \Omega \to \mathbb{R}, \eta \mapsto G_\xi(\eta) = G(\xi \cdot \eta), \quad \eta \in \Omega, \tag{2.105}$$

is called a *ξ-zonal function* on Ω (or ξ-axial radial basis function). Zonal functions are constant on the sets

$$\Omega(\xi; h) = \{\eta \in \Omega | \xi \cdot \eta = h\}, \ h \in [-1, 1]. \tag{2.106}$$

The set of all ξ-zonal functions is isomorphic to the set of functions $G : [-1,1] \to \mathbb{R}$. This allows us to interpret $\mathrm{C}[-1,1]$ and $\mathrm{L}^p[-1,1]$ (with norms defined correspondingly) as subspaces of $\mathrm{C}(\Omega)$ and $\mathrm{L}^p(\Omega)$. Obviously,

$$\|G\|_{\mathrm{C}[-1,1]} = \|G(\varepsilon^3 \cdot)\|_{\mathrm{C}(\Omega)}, \tag{2.107}$$

and we define

$$\begin{aligned} \|G\|_{\mathrm{L}^p[-1,1]} &= \|G(\varepsilon^3 \cdot)\|_{\mathrm{L}^p(\Omega)} \\ &= \left(\int_\Omega |G(\eta \cdot \varepsilon^3)|^p \, d\omega(\eta) \right)^{1/p} \\ &= \left(2\pi \int_{-1}^{1} |G(t)|^p \, dt \right)^{1/p} . \end{aligned} \tag{2.108}$$

Analogously, we define the inner product in $\mathrm{L}^2[-1,1]$ by

$$(F,G)_{\mathrm{L}^2[-1,1]} = 2\pi \int_{-1}^{1} F(t)G(t) \, dt, \tag{2.109}$$

$F, G \in \mathrm{L}^2[-1,1]$.

Next, we give some preliminaries for the study of vector fields defined on the unit sphere Ω. Using the canonical orthonormal basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ of $\mathbb{R}^3$, we may write any vector field $f : \Omega \to \mathbb{R}^3$ in the form

$$f(\xi) = \sum_{i=1}^{3} F_i(\xi)\varepsilon^i, \quad \xi \in \Omega, \tag{2.110}$$

where the component functions F_i are given by $F_i(\xi) = f(\xi) \cdot \varepsilon^i, \xi \in \Omega$.

$\mathrm{l}^2(\Omega)$ denotes the space consisting of all square-integrable vector fields on Ω. In connection with the inner product

$$(f,g)_{\mathrm{l}^2(\Omega)} = \int_\Omega f(\xi) \cdot g(\xi) \, d\omega(\xi), \quad f,g \in \mathrm{l}^2(\Omega), \tag{2.111}$$

$\mathrm{l}^2(\Omega)$ is a Hilbert space. The space $\mathrm{c}^{(p)}(\Omega)$, $0 \le p \le \infty$, consists of all p-times continuously differentiable vector fields on Ω. For brevity, we usually write $\mathrm{c}(\Omega) = \mathrm{c}^{(0)}(\Omega)$. The space $\mathrm{c}(\Omega)$ is complete with respect to the norm

$$\|f\|_{\mathrm{c}(\Omega)} = \sup_{\xi \in \Omega} |f(\xi)|, \quad f \in \mathrm{c}(\Omega). \tag{2.112}$$

Furthermore,

$$\overline{\mathrm{c}(\Omega)}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}} = \mathrm{l}^2(\Omega). \tag{2.113}$$

In analogy to (2.103), we have for all $f \in \mathrm{c}(\Omega)$ the *norm estimate*

$$\|f\|_{\mathrm{l}^2(\Omega)} \leq \sqrt{4\pi}\, \|f\|_{\mathrm{c}(\Omega)}. \tag{2.114}$$

The generalization of the preceding settings to tensor fields of rank two is straightforward. A tensor field is said to be of class $\mathbf{c}^{(k)}(\Omega)$, $0 \leq k \leq \infty$, if its component functions with respect to the basis $\{\varepsilon^i \otimes \varepsilon^k\}_{i,k=1,2,3}$ are in $\mathrm{C}^{(k)}(\Omega)$. The space $\mathbf{c}(\Omega)(= \mathbf{c}^{(0)}(\Omega))$ equipped with the norm $\|\cdot\|_{\mathbf{c}(\Omega)}$ defined by

$$\|\mathbf{f}\|_{\mathbf{c}(\Omega)} = \sup_{\xi \in \Omega} |\mathbf{f}(\xi)|, \quad \mathbf{f} \in \mathbf{c}(\Omega), \tag{2.115}$$

is a Banach space. By $\mathbf{l}^2(\Omega)$, we denote the Hilbert space of square-integrable tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ with inner product

$$(\mathbf{f}, \mathbf{g})_{\mathbf{l}^2(\Omega)} = \int_\Omega \mathbf{f}(\xi) \cdot \mathbf{g}(\xi)\, d\omega(\xi), \quad \mathbf{f}, \mathbf{g} \in \mathbf{l}^2(\Omega), \tag{2.116}$$

and associated norm $\|\cdot\|_{\mathbf{l}^2(\Omega)}$. The space $\mathbf{l}^2(\Omega)$ is the completion of $\mathbf{c}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathbf{l}^2(\Omega)}$.

2.5 Differential Calculus

In order to introduce a system of triads on spheres, we define the vector function

$$\Phi : [0, \infty) \times [0, 2\pi) \times [-1, 1] \to \mathbb{R}^3 \tag{2.117}$$

by

$$\Phi(r, \varphi, t) = \begin{pmatrix} r\sqrt{1-t^2}\cos\varphi \\ r\sqrt{1-t^2}\sin\varphi \\ rt \end{pmatrix}. \tag{2.118}$$

Setting $r = 1$ we already know that a local coordinate system is obtainable on the unit sphere. In other words, instead of denoting any element of Ω by its vectorial representation ξ, we may also use its coordinates (φ, t) in accordance with (2.94). Calculating the derivatives of Φ and setting $r = 1$, the corresponding set of orthonormal unit vectors in the directions r, φ, and

t is easily determined to be

$$\varepsilon^r(\varphi,t) = \begin{pmatrix} \sqrt{1-t^2}\cos\varphi \\ \sqrt{1-t^2}\sin\varphi \\ t \end{pmatrix}, \tag{2.119}$$

$$\varepsilon^\varphi(\varphi,t) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \tag{2.120}$$

$$\varepsilon^t(\varphi,t) = \begin{pmatrix} -t\cos\varphi \\ -t\sin\varphi \\ \sqrt{1-t^2} \end{pmatrix} . \tag{2.121}$$

Obviously,

$$\varepsilon^t(\varphi,t) = \varepsilon^r(\varphi,t) \wedge \varepsilon^\varphi(\varphi,t) \ . \tag{2.122}$$

The vectors ε^φ and ε^t mark the tangential directions. Since we associate ξ with its representations using the local coordinates φ and t, we identify $\varepsilon^r(\xi)$ with $\varepsilon^r(\varphi,t)$, etc (cf. Fig. 2.2).

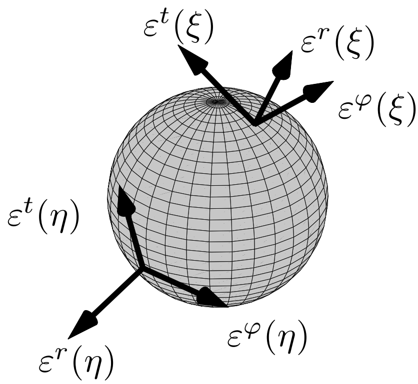


Fig. 2.2: The local triad ε^r, ε^φ, ε^t with respect to two different points ξ and η on the unit sphere.

From (2.119) to (2.121), we immediately obtain a representation of the cartesian unit vectors in terms of the spherical ones:

$$\varepsilon^1 = \sqrt{1-t^2}\cos\varphi\varepsilon^r(\varphi,t) - \sin\varphi\varepsilon^\varphi(\varphi,t) - t\cos\varphi\varepsilon^t(\varphi,t), \tag{2.123}$$

$$\varepsilon^2 = \sqrt{1-t^2}\sin\varphi\varepsilon^r(\varphi,t) + \cos\varphi\varepsilon^\varphi(\varphi,t) - t\sin\varphi\varepsilon^t(\varphi,t), \tag{2.124}$$

$$\varepsilon^3 = t\varepsilon^r(\varphi,t) + \sqrt{1-t^2}\varepsilon^t(\varphi,t) \ . \tag{2.125}$$

The system $\{\varepsilon^\varphi, \varepsilon^t\}$ enables us to formulate a *vector differential calculus.*

Gradient fields ∇F can be decomposed into a radial and a tangential component. More explicitly, the *surface gradient* ∇^* contains the tangential derivatives of the gradient ∇ as follows:

$$\nabla = \varepsilon^r \frac{\partial}{\partial r} + \frac{1}{r}\nabla^*. \tag{2.126}$$

Letting $x = r\xi$, $r = |x|$, $\xi \in \Omega$, we find with $\eta \in \Omega$

$$\nabla_x (x \cdot \eta) = \eta = \varepsilon^r (\xi \cdot \eta) + \nabla^*_\xi (\xi \cdot \eta), \tag{2.127}$$

such that

$$\nabla^*_\xi (\xi \cdot \eta) = \eta - (\xi \cdot \eta)\xi. \tag{2.128}$$

The *surface curl gradient* L^* is defined by

$$\mathrm{L}^*_\xi F(\xi) = \xi \wedge \nabla^*_\xi F(\xi), \qquad \xi \in \Omega, \tag{2.129}$$

$F \in \mathrm{C}^{(1)}(\Omega)$. According to its definition (2.129), $\mathrm{L}^* F$ is a tangential vector field perpendicular to $\nabla^* F$, i.e.,

$$\nabla^*_\xi F(\xi) \cdot \mathrm{L}^*_\xi F(\xi) = 0, \quad \xi \in \Omega. \tag{2.130}$$

$\nabla^* \cdot = \mathrm{div}^*$ and $\mathrm{L}^* \cdot = \mathrm{curl}^*$, respectively, denote the *surface divergence* and the *surface curl* given by

$$\nabla^*_\xi \cdot f(\xi) = \sum_{i=1}^{3} \nabla^*_\xi F_i(\xi) \cdot \varepsilon^i \tag{2.131}$$

and

$$\mathrm{L}^*_\xi \cdot f(\xi) = \sum_{i=1}^{3} \mathrm{L}^*_\xi F_i(\xi) \cdot \varepsilon^i \ . \tag{2.132}$$

Note that the surface curl as defined by (2.132), i.e.,

$$\xi \mapsto \mathrm{L}^*_\xi \cdot f(\xi) = \mathrm{curl}^*_\xi f(\xi) = \mathrm{div}^*_\xi (f(\xi) \wedge \xi) = \nabla^*_\xi \cdot (f(\xi) \wedge \xi), \quad \xi \in \Omega, \tag{2.133}$$

represents a scalar-valued function on the unit sphere Ω in $\mathbb{R}^3$.

The aforementioned relations can be understood from the well-known role of the Beltrami operator Δ^* in the representation of the *Laplace operator* Δ:

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi \ . \tag{2.134}$$

In spherical coordinates, the operators $\Delta^*, \mathrm{L}^*, \nabla^*$, respectively, read as follows:

$$\Delta^*_\xi = \frac{\partial}{\partial t}\left(1 - t^2\right)\frac{\partial}{\partial t} + \frac{1}{1 - t^2}\left(\frac{\partial}{\partial \varphi}\right)^2, \tag{2.135}$$

$$\nabla^*_\xi = \varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi} + \varepsilon^t \sqrt{1-t^2} \frac{\partial}{\partial t}, \tag{2.136}$$

$$\mathrm{L}^*_\xi = -\varepsilon^\varphi \sqrt{1-t^2} \frac{\partial}{\partial t} + \varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi} \ . \tag{2.137}$$

An easy calculation using (2.119)–(2.121) shows that

$$\begin{aligned} \nabla^*_\xi &= \frac{1}{\sqrt{1-t^2}} \left(-\sin\varphi \varepsilon^1 + \cos\varphi \varepsilon^2\right) \frac{\partial}{\partial \varphi} \\ &\quad + \sqrt{1-t^2} \left(-t \cos\varphi \varepsilon^1 - t \sin\varphi \varepsilon^2 + \sqrt{1-t^2} \varepsilon^3\right) \frac{\partial}{\partial t}, \end{aligned} \tag{2.138}$$

$$\begin{aligned} \mathrm{L}^*_\xi &= \sqrt{1-t^2} \left(\sin\varphi \varepsilon^1 - \cos\varphi \varepsilon^2\right) \frac{\partial}{\partial t} \\ &\quad + \frac{1}{\sqrt{1-t^2}} \left(-t \cos\varphi \varepsilon^1 - t \sin\varphi \varepsilon^2 + \sqrt{1-t^2} \varepsilon^3\right) \frac{\partial}{\partial \varphi}. \end{aligned} \tag{2.139}$$

For the convenience of the reader, a list of the spherical operators is included (see Table 2.1).

Table 2.1: Spherical differential operators.

Symbol	**Differential Operator**
∇^*_ξ	surface gradient at ξ
$\mathrm{L}^*_\xi = \xi \wedge \nabla^*_\xi$	surface curl gradient at ξ
$\nabla^*_\xi \cdot$	surface divergence at ξ
$\mathrm{L}^*_\xi \cdot$	surface curl at ξ
$\Delta^*_\xi = \nabla^*_\xi \cdot \nabla^*_\xi$	Beltrami operator at ξ
$\Delta^*_\xi = \mathrm{L}^*_\xi \cdot \mathrm{L}^*_\xi$	Beltrami operator at ξ

It should be mentioned that the operators $\nabla^*, \mathrm{L}^*, \Delta^*$ will always be used here in coordinate-free representation, thereby avoiding any singularity at the poles.

Since the operators ∇^*, L^* and $\nabla^* \cdot, \mathrm{L}^* \cdot$ are of particular interest throughout this work, we list some of their properties: If $\xi \in \Omega$, then

$$\begin{aligned}
\nabla^*_\xi \cdot \nabla^*_\xi F(\xi) &= \Delta^*_\xi F(\xi), && (2.140)\\
\mathrm{L}^*_\xi \cdot \mathrm{L}^*_\xi F(\xi) &= \Delta^*_\xi F(\xi), && (2.141)\\
\nabla^*_\xi \cdot \mathrm{L}^*_\xi F(\xi) &= 0, && (2.142)\\
\mathrm{L}^*_\xi \cdot \nabla^*_\xi F(\xi) &= 0, && (2.143)\\
\nabla^*_\xi F(\xi) \cdot \mathrm{L}^*_\xi F(\xi) &= 0, && (2.144)\\
\nabla^*_\xi \cdot (F(\xi) f(\xi)) &= (\nabla^*_\xi F(\xi)) \cdot f(\xi) + F(\xi)(\nabla^*_\xi \cdot f(\xi)), && (2.145)\\
\nabla^*_\xi \cdot \xi &= 2, && (2.146)
\end{aligned}$$

$\xi \in \Omega$. Moreover, we have

$$\nabla^*_\xi \wedge (F(\xi) f(\xi)) = \nabla^*_\xi F(\xi) \wedge f(\xi) + F(\xi) \nabla^*_\xi \wedge f(\xi), \ \xi \in \Omega. \qquad (2.147)$$

For a given function $F \in \mathrm{C}^{(1)}(\Omega)$, the triple $F(\xi)\xi, \nabla^*_\xi F(\xi), \mathrm{L}^*_\xi F(\xi)$, $\xi \in \Omega$, supplies us with a system of three orthogonal vectors at each point $\xi \in \Omega$, provided that $F(\xi) \neq 0$ and $\nabla^*_\xi F(\xi) \neq 0$.

Let $\eta \in \Omega$ be fixed, then it is not difficult to see (cf. (2.127)) that, for $\xi \in \Omega$,

$$\begin{aligned}
\nabla^*_\xi (\xi \cdot \eta) &= \eta - (\xi \cdot \eta)\xi, && (2.148)\\
\mathrm{L}^*_\xi (\xi \cdot \eta) &= \xi \wedge \nabla^*_\xi (\xi \cdot \eta) = \xi \wedge \eta, && (2.149)
\end{aligned}$$

and

$$\Delta^*_\xi (\xi \cdot \eta) = -2(\xi \cdot \eta). \qquad (2.150)$$

More generally, if F is of class $\mathrm{C}^{(1)}[-1,1]$ and $F' \in \mathrm{C}[-1,1]$ is its (one-dimensional) derivative, then

$$\begin{aligned}
\nabla^*_\xi F(\xi \cdot \eta) &= F'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi), && (2.151)\\
\mathrm{L}^*_\xi F(\xi \cdot \eta) &= F'(\xi \cdot \eta)(\xi \wedge \eta), && (2.152)
\end{aligned}$$

whereas, for $F \in \mathrm{C}^{(2)}[-1,1]$,

$$\Delta^*_\xi F(\xi \cdot \eta) = -2(\xi \cdot \eta) F'(\xi \cdot \eta) + (1 - (\xi \cdot \eta)^2) F''(\xi \cdot \eta). \qquad (2.153)$$

2.6 Integral Calculus

Having formulated the development of a vector differential calculus, we now come to the *integral calculus*: Let Γ be a subset of Ω with (sufficiently smooth) boundary curve $\partial\Gamma$ (see Fig. 2.3). Moreover, denote by ν and τ unit surface vectors normal (outward of Γ) and tangential to $\partial\Gamma$, respectively. Let σ denote the arc length along $\partial\Gamma$.

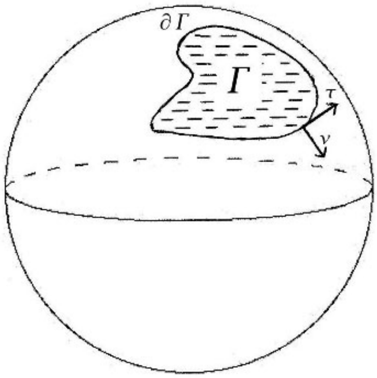


Fig. 2.3: Γ as subset of the unit sphere Ω.

Then the *surface theorem of Gauß* reads

$$\int_\Gamma \nabla^*_\xi \cdot f(\xi)\, d\omega(\xi) = \int_{\partial\Gamma} \nu_\xi \cdot f(\xi)\, d\sigma(\xi), \tag{2.154}$$

while the *surface theorem of Stokes* takes the form

$$\int_\Gamma \mathrm{L}^*_\xi \cdot f(\xi)\, d\omega(\xi) = \int_{\partial\Gamma} \tau_\xi \cdot f(\xi)\, d\sigma(\xi) \tag{2.155}$$

provided that f is a continuously differentiable field on $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ such that $f(\xi)\cdot\xi = 0$, $\xi \in \bar{\Gamma}$.

Applying the Gauß formula to $f = F\ \nabla^* G$ with suitable F, G we obtain the *First Green Surface Theorem*

$$\begin{aligned} &\int_\Gamma \nabla^*_\xi G(\xi)\cdot\nabla^*_\xi F(\xi)\, d\omega(\xi) + \int_\Gamma F(\xi)\Delta^*_\xi G(\xi)\, d\omega(\xi) \\ &= \int_{\partial\Gamma} F(\xi)\frac{\partial}{\partial\nu_\xi}G(\xi)\, d\sigma(\xi). \end{aligned} \tag{2.156}$$

In a similar way, applying the Stokes formula to $f = F\mathrm{L}^* G$ we get

$$\begin{aligned} &\int_\Gamma \mathrm{L}^*_\xi G(\xi)\cdot \mathrm{L}^*_\xi F(\xi)\ d\omega(\xi) + \int_\Gamma F(\xi)\Delta^*_\xi G(\xi)\ d\omega(\xi) \\ &\quad = \int_{\partial\Gamma} F(\xi)\frac{\partial}{\partial\tau_\xi}G(\xi) d\sigma(\xi) \end{aligned} \tag{2.157}$$

(observe that $\partial/\partial\nu_\xi = \nu_\xi \cdot \nabla^*_\xi$ and $\partial/\partial\tau_\xi = \tau_\xi \cdot \mathrm{L}^*_\xi$). Interchanging F, G in (2.157) and subtracting (2.156) yields the *Second Green Surface Theorem*

$$\begin{aligned}\int_\Gamma \{F(\xi)\Delta^*_\xi G(\xi) - G(\xi)\Delta^*_\xi F(\xi)\}\, d\omega(\xi)\\ = \int_{\partial\Gamma}\left\{F(\xi)\frac{\partial}{\partial\nu_\xi}G(\xi) - G(\xi)\frac{\partial}{\partial\nu_\xi}F(\xi)\right\} d\sigma(\xi) \qquad (2.158)\\ = \int_{\partial\Gamma}\left\{F(\xi)\frac{\partial}{\partial\tau_\xi}G(\xi) - G(\xi)\frac{\partial}{\partial\tau_\xi}F(\xi)\right\} d\sigma(\xi).\end{aligned}$$

There are immediate consequences of the above formulas due to the fact that the integral identities also hold true on $\Omega\backslash\bar{\Gamma}$ (under suitable assumptions on the integrands). For the whole sphere Ω, this leads to

$$\int_\Omega f(\xi)\cdot\nabla^*_\xi F(\xi)\, d\omega(\xi) = -\int_\Omega F(\xi)\nabla^*_\xi\cdot f(\xi)\, d\omega(\xi), \qquad (2.159)$$

$$\int_\Omega f(\xi)\cdot\mathrm{L}^*_\xi F(\xi)\, d\omega(\xi) = -\int_\Omega F(\xi)\mathrm{L}^*_\xi\cdot f(\xi)\, d\omega(\xi), \qquad (2.160)$$

$$\begin{aligned}\int_\Omega \nabla^*_\xi F(\xi)\cdot\nabla^*_\xi G(\xi)\, d\omega(\xi) &= -\int_\Omega F(\xi)\Delta^*_\xi G(\xi)\, d\omega(\xi) \qquad (2.161)\\ &= -\int_\Omega G(\xi)\Delta^*_\xi F(\xi)\, d\omega(\xi).\end{aligned}$$

Furthermore,

$$\int_\Omega \nabla^*_\xi\cdot f(\xi)\, d\omega(\xi) = 0, \qquad (2.162)$$

$$\int_\Omega \mathrm{L}^*_\xi F(\xi)\cdot\mathrm{L}^*_\xi G(\xi)\, d\omega(\xi) = -\int_\Omega F(\xi)\Delta^*_\xi G(\xi)\, d\omega(\xi), \qquad (2.163)$$

$$\int_\Omega \nabla^*_\xi\cdot(f(\xi)\wedge\xi)\, d\omega(\xi) = 0, \qquad (2.164)$$

provided that $F : \Omega \to \mathbb{R}$ (resp. $f : \Omega \to \mathbb{R}^3$) are sufficiently often continuously differentiable.

Let us consider a *spherical vector field* f of class $\mathrm{c}(\Omega)$. Of course, f can be decomposed by using the three basis vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$:

$$f(\xi) = \sum_{i=1}^{3}\left(f(\xi)\cdot\varepsilon^i\right)\varepsilon^i = \sum_{i=1}^{3} F_i(\xi)\varepsilon^i, \quad \xi\in\Omega, \qquad (2.165)$$

where $F_i : \Omega \to \mathbb{R}$ are differentiable functions with $F_i(\xi) = f(\xi)\cdot\varepsilon^i$, $\xi\in\Omega$, $i = 1, 2, 3$. The representation (2.165) can be used to reduce vectorial differential or integral equations, but it has the drawback that essential properties (for example, surface divergence, surface curl, spherical symmetry, etc) of vector fields are ignored. This problem can be overcome by the

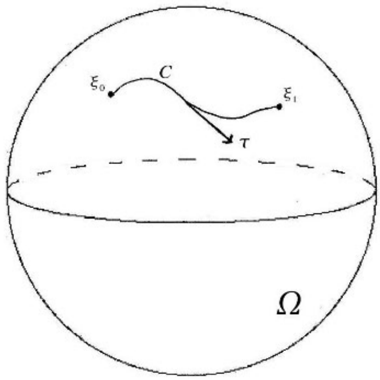


Fig. 2.4: A curve C on Ω connecting two points ξ_0 and ξ_1.

Helmholtz decomposition formula (for more details the reader is referred to Section 5.2). To be more specific, the decomposition (2.165) of vector fields using the unit vectors ε^i, $i \in \{1,2,3\}$, is no longer adequate for a large class of problems, since none of them reflects either the tangential or the normal direction on the sphere. A first hint for a system of unit vectors that is more suitable to a physically motivated situation can be given by the representation:

$$f(\xi) = f_{\mathrm{nor}}(\xi) + f_{\mathrm{tan}}(\xi), \tag{2.166}$$

where

$$f_{\mathrm{nor}}(\xi) = (f(\xi) \cdot \xi)\xi. \tag{2.167}$$

The vector $\xi \in \Omega$ points into the normal direction. Thus, we have to construct for $f_{\mathrm{tan}}(\xi)$ in each point $\xi \in \Omega$ two unit vectors perpendicular to ξ (that have to be of physical relevance).

Clearly, for a continuous vector field $f : \Omega \to \mathbb{R}^3$, we call

$$\xi \mapsto f_{\mathrm{nor}}(\xi) = (f(\xi) \cdot \xi)\xi, \quad \xi \in \Omega, \tag{2.168}$$

the *normal field* of f, while

$$\xi \mapsto f_{\mathrm{tan}}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi, \quad \xi \in \Omega, \tag{2.169}$$

is called the tangential field of f. Obviously, the identity (2.166) is valid and the normal field of f is orthogonal to the tangential field of f, i.e., for all $\xi \in \Omega$

$$\begin{aligned} ((f(\xi) \cdot \xi)\xi) \cdot ((f(\xi) - (f(\xi) \cdot \xi)\, \xi)) &= (f(\xi) \cdot \xi)^2 - (f(\xi) \cdot \xi)^2 \\ &= 0. \end{aligned} \tag{2.170}$$

Furthermore, for $f, g \in c(\Omega)$ and $\xi \in \Omega$,

$$f(\xi) \cdot g(\xi) = f_{\mathrm{nor}}(\xi) \cdot g_{\mathrm{nor}}(\xi) + f_{\mathrm{tan}}(\xi) \cdot g_{\mathrm{tan}}(\xi). \tag{2.171}$$

Lemma 2.5. *The tangential field of f vanishes i.e., $f_{\mathrm{tan}}(\xi) = 0$, $\xi \in \Omega$, if and only if $f(\xi) \cdot \hat{\tau}(\xi) = 0$ for every unit vector $\hat{\tau}(\xi)$ that is perpendicular to ξ, i.e., for which $\xi \cdot \hat{\tau}(\xi) = 0$, $\xi \in \Omega$.*

Proof. First, assume $f_{\mathrm{tan}} = 0$. For all $\xi \in \Omega$ we have

$$\begin{aligned} f(\xi) \cdot \hat{\tau}(\xi) &= (f(\xi) \cdot \xi) \underbrace{(\xi \cdot \hat{\tau}(\xi))}_{=0} \\ &\quad + \underbrace{(f(\xi) - (f(\xi) \cdot \xi)\xi)}_{=0} \cdot \hat{\tau}(\xi) \\ &= 0. \end{aligned} \tag{2.172}$$

Conversely, assume that the tangential field is non-vanishing, i.e.,

$$f_{\mathrm{tan}}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi \neq 0. \tag{2.173}$$

Then it follows that $f_{\mathrm{tan}}(\xi)|f_{\mathrm{tan}}(\xi)|^{-1}$ is a unit vector field perpendicular to ξ. Hence, by our hypothesis,

$$f_{\mathrm{tan}}(\xi) \cdot \frac{f_{\mathrm{tan}}(\xi)}{|f_{\mathrm{tan}}(\xi)|} = 0. \tag{2.174}$$

This implies

$$|f_{\mathrm{tan}}(\xi)| = 0, \tag{2.175}$$

which is a contradiction. Thus it follows that $f_{\mathrm{tan}}(\xi) = 0$, as required. □

Lemma 2.6. *Suppose that f is continuous on Ω. Moreover, let*

$$\int_{\mathcal{C}} \tau_\xi \cdot f(\xi) \, d\sigma(\xi) = 0 \tag{2.176}$$

for every curve $\mathcal{C}$ lying on Ω. Then

$$f_{\mathrm{tan}}(\xi) = 0 \tag{2.177}$$

for all $\xi \in \Omega$, i.e., the tangential field of f vanishes for all $\xi \in \Omega$.

Proof. Choose any point $\xi_0 \in \Omega$. Let τ_{ξ_0} be any unit vector satisfying $\tau_{\xi_0} \cdot \xi_0 = 0$. Then, there is a curve $\mathcal{C}$ on Ω passing through ξ_0 whose unit tangent vector at ξ_0 is just τ_{ξ_0}. Let $\mathcal{C}_{\text{sub}}^{\xi_0}$ be any subset of $\mathcal{C}$ containing ξ_0. Then, in accordance with our assumption,

$$\int_{\mathcal{C}_{\text{sub}}^{\xi_0}} \tau_\xi \cdot f(\xi)\ d\sigma(\xi) = 0. \tag{2.178}$$

Hence, letting the length of $\mathcal{C}_{\text{sub}}^{\xi_0}$ tend to zero we find $\tau_{\xi_0} \cdot f(\xi_0) = 0$. Lemma 2.5 then yields $f_{\tan}(\xi_0) = f(\xi_0) - (f(\xi_0) \cdot \xi_0)\xi_0 = 0$. Since ξ_0 can be any point on the sphere Ω, we have $f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi = 0$ for all $\xi \in \Omega$. □

The surface gradient acts like an ordinary gradient in $\mathbb{R}^3$ when we integrate it along lines on Ω. In more detail, suppose F is continuously differentiable in an open set in $\mathbb{R}^3$ containing Ω, and $\mathcal{C}$ is any curve lying on Ω, starting at ξ_0 and ending at ξ_1 (see Fig. 2.4). Suppose that τ_ξ is the unit tangent vector at ξ on $\mathcal{C}$ pointing from ξ_0 to ξ_1. Then

$$F(\xi_1) - F(\xi_0) = \int_{\mathcal{C}} \tau_\xi \cdot \nabla_\xi^* F(\xi)\ d\sigma(\xi) \tag{2.179}$$

(observe that $\tau_\xi \cdot \nabla_\xi F(\xi) = \tau_\xi \cdot \nabla_\xi^* F(\xi)$, $\xi \in \Omega$, (cf. C. Müller (1969))). This result enables us to show the following lemma.

Lemma 2.7. *Let F be of class $\mathrm{C}^{(1)}(\Omega)$. Assume that $\nabla_\xi^* F(\xi) = 0$ for all $\xi \in \Omega$, then F is constant, and conversely.*

Proof. If $\nabla_\xi^* F(\xi) = 0$, then we obtain, in connection with (2.179), $F(\xi_1) = F(\xi_0)$ for any ξ_0, ξ_1 on Ω.

Conversely, if F is constant, the identity (2.179) shows that $f = \nabla^* F$ fulfills

$$\int_{\mathcal{C}} \tau_\xi \cdot f(\xi)\ d\sigma(\xi) = 0 \tag{2.180}$$

for every curve $\mathcal{C}$ lying on Ω. Consequently, following Lemma 2.6,

$$f_{\tan}(\xi) = 0 \tag{2.181}$$

for all $\xi \in \Omega$. This shows that

$$f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi) \cdot \xi = f(\xi) = \nabla_\xi^* F(\xi) = 0 \tag{2.182}$$

for all $\xi \in \Omega$. □

From Lemma 2.7 we are immediately able to deduce the following statement.

Lemma 2.8. *Let F be of class $\mathrm{C}^{(1)}(\Omega)$. Assume that $\mathrm{L}^*_\xi F(\xi) = 0$ for all $\xi \in \Omega$, then F is constant, and conversely.*

Proof. If $\mathrm{L}^*_\xi F(\xi) = 0$, i.e., $\xi \wedge \nabla^*_\xi F(\xi) = 0$ for all $\xi \in \Omega$. Then $\xi \wedge \xi \wedge \nabla^*_\xi F(\xi) = (\xi \cdot \nabla^*_\xi F(\xi))\xi - \nabla^*_\xi F(\xi)(\xi \cdot \xi) = -\nabla^*_\xi F(\xi) = 0$ for all $\xi \in \Omega$. Thus, by virtue of Lemma 2.7, we find $F = \mathrm{const}$.

Conversely, if F is constant, then $\mathrm{L}^*_\xi F(\xi) = \xi \wedge \nabla^*_\xi F(\xi) = \xi \wedge 0 = 0$ for all $\xi \in \Omega$. This proves Lemma 2.8. □

Next, we prove the following well-known result of spherical vector analysis (see, e.g., G.E. Backus et al. (1996)).

Lemma 2.9. *Let $f \in \mathrm{c}(\Omega)$ be a tangent vector field, i.e., $f(\xi) = f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi, \xi \in \Omega$. Furthermore, suppose that*

$$\int_{\mathcal{C}} \tau_\xi \cdot f(\xi)\ d\sigma(\xi) = 0 \tag{2.183}$$

for every closed curve on Ω.

Then, there is a scalar field P on Ω such that

$$f(\xi) = \nabla^*_\xi P(\xi), \quad \xi \in \Omega. \tag{2.184}$$

The field P is continuously differentiable and is unique up to a constant.

Proof. Take an arbitrary, but fixed $\xi_0 \in \Omega$. We let

$$P(\xi) = \int_{\xi_0}^{\xi} \tau_\zeta \cdot f(\zeta)\ d\sigma(\zeta), \tag{2.185}$$

the integral being along any curve $\mathcal{C}$ that starts at $\xi_0 \in \Omega$ and ends at $\xi \in \Omega$. Then, for any two points ξ_0, ξ on Ω and any curve $\mathcal{C}$ lying on Ω and starting at ξ_0 and ending at ξ_1,

$$P(\xi_1) - P(\xi_0) = \int_{\xi_0}^{\xi_1} \tau_\zeta \cdot f(\zeta)\ d\sigma(\zeta). \tag{2.186}$$

Observing (2.179) we find

$$P(\xi_1) - P(\xi_0) = \int_{\xi_0}^{\xi_1} \tau_\zeta \cdot \nabla^*_\zeta P(\zeta)\ d\sigma(\zeta). \tag{2.187}$$

Combining (2.186) and (2.187), we obtain

$$\int_{\xi_0}^{\xi_1} \tau_\zeta \cdot \left(f(\zeta) - \nabla^*_\zeta P(\zeta) \right) \, d\sigma(\zeta) = 0 \tag{2.188}$$

for any curve $\mathcal{C}$ on Ω. Lemma 2.6, therefore, tells us that

$$f(\xi) - \nabla^*_\xi P(\xi) = 0, \quad \xi \in \Omega. \tag{2.189}$$

The proof that P is continuously differentiable on Ω is omitted. The easiest way to construct such a proof is to take P constant on each straight line passing through Ω in the normal direction (see, e.g., G.E. Backus et al. (1996)).

In order to verify that P is unique up to a constant, we observe that $\nabla^*_\xi P_1(\xi) = \nabla^*_\xi P_2(\xi), \xi \in \Omega$, implies $\nabla^*_\xi (P_1 - P_2)(\xi) = 0$, $\xi \in \Omega$, i.e., by virtue of Lemma 2.7, $P_1 - P_2 = \text{const}$. □

Now we are able to formulate the following important theorem:

Theorem 2.10. *Let $f \in \mathrm{c}^{(1)}(\Omega)$ be a tangential field, i.e., $f(\xi) = f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi$ for all $\xi \in \Omega$.*

Then $\mathrm{L}^*_\xi \cdot f(\xi) = 0$, $\xi \in \Omega$, if and only if there is a scalar field P such that

$$f(\xi) = \nabla^*_\xi P(\xi), \quad \xi \in \Omega, \tag{2.190}$$

and P is unique up to an additive constant (P is called potential function for f).

Similarly, $\nabla^*_\xi \cdot f(\xi) = 0$, $\xi \in \Omega$, if and only if there is a scalar field S such that

$$f(\xi) = \mathrm{L}^*_\xi S(\xi), \quad \xi \in \Omega, \tag{2.191}$$

and S is unique up to an additive constant (S is called stream function for f).

Proof. The condition $f = \nabla^* P$ implies $\mathrm{L}^* \cdot f = 0$, and $f = \mathrm{L}^* S$ implies $\nabla^* \cdot f = 0$.

Conversely, assume that $\mathrm{L}^*_\xi \cdot f(\xi) = 0$, $\xi \in \Omega$. Then the surface theorem of Stokes implies

$$\int_{\mathcal{C}} \tau_\xi \cdot f(\xi) \, d\sigma(\xi) = 0 \tag{2.192}$$

for every closed curve $\mathcal{C}$ on Ω. From Lemma 2.9, it follows that there exists a scalar field P such that $f = \nabla^* P$. Furthermore, P is unique up to an additive constant.

Finally, suppose $\nabla^* \cdot f = 0$. Then $\mathrm{L}^*_\xi \cdot (\xi \wedge f(\xi)) = 0$, $\xi \in \Omega$. Hence, by the same arguments as above, there is a scalar field S, unique up to a constant, such that

$$-\xi \wedge f(\xi) = \nabla^*_\xi S(\xi), \quad \xi \in \Omega. \tag{2.193}$$

This is equivalent to

$$-\xi \wedge (\xi \wedge f(\xi)) = (\xi \wedge \nabla^*_\xi) S(\xi), \quad \xi \in \Omega, \tag{2.194}$$

or

$$f = \mathrm{L}^* S \tag{2.195}$$

on Ω. This proves Theorem 2.10. □

For tangential fields, the validity of *homogeneous "pre-Maxwell equations"* implies that the field under consideration vanishes identically. This is the content of the next theorem.

Theorem 2.11. *Let f be a continuously differentiable tangential vector field on Ω (i.e., $f(\xi) = f_{\mathrm{tan}}(\xi) = f(\xi) - (f(\xi)\cdot\xi)\xi$, $\xi \in \Omega$) such that*

$$\begin{aligned} \nabla^*_\xi \cdot f(\xi) &= 0, \quad \xi \in \Omega, \\ \mathrm{L}^*_\xi \cdot f(\xi) &= 0, \quad \xi \in \Omega. \end{aligned}$$

Then $f = 0$ on Ω.

Proof. From $\mathrm{L}^*_\xi \cdot f(\xi) = 0$ we get from Theorem 2.10 that there exists a scalar field P such that

$$f(\xi) = \nabla^*_\xi P(\xi), \quad \xi \in \Omega. \tag{2.196}$$

From $\nabla^*_\xi \cdot f(\xi) = 0$ we can therefore deduce that $\nabla^*_\xi \cdot \nabla^*_\xi P(\xi) = \Delta^*_\xi P(\xi) = 0$.

Together with (2.161), this leads to

$$\int_\Omega (\nabla^*_\xi P(\xi))^2 \, d\omega(\xi) = 0. \tag{2.197}$$

Consequently, it follows that $f(\xi) = \nabla^*_\xi P(\xi) = 0$. This is the required result. □

2.7 Orthogonal Invariance

Systems of equations which maintain their form when the coordinate axes are subjected to an arbitrary rotation are said to be rotationally, or orthogonally, invariant. The orthogonal invariance is, of course, closely related to the group $O(3)$ of all orthogonal transformations, i.e., the group of all $\mathbf{t} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ such that $\mathbf{t}\mathbf{t}^T = \mathbf{t}^T\mathbf{t} = \mathbf{i}$, $\mathbf{i} = (\delta_{ij})_{i,j=1,2,3}$. The set of all rotations, i.e., $SO(3) = \{\mathbf{t} \in O(3) \,|\, \det \mathbf{t} = 1\}$ is a subgroup called the *special orthogonal group* .

We briefly recapitulate some properties of these groups (see, e.g., C. Müller (1998), N.J. Vilenkin (1968) and many others):

1. Let ξ, η be members of Ω. Then, there exists an orthogonal transformation $\mathbf{t} \in O(3)$ with $\eta = \mathbf{t}\xi$ and an orthogonal transformation $\mathbf{s} \in SO(3)$ with $\eta = \mathbf{s}\xi$.
2. For every $\mathbf{t} \in O(3)$
$$\mathbf{t}\xi \cdot \mathbf{t}\eta = \xi \cdot \eta, \quad \xi, \eta \in \Omega. \tag{2.198}$$
3. Suppose that $\xi \in \Omega$. The set $O_\xi(3) = \{\mathbf{t} \in O(3) \,|\, \mathbf{t}\xi = \xi\}$ is a subgroup of $O(3)$. Analogously, the set $SO_\xi(3) = \{\mathbf{t} \in SO(3) \,|\, \mathbf{t}\xi = \xi\}$ is a subgroup of $SO(3)$.
4. For every $\mathbf{t} \in O(3)$, we have $\det \mathbf{t} = \pm 1$. If $\det \mathbf{t} = 1$, $\mathbf{t}$ is called a *rotation*, while for $\det \mathbf{t} = -1$, $\mathbf{t}$ is called a *reflection* .
5. Let $\mathbf{t}, \mathbf{t}' \in O(3)$ with $\det \mathbf{t} = 1$, $\det \mathbf{t}' = -1$. Then
$$\mathbf{t}\xi \wedge \mathbf{t}\eta = \mathbf{t}(\xi \wedge \eta), \quad \xi, \eta \in \Omega \tag{2.199}$$
$$\mathbf{t}'\xi \wedge \mathbf{t}'\eta = -\mathbf{t}'(\xi \wedge \eta), \quad \xi, \eta \in \Omega. \tag{2.200}$$
6. Let $\mathbf{t} \in O(3)$. Then, for the dyadic product, we get
$$\mathbf{t}(\xi \otimes \eta)\mathbf{t}^T = \mathbf{t}\xi \otimes \mathbf{t}\eta, \quad \xi, \eta \in \Omega. \tag{2.201}$$

The following definitions will prove useful for our later considerations.

Definition 2.12. Let $F \in \mathrm{L}^2(\Omega)$, $f \in \mathrm{l}^2(\Omega)$, $\mathbf{f} \in \mathbf{l}^2(\Omega)$ and suppose that $\mathbf{t} \in O(3)$. For scalar, vector, and tensor fields the operator $R_\mathbf{t}$ is defined by

$$\begin{aligned} R_\mathbf{t} &: \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \quad R_\mathbf{t}F(\xi) = F(\mathbf{t}\xi), \\ R_\mathbf{t} &: \mathrm{l}^2(\Omega) \to \mathrm{l}^2(\Omega), \quad R_\mathbf{t}f(\xi) = \mathbf{t}^T f(\mathbf{t}\xi), \\ R_\mathbf{t} &: \mathbf{l}^2(\Omega) \to \mathbf{l}^2(\Omega), \quad R_\mathbf{t}\mathbf{f}(\xi) = \mathbf{t}^T \mathbf{f}(\mathbf{t}\xi)\mathbf{t}, \end{aligned}$$

respectively. $R_\mathbf{t}F$, $R_\mathbf{t}f$, and $R_t\mathbf{f}$ are called the $\mathbf{t}$-transformed fields .

For examples illustrating how the operators $R_{\mathbf{t}}$ act on functions and vector fields, see Figs. 2.5 and 2.6, respectively.

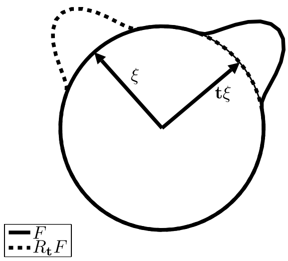


Fig. 2.5: The operator $R_{\mathbf{t}}$ acting on a function.

Definition 2.13. Let $\mathcal{F}$ be a subspace of $\mathrm{L}^2(\Omega)$ ($\mathrm{l}^2(\Omega)$ or $\mathbf{l}^2(\Omega)$). $\mathcal{F}$ is called *invariant with respect to orthogonal transformations* or, equivalently, *orthogonally invariant* if, for all $F \in \mathcal{F}$ and for all orthogonal transformations $\mathbf{t} \in O(3)$, the function $R_t F$ is of class $\mathcal{F}$.
An orthogonally invariant $\mathcal{F}$ is called *reducible* if there exists a proper subspace $\mathcal{F}' \subset \mathcal{F}$ which itself is invariant with respect to orthogonal transformations.

Note that the expressions *invariant with respect to rotations* and *invariant with respect to reflections* are understood in analogy to the aforementioned definition.

A linear, orthogonally invariant space which is not reducible is called *irreducible*. (It should be noted that each orthogonally invariant space of dimension 1 is irreducible).

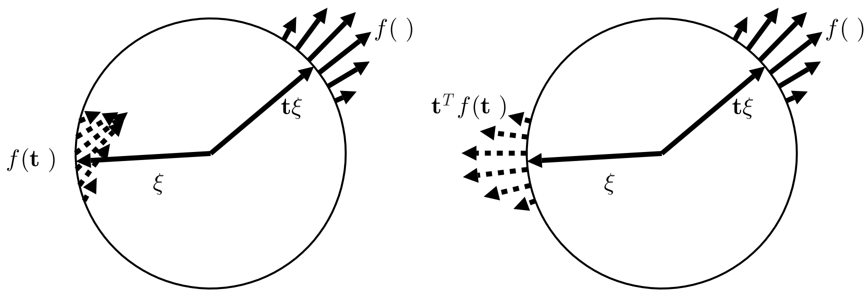


Fig. 2.6: The definition of the operator $R_{\mathbf{t}}$ for vector fields (note that it is necessary not only to substitute ξ by $\mathbf{t}\xi$, but also to transform the directions of the vectors).

Lemma 2.14. *Let $(\mathcal{F}, (\cdot,\cdot))$ be an orthogonally invariant Hilbert subspace of $\mathrm{L}^2(\Omega)$. Let $\mathcal{F}_1$ be an orthogonally invariant subspace of $\mathcal{F}$. Then, the orthogonal complement $\mathcal{F}_1^{\perp}$ of $\mathcal{F}_1$ is orthogonally invariant, as well.*

Proof. For all $F \in \mathcal{F}_1$, $F^{\perp} \in \mathcal{F}_1^{\perp}$ and for all orthogonal transformations $\mathbf{t} \in O(3)$, we have

$$\begin{aligned} (F, R_{\mathbf{t}}F^{\perp}) &= \int_{\Omega} F(\xi) R_{\mathbf{t}} F^{\perp}(\xi)\, d\omega(\xi) \\ &= (\det \mathbf{t}) \int_{\mathbf{t}\Omega} F(\xi) R_{\mathbf{t}} F^{\perp}(\xi)\, d\omega(\xi) \\ &= (\det \mathbf{t})^2 \int_{\Omega} R_{\mathbf{t}^T} F(\xi) F^{\perp}(\xi)\, d\omega(\xi) \\ &= 0, \end{aligned} \tag{2.202}$$

since $R_{\mathbf{t}^T}F \in \mathcal{F}_1$. This implies that $R_{\mathbf{t}}F^{\perp} \in \mathcal{F}_1^{\perp}$ and, therefore, $\mathcal{F}_1^{\perp}$ is invariant with respect to orthogonal transformations. □

Analogous results can be formulated for Hilbert spaces of square-integrable vector and tensor fields. Lemma 2.14 shows that each orthogonally invariant Hilbert-space can be completely decomposed into invariant parts.

In view of the last result, we are particularly interested in irreducible spaces, i.e., spaces that definitely provide us with elements that are invariant with respect to certain orthogonal transformations. The following results (see, e.g., T. Gervens (1989)) help us to analyze the structure of such rotationally invariant functions.

Lemma 2.15. *Let F be a function of class $\mathrm{C}(\Omega)$ with $R_{\mathbf{t}}F(\xi) = F(\xi)$ for all $\mathbf{t} \in SO(3)$ and all $\xi \in \Omega$. Then*

$$F = F(\varepsilon^3) = C = \text{const.}$$

Proof. For every $\xi \in \Omega$, there exists a rotation $\mathbf{t} \in SO(3)$ with $\mathbf{t}\xi = \varepsilon^3$. Consequently, for every $\xi \in \Omega$, we have $F(\xi) = R_{\mathbf{t}}F(\xi) = F(\mathbf{t}\xi) = F(\varepsilon^3) = C = \text{const.}$ □

Lemma 2.16. *Let $\eta \in \Omega$ be fixed. Furthermore, let $F \in \mathrm{C}(\Omega)$ with $R_{\mathbf{t}}F(\xi) = F(\xi)$ for all $\mathbf{t} \in SO_{\eta}(3)$ and for all $\xi \in \Omega$. Then, F can be represented in the form*

$$F(\xi) = \Phi(\xi \cdot \eta), \quad \xi \in \Omega,$$

Φ being a function $\Phi : [-1, 1] \to \mathbb{R}$.

Proof. Without loss of generality , let $\eta = \varepsilon^3$ (if this were not true, we could use the function $G(\xi) = R_{\mathbf{t}'} F(\xi)$, where $\mathbf{t}' \in O(3)$ with $\mathbf{t}'\varepsilon^3 = \eta$). With $\xi = t\varepsilon^3 + \sqrt{1-t^2}\eta'$ we have, by assumption, that

$$F(t\varepsilon^3 + \sqrt{1-t^2}\eta') = F(t\varepsilon^3 + \sqrt{1-t^2}\eta''), \tag{2.203}$$

for all points η', η'' of the unit circle. Hence, F depends only on $t = \xi \cdot \varepsilon^3$ and is, therefore, a function of t alone, as desired. □

Lemma 2.17. *Let $\eta \in \Omega$ be fixed. Let $F \in \mathrm{C}(\Omega)$ with $R_{\mathbf{t}} F(\xi) = (\det \mathbf{t})\ \mathrm{F}(\xi)$ for all $\mathbf{t} \in O_\eta(3)$ and all $\xi \in \Omega$. Then*

$$F = 0.$$

Proof. Suppose that ξ is an element of Ω. There exists a reflection $\mathbf{t} \in O_\eta(3)$ with $\mathbf{t}\xi = \xi$, but then — by assumption — we have $F(\xi) = R_{\mathbf{t}} F(\xi) = -F(\xi)$, hence, $F(\xi) = 0$. □

Note that in Lemma 2.15 and Lemma 2.16, the rotations can as well be replaced by reflections, i.e., in the scalar case, we need not distinguish between rotations and reflections. In the vectorial case, however, this is not true anymore.

In what follows, f is supposed to be a spherical vector field, i.e., $f : \Omega \to \mathbb{R}^3$. Let η be an element of Ω. In every point $\xi \neq \pm\eta$, we are able to introduce the so-called *moving triad at the point* ξ

$$\varepsilon^1_\xi = \xi, \tag{2.204}$$

$$\varepsilon^2_\xi = \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}(\eta - (\xi\cdot\eta)\xi), \tag{2.205}$$

$$\varepsilon^3_\xi = \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}\eta \wedge \xi, \tag{2.206}$$

such that there exist functions $F_1, F_2, F_3 : \Omega \to \mathbb{R}$ with

$$f = F_1\varepsilon^1_\xi + F_2\varepsilon^2_\xi + F_3\varepsilon^3_\xi. \tag{2.207}$$

For further investigations, the following lemma is helpful.

Lemma 2.18. *Let $\eta \in \Omega$ be fixed, and let the moving triad ε^i_ξ, $i = 1, 2, 3$, be defined as in (2.204)–(2.206). Then, for all $\mathbf{t} \in O(3)$,*

$$R_{\mathbf{t}}\varepsilon^i_\xi = \varepsilon^i_\xi, \quad i = 1, 2$$
$$R_{\mathbf{t}}\varepsilon^3_\xi = (\det \mathbf{t})\ \varepsilon^3_\xi.$$

Proof. For $\mathbf{t} \in O_\eta(3)$,

$$R_{\mathbf{t}}\varepsilon^1_\xi = \mathbf{t}^T \varepsilon^1_{\mathbf{t}\xi} = \mathbf{t}^T\mathbf{t}\,\xi = \xi = \varepsilon^1_\xi. \tag{2.208}$$

For the tangential fields, we only show the case $i = 2$ (the case $i = 3$ follows similarly). We have

$$\begin{aligned} R_{\mathbf{t}}\varepsilon^2_\xi = \mathbf{t}^T\varepsilon^2_{\mathbf{t}\xi} &= \frac{1}{\sqrt{1-(\mathbf{t}\xi\cdot\eta)^2}}\mathbf{t}^T(\eta - (\mathbf{t}\xi\cdot\eta)\mathbf{t}\xi) \\ &= \frac{1}{\sqrt{1-(\xi\cdot\mathbf{t}^T\eta)^2}}(\mathbf{t}^T\eta - (\xi\cdot\mathbf{t}^T\eta)\mathbf{t}^T\mathbf{t}\xi) \\ &= \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}(\eta - (\xi\cdot\eta)\xi) \\ &= \varepsilon^2_\xi. \end{aligned} \tag{2.209}$$

□

We now extend our results for rotationally invariant functions to the vector case .

Lemma 2.19. *Let $f \in \mathrm{c}(\Omega)$ with $R_{\mathbf{t}}f(\xi) = f(\xi)$ (or equivalently, $f(\mathbf{t}\xi) = \mathbf{t}f(\xi)$) for all $\mathbf{t} \in SO(3)$ and $\xi \in \Omega$. Then, there exists a constant $C \in \mathbb{R}$ such that*

$$f(\xi) = C\,\xi\,, \quad \xi \in \Omega.$$

Proof. Consider the orthogonal matrix

$$\mathbf{t} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.210}$$

Then $\mathbf{t}\varepsilon^3 = \varepsilon^3$ and, by assumption, $f(\varepsilon^3) = \mathbf{t}f(\varepsilon^3)$. Hence, in connection with (2.210), we have $f(\varepsilon^3) = C\varepsilon^3$, $C \in \mathbb{R}$. For $\xi \in \Omega$, there exists a rotation $\mathbf{t}'$ with $\mathbf{t}'\varepsilon^3 = \xi$. Consequently, we have

$$f(\xi) = f(\mathbf{t}'\varepsilon^3) = \mathbf{t}'f(\varepsilon^3) = C\mathbf{t}'\varepsilon^3 = C\xi. \tag{2.211}$$

□

Lemma 2.20. *Let $\eta \in \Omega$. Let $f \in c(\Omega)$ with $R_{\mathbf{t}}f(\xi) = f(\xi)$ for all $\mathbf{t} \in SO_\eta(3)$. Then, for $\xi \neq \pm\eta$, f has the representation:*

$$f(\xi) = \Phi_1(\xi\cdot\eta)\varepsilon^1_\xi + \Phi_2(\xi\cdot\eta)\varepsilon^2_\xi + \Phi_3(\xi\cdot\eta)\varepsilon^3_\xi,$$

where Φ_i, $i = 1,2,3$, are functions $\Phi_i : [-1,1] \to \mathbb{R}$.

Proof. From Lemma 2.18, it follows that the functions F_i in (2.207) fulfill

$$R_{\mathbf{t}} F_i(\xi) = F_i(\xi), \quad \xi \in \Omega, \tag{2.212}$$

provided that $\mathbf{t}\eta = \eta$. Therefore, via Lemma 2.16, we know that, for the functions F_i, we have

$$F_i(\xi) = \Phi_i(\xi \cdot \eta). \tag{2.213}$$

□

Lemma 2.21. *Suppose that $\eta \in \Omega$. Let f be of class* $\mathrm{c}(\Omega)$ *with $R_{\mathbf{t}} f(\xi) = f(\xi)$ for all $\mathbf{t} \in O_\eta(3)$. Then, for $\xi \neq \pm\eta$, f has the representation,*

$$f(\xi) \quad = \quad \Phi_1(\xi \cdot \eta)\varepsilon^1_\xi + \Phi_2(\xi \cdot \eta)\varepsilon^2_\xi,$$

Φ_i, $i = 1, 2$, being functions $\Phi_i : [-1, 1] \to \mathbb{R}$.

Proof. Starting from Lemma 2.20, we now have to consider reflections, as well. By our assumption and Lemma 2.18, we get $\Phi_3(\xi \cdot \eta) = -\Phi_3(\xi \cdot \eta)$ and, therefore, $\Phi_3(\xi \cdot \eta) = 0$. □

Lemma 2.22. *Suppose that $\eta \in \Omega$. Let f be of class* $\mathrm{c}(\Omega)$ *with $R_{\mathbf{t}}\ f(\xi) =$* $(\det \mathbf{t})\ \mathrm{f}(\xi)$ *for all $\mathbf{t} \in O_\eta(3)$. Then, for $\xi \neq \pm\eta$, the field f can be represented as follows*

$$f(\xi) = \Phi_3(\xi \cdot \eta)\varepsilon^3_\xi, \tag{2.214}$$

with Φ_3 being a function $\Phi_3 : [-1, 1] \to \mathbb{R}$.

Proof. Using the same reasoning as in the proof of Lemma 2.21, but now considering the change in sign under reflections, we end up with

$$\Phi_1(\xi \cdot \eta) = -\Phi_1(\xi \cdot \eta),\ \Phi_2(\xi \cdot \eta) = -\Phi_2(\xi \cdot \eta), \tag{2.215}$$

hence, $\Phi_1(\xi \cdot \eta) = \Phi_2(\xi \cdot \eta) = 0$. □

In order to extend our considerations to tensor fields of second rank , we assume $\mathbf{f}(\xi)$ to be a matrix constituting a linear vector function for each $\xi \in \Omega$. Furthermore, let η be an element of Ω. Then, to the moving triad (2.204)–(2.206), there exist scalar spherical functions $F_{i,j} : \Omega \to \mathbb{R}$ such that $\mathbf{f}$ can be represented via dyadic products of the unit vectors, i.e.,

$$\mathbf{f}(\xi) = \sum_{i=1}^{3} \sum_{j=1}^{3} F_{i,j}(\xi)\, \varepsilon^i_\xi \otimes \varepsilon^j_\xi, \quad \xi \neq \pm\eta. \tag{2.216}$$

It should be remarked that, for $F_{i,j} = \delta_{ij}$, (2.216) forms a partition of the unit matrix $\mathbf{i}$.

We now examine matrices with certain, rotationally invariant characteristics.

Lemma 2.23. *For every* $\xi \in \Omega$ *and every* $\mathbf{t} \in O(3)$*, let* $\mathbf{f}$ *be of class* $\mathbf{c}(\Omega)$ *with*

$$R_{\mathbf{t}}\mathbf{f}(\xi) \quad = \quad \mathbf{f}(\xi) \ \ (i.e., \ \ \mathbf{f}(\mathbf{t}\xi) = \mathbf{t}\mathbf{f}(\xi)\mathbf{t}^T), \tag{2.217}$$

$\xi \in \Omega$*. Then, there exist constants* $C_1, C_2 \in \mathbb{R}$ *with*

$$\mathbf{f}(\xi) = C_1 \mathbf{i} + C_2\, \xi \otimes \xi, \quad \xi \in \Omega, \tag{2.218}$$

$\mathbf{i}$ *being the unit matrix .*

Proof. We start with the determination of the matrix $\mathbf{f}(\varepsilon^3) = (F_{ij})_{i,j=1,2,3}$. Using the transformation

$$\mathbf{t}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.219}$$

(2.217) leads to $F_{12} = F_{13} = F_{21} = F_{31} = 0$. Analogously, the application of the transformation

$$\mathbf{t}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.220}$$

yields $F_{23} = F_{32} = 0$. Finally,

$$\mathbf{t}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.221}$$

leads to $F_{11} = F_{22}$. Consequently, there exist constants $C_1, C_2 \in \mathbb{R}$ such that

$$\mathbf{f}(\varepsilon^3) = C_1 \mathbf{i} + C_2\, \varepsilon^3 \otimes \varepsilon^3. \tag{2.222}$$

If $\xi \in \Omega$, then there exits an orthogonal transformation $\mathbf{t}$ with $\mathbf{t}\varepsilon^3 = \xi$. Thus, it follows that, for $\xi \in \Omega$,

$$\begin{aligned} \mathbf{f}(\xi) = \mathbf{f}(\mathbf{t}\varepsilon^3) &= \mathbf{t}\mathbf{f}(\varepsilon^3)\mathbf{t}^T \\ &= \mathbf{t}(C_1 \mathbf{i} + C_2\, \varepsilon^3 \otimes \varepsilon^3)\mathbf{t}^T \\ &= C_1 \mathbf{i} + C_2\, \xi \otimes \xi. \end{aligned} \tag{2.223}$$

□

Lemma 2.24. *Let* $\mathbf{f}$ *be of class* $\mathbf{c}(\Omega)$ *with* $R_{\mathbf{t}}\mathbf{f}(\xi) = \mathbf{t}^T f(\mathbf{t}\xi)\mathbf{t} = (\det \mathbf{t})\ \mathbf{f}(\xi)$ *for all* $\xi \in \Omega$ *and all* $\mathbf{t} \in O(3)$. *Then there exists a constant* $C \in \mathbb{R}$ *with*

$$\mathbf{f}(\xi) = C\,\mathbf{i}^*(\xi), \quad \xi \in \Omega,$$

where

$$\mathbf{i}^*(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \quad \xi = (\xi_1, \xi_2, \xi_3)^T.$$

Proof. In analogy to the proof of Lemma 2.23, we first determine $\mathbf{f}(\varepsilon^3)$ with the same transformations $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ as before. Now, our assumptions lead to $F_{11} = F_{22} = F_{23} = F_{32} = F_{33} = F_{13} = F_{31} = 0$, and $F_{12} = -F_{21}$. Therefore, we can find a $C \in \mathbb{R}$ such that

$$\mathbf{f}(\varepsilon^3) = \mathrm{C}(\varepsilon^2 \otimes \varepsilon^1 - \varepsilon^1 \otimes \varepsilon^2). \tag{2.224}$$

For every vector $a \in \mathbb{R}^3$, we obviously have $\mathbf{f}(\varepsilon^3)a = C\,\varepsilon^3 \wedge a$. If $\xi \in \Omega$ and if $\mathbf{t} \in O(3)$ with $\mathbf{t}\varepsilon^3 = \xi$, then

$$\mathbf{f}(\xi)a = \mathbf{f}(\mathbf{t}\varepsilon^3)a = (\det\ \mathbf{t})\ \mathbf{t}\mathbf{f}(\varepsilon^3)\mathbf{t}^T a = (\det\ \mathbf{t})\ C\,\mathbf{t}(\varepsilon^3 \wedge (\mathbf{t}^T a)) = \xi \wedge a. \tag{2.225}$$

The vector product $\xi \wedge a$ can easily be expressed by the antisymmetric matrix $\mathbf{i}^*(\xi)$, i.e., $\xi \wedge a = \mathbf{i}^*(\xi)a$. □

Lemma 2.25. *Suppose that* $\eta \in \Omega$. *For every* $\xi \in \Omega$ *and every* $\mathbf{t} \in SO_\eta(3)$, *let* $\mathbf{f}$ *be of class* $\mathbf{c}(\Omega)$ *with* $R_{\mathbf{t}}\mathbf{f}(\xi) = \mathbf{t}^T\mathbf{f}(\mathbf{t}\xi)\mathbf{t} = \mathbf{f}(\xi)$, $\xi \in \Omega$. *Then, for* $\xi \neq \pm\eta$, *we have*

$$\mathbf{f}(\xi) = \sum_{i=1}^{3}\sum_{j=1}^{3} \Phi_{i,j}(\xi \cdot \eta)\, \varepsilon^i_\xi \otimes \varepsilon^j_\xi, \tag{2.226}$$

with $\Phi_{i,j}$ *being functions* $\Phi_{i,j} : [-1, 1] \to \mathbb{R}$.

Proof. We start from (2.216) and let $F_{ij}(\xi) = \varepsilon^i_\xi \cdot (\mathbf{f}(\xi)\varepsilon^j_\xi)$. By assumption, we have $F_{ij}(\mathbf{t}\xi) = F_{ij}(\xi)$, for every $\mathbf{t} \in SO_\eta(3)$. Due to Lemma 2.16, we have $F_{ij} = \Phi_{i,j}(\xi \cdot \eta)$. This is the wanted result. □

Lemma 2.26. *Suppose that* η *is a point of* Ω. *For all* $\xi \in \Omega$ *and for all* $\mathbf{t} \in O_\eta(3)$, *let* $\mathbf{f}$ *be a of class* $\mathbf{c}(\Omega)$ *with* $R_{\mathbf{t}}\mathbf{f}(\xi) = \mathbf{f}(\xi)$. *Then, for* $\xi \neq \pm\eta$, $\mathbf{f}(\xi)$ *can be written as follows*

$$\begin{aligned}\mathbf{f}(\xi) \quad = \quad & \Phi_{1,1}(\xi\cdot\eta)\,\varepsilon^1_\xi \otimes \varepsilon^1_\xi + \Phi_{1,2}(\xi\cdot\eta)\,\varepsilon^1_\xi \otimes \varepsilon^2_\xi + \Phi_{2,1}(\xi\cdot\eta)\,\varepsilon^2_\xi \otimes \varepsilon^1_\xi \\ & + \Phi_{2,2}(\xi\cdot\eta)\,\varepsilon^2_\xi \otimes \varepsilon^2_\xi + \Phi_{3,3}(\xi\cdot\eta)\,\varepsilon^3_\xi \otimes \varepsilon^3_\xi,\end{aligned}$$

with $\Phi_{i,j}$ *being functions* $\Phi_{i,j} : [-1, 1] \to \mathbb{R}$.

Proof. In contrast to Lemma 2.25, we also have to consider the use of reflections, i.e., we have to take into account that the transformation of cross-products by reflections leads to a change in sign. Consequently, $\Phi_{1,3}(\xi\cdot\eta) = \Phi_{2,3}(\xi\cdot\eta) = \Phi_{3,1}(\xi\cdot\eta) = \Phi_{3,2}(\xi\cdot\eta) = 0$. This proves Lemma 2.26. □

Lemma 2.27. *Let $\eta \in \Omega$. For all $\xi \in \Omega$ and for all $\mathbf{t} \in O_\eta(3)$ let $\mathbf{f}$ be of class $\mathbf{c}(\Omega)$ with $R_{\mathbf{t}}\mathbf{f}(\xi) = (\det\mathbf{t})\,\mathbf{f}(\xi)$. Then, for $\xi \neq \pm\eta$, $\mathbf{f}(\xi)$ can be written in the form*

$$\begin{aligned}\mathbf{f}(\xi) &= \Phi_{1,3}(\xi\cdot\eta)\,\varepsilon^1_\xi \otimes \varepsilon^3_\xi + \Phi_{3,1}(\xi\cdot\eta)\,\varepsilon^3_\xi \otimes \varepsilon^1_\xi \\ &\quad +\Phi_{2,3}(\xi\cdot\eta)\,\varepsilon^2_\xi \otimes \varepsilon^3_\xi + \Phi_{3,2}(\xi\cdot\eta)\,\varepsilon^3_\xi \otimes \varepsilon^2_\xi,\end{aligned}$$

with $\Phi_{i,j}$ being functions $\Phi_{i,j} : [-1,1] \to \mathbb{R}$.

Proof. Considering that the transformation of $\mathbf{f}(\xi)$ using reflections leads to a minus sign, we get that in (2.226) the terms with $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$, $\Phi_{2,2}$ and $\Phi_{3,3}$ vanish. □

Remark 2.28. For $\xi \neq \pm\eta$, another basis system is given by

$$\varepsilon^1_\eta = \eta, \tag{2.227}$$

$$\varepsilon^2_\eta = \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}(\xi - (\xi\cdot\eta)\eta), \tag{2.228}$$

$$\varepsilon^3_\eta = \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}\eta \wedge \xi. \tag{2.229}$$

This shows us that analogous results to Lemma 2.27 can be based on (2.227), (2.228), (2.229). For example, under the assumptions of Lemma 2.26, we find

$$\begin{aligned}f(\xi) &= \Phi_{1,1}(\xi\cdot\eta)\varepsilon^1_\xi \otimes \varepsilon^1_\eta + \Phi_{1,2}(\xi\cdot\eta)\varepsilon^1_\xi \otimes \varepsilon^2_\eta \\ &\quad + \Phi_{2,1}(\xi\cdot\eta)\varepsilon^2 \otimes \varepsilon^1_\eta + \Phi_{2,2}(\xi\cdot\eta)\varepsilon^2_\xi \otimes \varepsilon^2_\eta + \Phi_{3,3}(\xi\cdot\eta)\varepsilon^3_\eta \otimes \varepsilon^3_\eta,\end{aligned} \tag{2.230}$$

$\xi \in \Omega$.

3 Scalar Spherical Harmonics

In this chapter, we deal with the theory of scalar spherical harmonics. As already mentioned, our scalar approach is essentially based on the work due to C. Müller (1952, 1966, 1998) and W. Freeden (1979a); W. Freeden (1981b). In fact, it is led by the observation (see H. Weyl (1934, 1946, 1965)) that spherical harmonics must be more than a fortunate guess in Fourier (orthogonal) expansions for providing tables of potential coefficients for geophysical quantities. This opinion arose from the occupation with theoretical physics (in particular, gravitational theory, electromagnetism, quantum mechanics, and general relativity) and was supported by many physicists during the last century. Today, even problems in medicine, e.g., the electroencephalographic description of scalp potential fields, can be tackled appropriately in terms of spherical harmonics.

The layout of this chapter is as follows: The scalar spherical harmonics are introduced as the restrictions of the homogeneous harmonic polynomials to the unit sphere. In consequence, the addition theorem of homogeneous harmonic polynomials canonically goes over to the theory of scalar spherical harmonics. Maxwell's representation formula shows that the (one-dimensional) Legendre polynomials may be obtained by repeated differentiation of the fundamental solutions of the Laplace operator. The closure and completeness of orthonormal systems in the space $L^2(\Omega)$ is fundamental for approximating square-integrable functions on the sphere by Fourier (spherical harmonic) expansions. The closure in $L^2(\Omega)$ can be derived from Bernstein or Abel-Poisson summability. The Funk–Hecke formula establishes the close connection between the orthogonal invariance of the sphere and the addition theorem. It turns out that any spherical harmonic is an eigenfunction of the Beltrami operator. The angular derivatives, i.e., the operators of the longitude and latitude, are shown to act as anisotropic operators within the framework of scalar spherical harmonics. Finally, the usually (in geosciences) used $L^2(\Omega)$-orthonormal system of scalar spherical harmonics involving associated Legendre functions is introduced; its representation in terms of trigonometric functions is discussed in more detail. Associated Legendre harmonics are generated exactly entirely by integer operations.

3.1 Homogeneous Harmonic Polynomials

Let Hom_n (more accurately: $\mathrm{Hom}_n(\mathbb{R}^3)$) consist of all polynomials H_n in three variables which are homogeneous of degree n (i.e., $H_n(\lambda x) = \lambda^n H_n(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^3$). Thus, if $H_n \in \mathrm{Hom}_n$, then there exist real numbers $C_\alpha = C_{\alpha_1\alpha_2\alpha_3}$ such that

$$H_n(x) = \sum_{[\alpha]=n} C_\alpha \, x^\alpha. \tag{3.1}$$

In cartesian coordinates,

$$H_n(x_1, x_2, x_3) = \sum_{\alpha_1+\alpha_2+\alpha_3=n} C_{\alpha_1\alpha_2\alpha_3} \, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \tag{3.2}$$

It is obvious that the set of monomials $x \mapsto x^\alpha$, $[\alpha] = n$, is a basis for the space Hom_n. The number of such monomials is precisely the number of ways a triple can be chosen so that we have $[\alpha] = n$, i.e., the number of ways of selecting 2 elements out of a collection of $n + 2$. This means that the dimension $d(\mathrm{Hom}_n)$ of Hom_n is equal to

$$d(\mathrm{Hom}_n) = \frac{(n+1)(n+2)}{2} = \binom{n+2}{2}. \tag{3.3}$$

Let $H_n(\nabla_x)$ be the differential operator associated to $H_n(x)$ (i.e., replace x^α formally by $(\nabla_x)^\alpha$ in the expression of $H_n(x)$):

$$H_n(\nabla_x) = \sum_{\alpha_1+\alpha_2+\alpha_3=n} C_{\alpha_1\alpha_2\alpha_3} \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} = \sum_{[\alpha]=n} C_\alpha (\nabla_x)^\alpha \, . \tag{3.4}$$

If such an operator is applied to a homogeneous polynomial U_n of the same degree

$$U_n(x) = \sum_{[\beta]=n} D_\beta \, x^\beta, \tag{3.5}$$

we obtain as result a real number:

$$\begin{aligned}
&(H_n(\nabla_x)) \, U_n(x) \\
&= \sum_{[\alpha]=n} \sum_{[\beta]=n} C_\alpha \, D_\beta \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} x_1^{\beta_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} x_2^{\beta_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3} x_3^{\beta_3} \\
&= \sum_{[\alpha]=n} C_\alpha \, D_\alpha \, \alpha! \, ,
\end{aligned} \tag{3.6}$$

where the factorial of a multi-index is defined as $\alpha! = \alpha_1!\alpha_2!\alpha_3!$. Clearly, we find

$$\begin{aligned} (H_n(\nabla_x))\ U_n(x) &= (U_n(\nabla_x))\ H_n(x), \\ (H_n(\nabla_x))\ H_n(x) &\geq 0. \end{aligned} \tag{3.7}$$

This enables us to introduce an inner product $(\cdot,\cdot)_{\mathrm{Hom}_n}$ on the space Hom_n by letting

$$(H_n, U_n)_{\mathrm{Hom}_n} = (H_n(\nabla_x))\ U_n(x). \tag{3.8}$$

The space Hom_n equipped with the inner product $(\cdot,\cdot)_{\mathrm{Hom}_n}$ is a finite-dimensional Hilbert space. The set of monomials

$$\{x \mapsto (\alpha!)^{-1/2} x^\alpha \mid [\alpha] = n\}$$

forms an orthonormal system in the space Hom_n. For each $H_n \in \mathrm{Hom}_n$, we have in connection with (3.4)

$$\begin{aligned} H_n(x) &= \sum_{[\alpha]=n} \frac{1}{\alpha!}\ (H_n(\nabla_y))\ y^\alpha\ x^\alpha \\ &= (H_n(\nabla_y))\ \frac{1}{n!} \sum_{[\alpha]=n} \frac{n!}{\alpha!}\ x^\alpha y^\alpha \\ &= (H_n(\nabla_y))\ \frac{(x\cdot y)^n}{n!} \\ &= \frac{1}{n!}\ (x\cdot\nabla_y)^n\ H_n(y). \end{aligned} \tag{3.9}$$

In other words,

$$H_n(x) = (\frac{(x\cdot\ )^n}{n!}, H_n)_{\mathrm{Hom}_n}. \tag{3.10}$$

Theorem 3.1. Hom_n *equipped with the inner product* $(\cdot,\cdot)_{\mathrm{Hom}_n}$ *is a finite-dimensional Hilbert space of dimension* $\frac{(n+1)(n+2)}{2}$ *with the reproducing kernel*

$$K_{\mathrm{Hom}_n}(x,y) = \frac{(x\cdot y)^n}{n!}, \quad x,y \in \mathbb{R}^3, \tag{3.11}$$

i.e.,

(i) *for every fixed* y*, the function* $K_{\mathrm{Hom}_n}(\cdot,y)$ *belongs to* Hom_n*,*

(ii) *for any* $H_n \in \mathrm{Hom}_n$ *and any point* x *the reproducing property*

$$H_n(x) = (K_{\mathrm{Hom}_n}(x,\cdot), H_n)_{\mathrm{Hom}_n}$$

is valid.

Let $\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)}$, $\{U_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)}$ be two orthonormal systems in the space Hom_n:

$$\begin{aligned} (H_{n,j}, H_{n,k})_{\mathrm{Hom}_n} &= \delta_{jk}, \\ (U_{n,j}, U_{n,k})_{\mathrm{Hom}_n} &= \delta_{jk}, \end{aligned} \tag{3.12}$$

where δ_{jk} is the usual Kronecker symbol. Then, for $j = 1, ..., d(\mathrm{Hom}_n)$, we have

$$\begin{aligned} H_{n,j} &= \sum_{k=1}^{d(\mathrm{Hom}_n)} (H_{n,j}, U_{n,k})_{\mathrm{Hom}_n} U_{n,k}, \\ U_{n,j} &= \sum_{k=1}^{d(\mathrm{Hom}_n)} (U_{n,j}, H_{n,k})_{\mathrm{Hom}_n} H_{n,k}. \end{aligned} \tag{3.13}$$

Therefore, it follows that

$$\sum_{j=1}^{d(\mathrm{Hom}_n)} H_{n,j}(x)\ H_{n,j}(y) = \sum_{j=1}^{d(\mathrm{Hom}_n)} U_{n,j}(x) U_{n,j}(y). \tag{3.14}$$

Hence, in particular, for the orthonormal system of monomials, we obtain the following result.

Theorem 3.2. *Let* $\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)}$ *be an orthonormal system in* Hom_n. *Then*

$$K_{\mathrm{Hom}_n}(x,y) = \frac{(x\cdot y)^n}{n!} = \sum_{j=1}^{d(\mathrm{Hom}_n)} H_{n,j}(x) H_{n,j}(y), \quad x,y \in \mathbb{R}^3. \tag{3.15}$$

$K_{\mathrm{Hom}_n}(\cdot,\cdot)$ *is the only reproducing kernel in* Hom_n.

Suppose that there are given $d(\mathrm{Hom}_n)$ points $x_1, ..., x_{d(\mathrm{Hom}_n)} \in \mathbb{R}^3$ and $d(\mathrm{Hom}_n)$-values $d_1, ..., d_{d(\mathrm{Hom}_n)} \in \mathbb{R}$. We are able to solve the Hom_n- interpolation problem

$$\sum_{j=1}^{d(\mathrm{Hom}_n)} b_j H_{n,j}(x_k) = d_k, \quad k = 1, ..., d(\mathrm{Hom}_n), \tag{3.16}$$

if and only if the matrix

$$\mathbf{matr}_{\{x_1,\ldots,x_{d(\mathrm{Hom}_n)}\}}(H_{n,1}, \ldots, H_{n,d(\mathrm{Hom}_n)}) \tag{3.17}$$

$$= \begin{pmatrix} H_{n,1}(x_1) & \ldots & H_{n,1}(x_{d(\mathrm{Hom}_n)}) \\ \vdots & \ddots & \vdots \\ H_{n,d(\mathrm{Hom}_n)}(x_1) & \ldots & H_{n,d(\mathrm{Hom}_n)}(x_{d(\mathrm{Hom}_n)}) \end{pmatrix}$$

is non-singular. A system of $d(\mathrm{Hom}_n)$ points $x_1, ..., x_{d(\mathrm{Hom}_n)}$ is called a *fundamental system relative to* Hom_n if the matrix (3.17) is non-singular.

In what follows, we guarantee the existence of a fundamental system relative to Hom_n (cf. C. Müller (1966)).

Lemma 3.3. *There exists a system* $\{x_1, \ldots, x_{d(\mathrm{Hom}_n)}\} \subset \mathbb{R}^3$ *such that (3.17) is non-singular.*

Proof. As orthonormal system, the functions $H_{n,1}, ..., H_{n,d(\mathrm{Hom}_n)}$ are linearly independent. Hence, there exists a point x_1 for which

$$H_{n,1}(x_1) \neq 0. \tag{3.18}$$

Now, there must also be a point x_2 such that

$$\begin{vmatrix} H_{n,1}(x_1) & H_{n,1}(x_2) \\ H_{n,2}(x_1) & H_{n,2}(x_2) \end{vmatrix} \neq 0, \tag{3.19}$$

for else, we would have a contradiction to the linear independence of $H_{n,1}$, $H_{n,2}$. In the same way, the existence of a point x_3 can be deduced by the requirement

$$\begin{vmatrix} H_{n,1}(x_1) & H_{n,1}(x_2) & H_{n,1}(x_3) \\ H_{n,2}(x_1) & H_{n,2}(x_2) & H_{n,2}(x_3) \\ H_{n,3}(x_1) & H_{n,3}(x_2) & H_{n,3}(x_3) \end{vmatrix} \neq 0. \tag{3.20}$$

Finally, by induction, we obtain a system of points $x_1, ..., x_{d(\mathrm{Hom}_n)}$ such that

$$\begin{vmatrix} H_{n,1}(x_1) & \ldots & H_{n,1}(x_{d(\mathrm{Hom}_n)}) \\ \vdots & \ddots & \vdots \\ H_{n,d(\mathrm{Hom}_n)}(x_1) & \ldots & H_{n,d(\mathrm{Hom}_n)}(x_{d(\mathrm{Hom}_n)}) \end{vmatrix} \neq \; 0, \tag{3.21}$$

i.e., $\{x_1, \ldots, x_{d(\mathrm{Hom}_n)}\}$ constitutes a fundamental system relative to Hom_n. □

To every $H_n \in \mathrm{Hom}_n$, there exist real numbers $b_1, ..., b_{d(\mathrm{Hom}_n)}$ such that

$$H_n \; = \; \sum_{k=1}^{d(\mathrm{Hom}_n)} b_k \; H_{n,k}. \tag{3.22}$$

Under the assumption that $\{x_1, ..., x_{d(\mathrm{Hom}_n)}\}$ is a fundamental system relative to Hom_n, the linear equations

$$\sum_{j=1}^{d(\mathrm{Hom}_n)} a_j \; H_{n,k}(x_j) \; = \; b_k, \quad k = 1, ..., d(\mathrm{Hom}_n) \; , \tag{3.23}$$

are uniquely solvable in the unknowns $a_1, ..., a_{d(\mathrm{Hom}_n)}$. Thus, we obtain

$$H_n = \sum_{k=1}^{d(\mathrm{Hom}_n)} \sum_{j=1}^{d(\mathrm{Hom}_n)} a_j \; H_{n,k}(x_j) \; H_{n,k}. \tag{3.24}$$

Theorem 3.4. *Let* $\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)}$ *be an orthonormal system in* Hom_n. *Assume that* $\{x_k\}_{k=1,\ldots,d(\mathrm{Hom}_n)}$ *is a fundamental system relative to* Hom_n. *Then, each* $H_n \in \mathrm{Hom}_n$ *is uniquely representable in the form*

$$H_n(x) = \sum_{j=1}^{d(\mathrm{Hom}_n)} a_j \; K_{\mathrm{Hom}_n}(x_j, x) = \sum_{j=1}^{d(\mathrm{Hom}_n)} a_j \frac{(x_j \cdot x)^n}{n!}. \tag{3.25}$$

Let Harm_n (more accurately: $\mathrm{Harm}_n(\mathbb{R}^3)$) be the class of all polynomials in Hom_n that are harmonic:

$$\mathrm{Harm}_n = \{H_n \in \mathrm{Hom}_n \mid \Delta_x H_n(x) = 0, \; x \in \mathbb{R}^3\}. \tag{3.26}$$

For $n < 2$, of course, all homogeneous polynomials are harmonic.

Any homogeneous harmonic polynomial of degree n can be represented in the form

$$H_n(x) = H_n(x_1, x_2, x_3) = \sum_{j=0}^{n} x_3^j A_{n-j}(x_1, x_2), \tag{3.27}$$

where A_{n-j} is a homogeneous polynomial of degree $n-j$ in the variables x_1, x_2. Application of the Laplace operator gives

$$\begin{aligned} 0 = \Delta_x H_n(x) &= \sum_{j=0}^{n} \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 \right) x_3^j A_{n-j}(x_1, x_2) \\ &= \sum_{j=0}^{n-2} x_3^j \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 \right) A_{n-j}(x_1, x_2) \\ &\quad + \sum_{j=0}^{n-2} x_3^j (j+2)(j+1) A_{n-j-2}(x_1, x_2), \end{aligned} \tag{3.28}$$

where we have used the facts that

$$\begin{aligned} \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 \right) A_0(x_1, x_2) &= 0, \\ \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 \right) A_1(x_1, x_2) &= 0. \end{aligned} \tag{3.29}$$

Thus, the functions $A_{n-j} : \mathbb{R}^2 \to \mathbb{R}$ satisfy the recursion relation

$$\begin{aligned}\left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right) A_{n-j}(x_1, x_2) + (j+2)(j+1)A_{n-j-2}(x_1, x_2) \\ = 0 \end{aligned} \tag{3.30}$$

for $j = 0, 1, ..., n-2$. Therefore, all polynomials A_{n-j} are determined if we know A_n and A_{n-1}.

Theorem 3.5. *Let A_n and A_{n-1} be homogeneous polynomials of degree n and $n-1$ in $\mathbb{R}^2$, respectively. For $j = 0, ..., n-2$ we set recursively*

$$A_{n-j-2}(x_1, x_2) = -\frac{1}{(j+1)(j+2)} \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right) A_{n-j}(x_1, x_2). \tag{3.31}$$

Then $H_n : \mathbb{R}^3 \to \mathbb{R}$ given by

$$H_n(x_1, x_2, x_3) = \sum_{j=0}^{n} x_3^j A_{n-j}(x_1, x_2) \tag{3.32}$$

is a homogeneous harmonic polynomial of degree n in $\mathbb{R}^3$, i.e., $H_n \in \mathrm{Harm}_n$. The number of linearly independent homogeneous harmonic polynomials is equal to the number of coefficients of A_n and A_{n-1}, i.e.,

$$d(\mathrm{Harm}_n) = n + n + 1 = 2n + 1.$$

Assume that n is an integer with $n \geq 2$. Let H_{n-2} be a homogeneous polynomial of degree $n-2$, i.e., $H_{n-2} \in \mathrm{Hom}_{n-2}$. Then, for each homogeneous harmonic polynomial K_n, we have

$$(|\cdot|^2 H_{n-2}, K_n)_{\mathrm{Hom}_n} = (H_{n-2}(\nabla_x))\Delta_x K_n(x) = 0 . \tag{3.33}$$

This means $|\cdot|^2 H_{n-2}$ is orthogonal to K_n in the sense of the inner product $(\cdot, \cdot)_{\mathrm{Hom}_n}$. Conversely, suppose that $K_n \in \mathrm{Hom}_n$ is orthogonal to all elements L_n of the form

$$L_n(x) = |x|^2 H_{n-2}(x) , \quad H_{n-2} \in \mathrm{Hom}_{n-2}. \tag{3.34}$$

Then it follows that

$$0 = (|\cdot|^2 H_{n-2}, K_n)_{\mathrm{Hom}_n} = (H_{n-2}(\nabla_x))\Delta_x K_n(x) = (H_{n-2}, \Delta K_n)_{\mathrm{Hom}_{n-2}} \tag{3.35}$$

for all $H_{n-2} \in \mathrm{Hom}_{n-2}$. This is true only if $\Delta K_n = 0$, i.e., K_n is a homogeneous harmonic polynomial.

Theorem 3.6. *(Decomposition Theorem of* Hom_n*)* Hom_n, $n \geq 2$, *is the orthogonal direct sum of* Harm_n *and* $\mathrm{Harm}_n^{\perp}$, *where* $\mathrm{Harm}_n^{\perp} = |\cdot|^2 \mathrm{Hom}_{n-2}$ *is the space of all* L_n *with* $L_n(x) = |x|^2 H_{n-2}(x)$, $H_{n-2} \in \mathrm{Hom}_{n-2}$. *Consequently, each homogeneous polynomial* H_n *of degree* n *can be uniquely decomposed in the form*

$$H_n(x) = K_n(x) + |x|^2 H_{n-2}(x) , \tag{3.36}$$

where K_n *is a homogeneous harmonic polynomial of degree* n *and* H_{n-2} *is a homogeneous polynomial of degree* $n-2$.

Denote by $\mathrm{Proj}_{\mathrm{Harm}_n}$ and $\mathrm{Proj}_{\mathrm{Harm}_n^{\perp}}$ the projection operators in Hom_n onto Harm_n and $\mathrm{Harm}_n^{\perp}$, respectively. Then

$$H_n = \mathrm{Proj}_{\mathrm{Harm}_n} H_n + \mathrm{Proj}_{\mathrm{Harm}_n^{\perp}} H_n . \tag{3.37}$$

In other words,

$$K_n(x) = \mathrm{Proj}_{\mathrm{Harm}_n} H_n(x), \tag{3.38}$$
$$|x|^2 H_{n-2}(x) = \mathrm{Proj}_{\mathrm{Harm}_n^{\perp}} H_n(x) . \tag{3.39}$$

For all $H_n, U_n \in \mathrm{Hom}_n$,

$$(\mathrm{Proj}_{\mathrm{Harm}_n} H_n, U_n)_{\mathrm{Hom}_n} = (H_n, \mathrm{Proj}_{\mathrm{Harm}_n} U_n)_{\mathrm{Hom}_n} . \tag{3.40}$$

Moreover, we have $\mathrm{Proj}_{\mathrm{Harm}_n} H_n = \mathrm{Proj}_{\mathrm{Harm}_n} K_n = K_n$. Observe that

$$\begin{aligned} d(\mathrm{Harm}_n) &= d(\mathrm{Hom}_n) - d(\mathrm{Harm}_n^{\perp}) \\ &= d(\mathrm{Hom}_n) - d(\mathrm{Hom}_{n-2}) \\ &= \binom{n+2}{2} - \binom{n}{2} = 2n+1. \end{aligned} \tag{3.41}$$

If we apply Theorem 3.6 recursively to H_{n-2}, $H_{n-4}, \ldots$, we obtain the following result.

Theorem 3.7. *Each homogeneous polynomial of degree* n *can be uniquely decomposed in the form*

$$H_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} |x|^{2i} K_{n-2i}(x), \quad K_{n-2i} \in \mathrm{Harm}_{n-2i}, \quad x \in \mathbb{R}^3, \tag{3.42}$$

where $\lfloor n/2 \rfloor$ *is the largest integer which is less than or equal to* $n/2$.

In other words, Hom_n admits the direct sum decomposition

$$\mathrm{Hom}_n(\mathbb{R}^3) = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} |\cdot|^{2i} \mathrm{Harm}_{n-2i}(\mathbb{R}^3). \tag{3.43}$$

This result gives rise to the following corollary.

Corollary 3.8. *For* $n = 0, 1, \ldots$

$$\mathrm{Hom}_n(\mathbb{R}^3)|\Omega = \mathrm{Hom}_n(\Omega) = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{Harm}_{n-2i}(\mathbb{R}^3)|\Omega.$$

Since the space $\mathrm{Pol}_{0,\ldots,n}(\mathbb{R}^3)$ of polynomials in three variables of degree $\leq n$ can be written as direct sum decomposition of $\mathrm{Hom}_n(\mathbb{R}^3)$ and $\mathrm{Hom}_{n-1}(\mathbb{R}^3)$, when restricted to Ω, i.e.,

$$\mathrm{Pol}_{0,\ldots,n}(\mathbb{R}^3)|\Omega = (\mathrm{Hom}_n(\mathbb{R}^3)|\Omega) \oplus (\mathrm{Hom}_{n-1}(\mathbb{R}^3)|\Omega) \tag{3.44}$$

we finally obtain the following corollary.

Corollary 3.9. *For* $n = 0, 1, \ldots$

$$\mathrm{Pol}_{0,\ldots,n}(\mathbb{R}^3)|\Omega = \bigoplus_{i=0}^{n} \mathrm{Harm}_i(\mathbb{R}^3)|\Omega.$$

In other words, the restriction to the unit sphere Ω of any polynomial of three variables is a sum of restrictions to Ω of homogeneous harmonic polynomials.

3.2 Addition Theorem

We are now interested in giving the explicit representation of the orthogonal projection $\mathrm{Proj}_{\mathrm{Harm}_n} H_n$ of a given homogeneous polynomial H_n. For that purpose, we need some preliminaries. By induction, we are able to prove that for $i = 1, 2, 3$ and $|x| \neq 0$ (cf. E.W. Hobson (1955))

$$\begin{aligned} &\left(\frac{\partial}{\partial x_i}\right)^n \frac{1}{|x|} \qquad (3.45) \\ &= (-1)^n \frac{(2n)!}{n! 2^n} \frac{1}{|x|^{2n+1}} \left(\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right) x_i^n . \end{aligned}$$

In other words, we find

$$\begin{aligned} & (\varepsilon^i \cdot \nabla_x)^n \frac{1}{|x|} \qquad (3.46) \\ = \; & (-1)^n \frac{(2n)!}{n!2^n} \frac{1}{|x|^{2n+1}} \left(\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right) \left(\varepsilon^i \cdot x\right)^n \end{aligned}$$

$(i = 1,2,3)$. Since the differential operator Δ is invariant with respect to orthogonal transformations, it is easy to see that

$$\begin{aligned} & (y \cdot \nabla_x)^n \frac{1}{|x|} \qquad (3.47) \\ = \; & (-1)^n \frac{(2n)!}{n!2^n} \frac{1}{|x|^{2n+1}} \left(\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right) (y \cdot x)^n \end{aligned}$$

is valid for every $y \in \mathbb{R}^3$. Now, as we have seen in Theorem 3.4, each $H_n \in \mathrm{Hom}_n$ may be represented in the form

$$H_n(x) = \sum_{j=1}^{d(\mathrm{Hom}_n)} c_j (x_j \cdot x)^n, \quad x \in \mathbb{R}^3, \qquad (3.48)$$

where $c_j,\ j = 1, ..., d(\mathrm{Hom}_n)$, are suitable coefficients and $x_1, ..., x_{d(\mathrm{Hom}_n)}$ is a fundamental system relative to Hom_n.

Consequently, we have the following result:

Theorem 3.10. *Let H_n be a homogeneous polynomial of degree n. Then, for each $x \in \mathbb{R}^3$, $|x| \neq 0$,*

$$\begin{aligned} & (H_n(\nabla_x)) \frac{1}{|x|} \\ = \; & (-1)^n \frac{(2n)!}{n!2^n} \frac{1}{|x|^{2n+1}} \left(\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right) H_n(x). \end{aligned}$$

Using the decomposition (3.36) as in Theorem 3.6, it follows that

$$(H_n(\nabla_x)) \frac{1}{|x|} = (K_n(\nabla_x)) \frac{1}{|x|} + (H_{n-2}(\nabla_x)) \Delta_x \frac{1}{|x|}, \quad |x| \neq 0. \qquad (3.49)$$

Thus, in connection with

$$\begin{aligned} \Delta_x \frac{1}{|x|} &= 0, \quad |x| \neq 0, \\ \Delta_x K_n(x) &= 0, \quad x \in \mathbb{R}^3, \end{aligned} \qquad (3.50)$$

we obtain for $|x| \neq 0$

$$(H_n(\nabla_x))\frac{1}{|x|} = (K_n(\nabla_x))\frac{1}{|x|} = (-1)^n\frac{(2n)!}{n!2^n}\frac{1}{|x|^{2n+1}}K_n(x). \tag{3.51}$$

By solving (3.51) for $K_n(x)$, we get from Theorem 3.10 the following lemma.

Lemma 3.11. *Let H_n be a homogeneous polynomial of degree n. Then*

$$\mathrm{Proj}_{\mathrm{Harm}_n} H_n(x) = \left(\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!}|x|^{2s}\Delta^s\right) H_n(x) \tag{3.52}$$

such that

$$\begin{aligned} \mathrm{Proj}_{\mathrm{Harm}_n^\perp} H_n(x) &= H_n(x) - \mathrm{Proj}_{\mathrm{Harm}_n} H_n(x) \\ &= \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s-1} \frac{n!(2n-2s)!}{(2n)!(n-s)!s!}|x|^{2s}\Delta^s H_n(x). \end{aligned} \tag{3.53}$$

The differential operator $\mathrm{Proj}_{\mathrm{Harm}_n^\perp}$ given by (3.53) is called the *Clebsch projection* (see E.W. Hobson (1955)). It forms a mapping from $H_n \in \mathrm{Hom}_n$ to $K_n \in \mathrm{Harm}_n$ such that

$$K_n(x) = H_n(x) - \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s-1} \frac{n!(2n-2s)!}{(2n)!(n-2)!s!}|x|^{2s}\Delta^s H_n(x). \tag{3.54}$$

Remark 3.12. Note that

$$|x|^2 H_{n-2}(x) = \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^{s-1} \frac{n!(2n-2s)!}{(2n)!(n-s)!s!}|x|^{2s}\Delta^s H_n(x), \tag{3.55}$$

hence, the Clebsch projection can be regarded as a mechanism for the division by $|x|^2$.

Observing

$$\Delta_x (x \cdot y)^n = n(n-1)|y|^2 (x\cdot y)^{n-2}, \quad y \in \mathbb{R}^3, \tag{3.56}$$

we obtain,

$$\mathrm{Proj}_{\mathrm{Harm}_n}\left(\frac{(x\cdot y)^n}{n!}\right) \tag{3.57}$$
$$= \frac{1}{n!}\sum_{s=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^s \frac{(2n-2s)!(n!)^2}{(n-2s)!(n-s)!s!(2n)!}|x|^{2s}|y|^{2s}(x\cdot y)^{n-2s}.$$

Thus, we find by using $x=|x|\xi$, $y=|y|\eta$, $\xi,\eta\in\Omega$, the equation

$$\mathrm{Proj}_{\mathrm{Harm}_n}\left(\frac{(x\cdot y)^n}{n!}\right) \tag{3.58}$$
$$= \frac{(2n+1)2^n\cdot n!}{(2n+1)!}\,(|x|\;|y|)^n\sum_{s=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^s \frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!}(\xi\cdot\eta)^{n-2s}.$$

Suppose that $\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Harm}_n)}$ is an orthonormal system in Harm_n with respect to $(\cdot,\cdot)_{\mathrm{Hom}_n}$. Let $\{U_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)-d(\mathrm{Harm}_n)}$ be an orthonormal system in $\mathrm{Harm}_n^{\perp}$. Then, the union of both systems

$$\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Harm}_n)}\cup\{U_{n,j}\}_{j=1,\ldots,d(\mathrm{Hom}_n)-d(\mathrm{Harm}_n)} \tag{3.59}$$

forms an orthonormal system in Hom_n. Therefore, it follows that

$$\frac{(x\cdot y)^n}{n!} \tag{3.60}$$
$$= \sum_{j=1}^{d(\mathrm{Harm}_n)} H_{n,j}(x)\;H_{n,j}(y) + \sum_{j=1}^{d(\mathrm{Hom}_n)-d(\mathrm{Harm}_n)} U_{n,j}(x)\;U_{n,j}(y)$$

for any pair $x,y\in\mathbb{R}^3$. On the one hand, in view of the definition of the projection operator $\mathrm{Proj}_{\mathrm{Harm}_n}$, we get

$$\mathrm{Proj}_{\mathrm{Harm}_n}\left(\sum_{j=1}^{d(\mathrm{Harm}_n)} H_{n,j}(x)H_{n,j}(y) + \sum_{j=1}^{d(\mathrm{Hom}_n)-d(\mathrm{Harm}_n)} U_{n,j}(x)U_{n,j}(y)\right)$$
$$= \sum_{j=1}^{d(\mathrm{Harm}_n)} H_{n,j}(x)H_{n,j}(y). \tag{3.61}$$

On the other hand, as we have shown above,

$$\mathrm{Proj}_{\mathrm{Harm}_n}\left(\frac{(x\cdot y)^n}{n!}\right) \tag{3.62}$$
$$= \frac{(2n+1)2^n n!}{(2n+1)!}|x|^n|y|^n\sum_{s=0}^{\lfloor n/2\rfloor}(-1)^s \frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!}(\xi\cdot\eta)^{n-2s}.$$

By comparison of (3.61) and (3.62), we obtain the *addition theorem of homogeneous harmonic polynomials in* $\mathbb{R}^3$.

Theorem 3.13. *Let* $\{H_{n,j}\}_{j=1,\ldots,d(\mathrm{Harm}_n)}$, $d(\mathrm{Harm}_n) = 2n+1$, *be an orthonormal system in* Harm_n *with respect to* $(\cdot,\cdot)_{\mathrm{Hom}_n}$. *Then, for* $x, y \in \mathbb{R}^3$, $x = |x|\xi$, $y = |y|\eta$, *we have*

$$\sum_{j=1}^{2n+1} H_{n,j}(x)\ H_{n,j}(y) = \frac{2^n n!}{(2n)!}|x|^n|y|^n P_n(\xi\cdot\eta),$$

where we have used the abbreviation

$$P_n(t) \;=\; \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^s \frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!}t^{n-2s}, \quad t \in [-1,1]. \tag{3.63}$$

Remark 3.14. P_n is known as the *Legendre polynomial of degree* n (see Section 3.5 for a detailed description).

Next, we discuss the important question of how, for any pair of elements $H_n \in \mathrm{Harm}_n$, $K_n \in \mathrm{Harm}_n$, the inner product $(\cdot,\cdot)_{\mathrm{Hom}_n}$ defined by (3.8) is related to the (usually used) inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$.

Theorem 3.15. *For* $H_m \in \mathrm{Harm}_m, K_n \in \mathrm{Harm}_n$,

$$(H_m, K_n)_{\mathrm{L}^2(\Omega)} \;=\; \frac{\delta_{nm}}{\mu_n}(H_m(\nabla_x))K_n(x), \tag{3.64}$$

where μ_n *is given by*

$$\mu_n = \frac{(2n+1)!}{4\pi 2^n n!} = \frac{1\cdot 3\cdot \ldots \cdot (2n+1)}{4\pi}. \tag{3.65}$$

Proof. By virtue of the Third Green Theorem of potential theory (see, Theorem 2.4), we find

$$K_n(x) \;=\; \frac{1}{4\pi}\int_\Omega \left\{ \frac{1}{|x-y|}\frac{\partial}{\partial\nu_y}K_n(y) - K_n(y)\frac{\partial}{\partial\nu_y}\frac{1}{|x-y|}\right\} d\omega(y) \tag{3.66}$$

for all $x \in \mathbb{R}^3$ with $|x| < 1$, where $\partial/\partial\nu$ denotes the derivative in the direction of the outer normal to Ω. Therefore, we find

$$\begin{aligned}(H_m(\nabla_x))K_n(x) = \frac{1}{4\pi}\int_\Omega \Big\{ &(H_m(\nabla_x))\frac{1}{|x-y|}\frac{\partial}{\partial\nu_y}K_n(y) \\ - \;\; &K_n(y)\frac{\partial}{\partial\nu_y}(H_m(\nabla_x))\frac{1}{|x-y|}\Big\}\, d\omega(y).\end{aligned} \tag{3.67}$$

For $x \neq y$, we get from (3.51)

$$(H_m(\nabla_x))\frac{1}{|x-y|} = (-1)^m \frac{(2m)!}{m!2^m}\frac{H_m(x-y)}{|x-y|^{2m+1}}. \tag{3.68}$$

Because H_m is homogeneous, this is equivalent to

$$(H_m(\nabla_x))\frac{1}{|x-y|} = \frac{(2m)!}{m!2^m}\frac{H_m(y-x)}{|x-y|^{2m+1}}. \tag{3.69}$$

Inserting (3.69) into (3.67) gives

$$\begin{aligned}(H_m(\nabla_x))K_n(x) &= \frac{(2m)!}{(m!)2^m}\frac{1}{4\pi}\int_\Omega \left\{ \frac{H_m(y-x)}{|x-y|^{2m+1}}\frac{\partial}{\partial\nu_y}K_n(y) \right. \\ &\quad \left. -K_n(y)\frac{\partial}{\partial\nu_y}\frac{H_m(y-x)}{|x-y|^{2m+1}}\right\} d\omega(y). \end{aligned}\tag{3.70}$$

It is easy to see that for $m \neq n$

$$(H_m(\nabla_x))K_n(x)\,|_{x=0} = 0, \tag{3.71}$$

while for $m = n$

$$(H_m(\nabla_x))K_n(x)\,|_{x=0} = (H_m(\nabla_x))K_n(x) = (H_m, K_n)_{\mathrm{Hom}_n}. \tag{3.72}$$

Therefore, we obtain

$$\begin{aligned}&\frac{1}{4\pi}\int_\Omega \left\{ \frac{H_m(y)}{|y|^{2m+1}}\frac{\partial}{\partial\nu_y}K_n(y) - K_n(y)\frac{\partial}{\partial\nu_y}\frac{H_m(y)}{|y|^{2m+1}}\right\} d\omega(y) \\ &= \begin{cases} 0 & \text{for} \quad m \neq n \\ \left(\frac{2^m m!}{(2m)!}\right)(H_m, K_n)_{\mathrm{Hom}_n} & \text{for} \quad m = n \end{cases}.\end{aligned}\tag{3.73}$$

Since the normal derivatives of K_n and H_m are equal to

$$\frac{\partial}{\partial r}K_n(r\xi)\,|_{r=1} = nK_n(\xi)\,, \qquad \frac{\partial}{\partial r}H_m(r\xi)\,|_{r=1} = mH_m(\xi), \tag{3.74}$$

respectively, it follows that

$$\begin{aligned}&\frac{1}{4\pi}\int_\Omega \left\{ \frac{H_m(y)}{|y|^{2m+1}}\frac{\partial}{\partial\nu_y}K_n(y) - K_n(y)\frac{\partial}{\partial\nu_y}\frac{H_m(y)}{|y|^{2m+1}}\right\} d\omega(y) \\ &= \frac{1}{4\pi}\int_\Omega \{nH_m(\xi)K_n(\xi) + (m+1)H_m(\xi)K_n(\xi)\}\, d\omega(\xi) \\ &= \frac{n+m+1}{4\pi}\int_\Omega H_m(\xi)K_n(\xi)\, d\omega(\xi).\end{aligned}\tag{3.75}$$

Thus, by combination of (3.73) and (3.75), we finally obtain the desired result. □

In other words, to any orthonormal system $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ in Harm_n with respect to $(\cdot,\cdot)_{\mathrm{Hom}_n}$ there corresponds the $\mathrm{L}^2(\Omega)$-orthonormal system $\{\sqrt{\mu_n} H_{n,j}\}_{j=1,\ldots,2n+1}$, and vice versa.

Finally, we are led to the following reformulation of the addition theorem.

Theorem 3.16. $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ *is an orthonormal system in* Harm_n *with respect to* $(\cdot,\cdot)_{\mathrm{Hom}_n}$ *if and only if* $\{\sqrt{\mu_n} H_{n,j}\}_{j=1,\ldots 2n+1}$ *is an orthonormal system in* Harm_n *with respect to* $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$*. For* $x, y \in \mathbb{R}^3$*, we have*

$$\sum_{j=1}^{2n+1} \sqrt{\mu_n} H_{n,j}(x) \sqrt{\mu_n} H_{n,j}(y) = \frac{2n+1}{4\pi} |x|^n |y|^n P_n(\xi \cdot \eta).$$

where

$$\mu_n = \frac{(2n+1)!}{4\pi 2^n n!} = \frac{1 \cdot 3 \cdot \ldots \cdot (2n+1)}{4\pi}.$$

We summarize the relationship between the two topologies in Harm_n in Table 3.1.

Table 3.1: Comparison of inner products.

Topologies in Harm_n		
$(H_n, K_n)_{\mathrm{Hom}_n} = H_n(\nabla_x) K_n(x)$	|	$(H_n, K_n)_{\mathrm{L}^2(\Omega)} = \int_\Omega H_n(\xi) K_n(\xi)\, d\omega(\xi)$
$\frac{1}{\mu_n} (H_n, K_n)_{\mathrm{Hom}_n}$	$=$	$(H_n, K_n)_{\mathrm{L}^2(\Omega)}$

3.3 Exact Computation of Homogeneous Harmonic Polynomials

Our purpose is to explain how a maximal linearly independent system of homogeneous harmonic polynomials of degree n can be generated exactly (see W. Freeden, R. Reuter (1984)). The concept is based on the observation that any linearly independent system $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ of homogeneous harmonic polynomials of degree n

$$\begin{aligned} H_{n,1}(x) &= \sum_{[\alpha]=n} C^1_\alpha x^\alpha \\ \vdots\ \vdots & \quad\ \vdots\ \vdots \\ H_{n,2n+1}(x) &= \sum_{[\alpha]=n} C^{2n+1}_\alpha x^\alpha \end{aligned} \tag{3.76}$$

can be calculated by *exact computation* of the coefficients C^j_α, $j = 1, \ldots,$ $2n+1$, i.e., entirely by integer operations (note that we briefly write C^j_α instead of $C^{n,j}_\alpha$ when confusion is not likely to arise). In other words, we want to show that the coefficients C^j_α, $j = 1, ..., 2n+1$, in (3.76) can be expressed as integers.

Let H_n be a homogeneous polynomial of the form $H_n = \Sigma_{[\alpha]=n} C_\alpha x^\alpha$, $x \in \mathbb{R}^3$, $n \geq 2$. Assuming that H_n is harmonic, i.e., $\Delta_x H_n(x) = 0$, $x \in \mathbb{R}^3$, we obtain

$$\Delta_x H_n(x) = \Delta_x \sum_{[\alpha]=n} C_\alpha x^\alpha = \sum_{[\alpha]=n} C_\alpha \Delta_x(x^\alpha) = 0. \tag{3.77}$$

Thus, it follows that

$$\sum_{\alpha_1+\alpha_2+\alpha_3=n} C_\alpha \big(\alpha_1(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2}x_3^{\alpha_3} + \alpha_2(\alpha_2-1)x_2^{\alpha_2-2}x_1^{\alpha_1}x_3^{\alpha_3} + \alpha_3(\alpha_3-1)x_3^{\alpha_3-2}x_1^{\alpha_1}x_2^{\alpha_2}\big) = 0. \tag{3.78}$$

We discuss the terms

$$\begin{aligned} &\alpha_1(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2}x_3^{\alpha_3},\ \alpha_1+\alpha_2+\alpha_3 = n, \\ &\alpha_2(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2}x_3^{\alpha_3},\ \alpha_1+\alpha_2+\alpha_3 = n, \\ &\alpha_3(\alpha_3-1)x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-2},\ \alpha_1+\alpha_2+\alpha_3 = n \end{aligned} \tag{3.79}$$

in more detail. Every term in (3.79) with index $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ satisfying $[\alpha] = \alpha_1+\alpha_2+\alpha_3 = n$ is a homogeneous polynomial of degree $n-2$. Hence, the left hand side of (3.78) is a homogeneous polynomial of degree $n-2$. Therefore, ΔH_n can be represented in the form

$$\Delta_x H_n(x) = \sum_{[\beta]=n-2} D_\beta x^\beta. \tag{3.80}$$

The coefficients D_β are given by

$$D_\beta = \sum_{[\alpha]=n} C_\alpha m_{\beta\alpha}, \tag{3.81}$$

where $m_{\beta\alpha}$ is given by

$$m_{\beta\alpha} = \begin{cases} \alpha_1(\alpha_1 - 1), & \beta - \alpha = (-2,0,0)^T \\ \alpha_2(\alpha_2 - 1), & \beta - \alpha = (0,-2,0)^T \\ \alpha_3(\alpha_3 - 1), & \beta - \alpha = (0,0,-2)^T \\ 0 & \text{otherwise.} \end{cases} \tag{3.82}$$

H_n is assumed to be harmonic, i.e., $\Delta_x H_n(x) = 0$ for all $x \in \mathbb{R}^3$. But this means that all numbers D_β are equal to 0. Therefore, it follows that

$$\sum_{[\alpha]=n} C_\alpha m_{\beta\alpha} = 0 \tag{3.83}$$

for all β with $[\beta] = n-2$. Now, (3.83) is a linear system of $\binom{n}{2}$ equations in the $\binom{n+2}{2}$ unknowns C_α, $[\alpha] = n$.

The matrix $\mathbf{m} = (m_{\beta\alpha})$ has $\binom{n}{2}$ rows and $\binom{n+2}{2}$ columns; $\mathbf{m}$ can be partitioned as follows:

$$\mathbf{m} = (\underbrace{\mathbf{l}}_{\binom{n}{2}} \;\vdots\; \underbrace{\mathbf{r}}_{\binom{n+2}{2}-\binom{n}{2}=2n+1}), \tag{3.84}$$

where $\mathbf{l} = (l_{\beta\delta})$ is a $\binom{n}{2}$ by $\binom{n}{2}$ matrix and $\mathbf{r} = (r_{\beta\delta})$ is a $\binom{n}{2}$ by $\binom{n+2}{2} - \binom{n}{2}$ matrix.

For the set of multi indices of degree n, we introduce a binary relation (lexicographical order) between elements

$$\alpha' = (\alpha_1', \alpha_2', \alpha_3')^T,\ \alpha'' = (\alpha_1'', \alpha_2'', \alpha_3'')^T \tag{3.85}$$

designated by '>' and defined as follows:

$$\alpha' > \alpha'' \tag{3.86}$$

if and only if one of the following relations is satisfied

$$\alpha_1' > \alpha_1'' \tag{3.87}$$

or

$$\alpha_1' = \alpha_1'',\ \alpha_2' > \alpha_2'' \tag{3.88}$$

or

$$\alpha_1' = \alpha_1'',\ \alpha_2' = \alpha_2'',\ \alpha_3' > \alpha_3''. \tag{3.89}$$

The binary relation '>' implies an ordering for the multi-indices α, $[\alpha] = n$, according to the mapping

$$(n,0,0) \quad \to 1 \qquad \} \, 1$$

$$\left.\begin{array}{ll}(n-1,1,0) & \to 2 \\ (n-1,0,1) & \to 3\end{array}\right\} 2$$

$$\left.\begin{array}{ll}(n-2,2,0) & \to 4 \\ (n-2,1,1) & \to 5 \\ (n-2,0,2) & \to 6\end{array}\right\} 3$$

$$\vdots \; \vdots \; \vdots$$

$$\left.\begin{array}{ll}(0,n,0) \to & \binom{n+2}{2}-n \\ & \vdots\;\vdots\;\vdots \\ (0,0,n) \to & \binom{n+2}{2}\end{array}\right\} n+1 \, .$$

In the same way, the set of multi-indices β, $[\beta]=n-2$, may be ordered by increasing integers $i, 1 \leq i \leq \binom{n}{2}$. Hence, in canonical manner, each pair (β,α) with $[\beta]=n-2$, $[\alpha]=n$, corresponds uniquely to a pair $(i,j), 1 \leq i \leq \binom{n}{2}, 1 \leq j \leq \binom{n+2}{2}$. In this notation, the matrix

$$\mathbf{m} \;=\; (m_{\beta\alpha}),\ [\beta] \;=\; n-2,\ [\alpha] \;=\; n \tag{3.90}$$

can be rewritten in the ordered form

$$\mathbf{m} = (m_{ij}),\ 1 \leq i \leq \binom{n}{2},\ 1 \leq j \leq \binom{n+2}{2}. \tag{3.91}$$

Analogously

$$\mathbf{l} = l_{\beta\gamma},\ [\beta]=n-2,\ [\gamma]=n-2 \tag{3.92}$$

becomes

$$\mathbf{l} = (l_{ij}),\ 1 \leq i \leq \binom{n}{2},\ 1 \leq j \leq \binom{n}{2}. \tag{3.93}$$

From (3.82), it can be deduced that

$$\begin{array}{lll} l_{ij} = 0 & \text{for} & i > j,\ i = 2,\ldots,\binom{n}{2}, \\ l_{ij} \neq 0 & \text{for} & i = j,\ i = 1,\ldots,\binom{n}{2}. \end{array} \tag{3.94}$$

But this shows that $\mathbf{l}$ is non-singular, hence, the matrix $\mathbf{m}$ is of maximal rank: $\binom{n}{2}$. Therefore we are able to find $\binom{n+2}{2}-\binom{n}{2}$, i.e., $2n+1$ linearly independent solution vectors (A^1_α) , ... , (A^{2n+1}_α), $[\alpha]=n$, of the homogeneous linear system (3.83). According to standard arguments of Linear Algebra, the $\binom{n+2}{2}$ by $2n+1$ matrix $\mathbf{a}$ consisting of the vectors $(A^1_\alpha),\ldots,(A^{2n+1}_\alpha)$

$$\mathbf{a} = \underbrace{\left((A^1_\alpha),\ldots,(A^{2n+1}_\alpha)\right)}_{2n+1} \Big\}\ \binom{n+2}{2} \tag{3.95}$$

may be partitioned in the following form

$$\mathbf{a} = \underbrace{\begin{pmatrix} \mathbf{u} \\ -\mathbf{i} \end{pmatrix}}_{2n+1}, \tag{3.96}$$

where $\mathbf{i}$ is the $(2n+1)$ by $(2n+1)$ unit matrix, and $\mathbf{u}$ is a $\binom{n+2}{2} - (2n+1)$ by $(2n+1)$ matrix. Then the linear system $\mathbf{m}\,\mathbf{a} = \mathbf{0}$ can be written as follows: $\mathbf{l}\,\mathbf{u} = \mathbf{r}$. Since $\mathbf{l}$ is a $(2n+1)$ by $(2n+1)$ upper triangular matrix, the unknown matrix $\mathbf{u}$ can be computed by $(2n+1)$-times backward substitution.

The elements of the matrix $\mathbf{m} = (m_{\beta\alpha})$ are all integers. Therefore, any solution of the linear system (3.83) is a column vector of rational components. Hence, there exists a matrix

$$\mathbf{c} = ((C^1_\alpha), ..., (C^{2n+1}_\alpha)), \ [\alpha] = n, \tag{3.97}$$

the elements of which are all integers (observe that if $(C_\alpha),\ [\alpha] = n$, is a solution of (3.83), then $k\ (C_\alpha),\ [\alpha] = n,\ k$ integer, is a solution, too).

In other words, the solution process can be performed strictly in the modulus of integers. *Exact computation* (without rounding errors) is possible *in integer mode* by use of integer operations (addition, subtraction, multiplication of integers). When the matrix $\mathbf{c}$ has been calculated, the homogeneous harmonic polynomials $H_{n,j}$ given by (3.76) form a (maximal) linearly independent system, i.e., a basis in Harm_n.

Finally, it should be emphasized that exact computation, i.e., addition, subtraction, multiplication in integer mode must be performed strictly in the available range of the integer constants. Helpful is an arithmetic for arbitrarily long integers whose implementation on a computer system operates with lists so that there is no restriction on the size of the integers worked with (this is a standard feature of computer algebra packages). Let us demonstrate the technique of calculating the matrix $\mathbf{c}$ with an example:

Example 3.17. We choose the degree $n = 3$. Then an elementary calculation yields

$$\binom{n+2}{2} = 10, \quad \binom{n}{2} = 3, \tag{3.98}$$

hence,

$$\binom{n+2}{2} - \binom{n}{2} = 7. \tag{3.99}$$

Every polynomial $H_3 \in \mathrm{Hom}_3$ may be represented in the form:

$$\begin{array}{rllllllll} H_3(x) = & C_{300} & x_1^3 & + & C_{210} & x_1^2 x_2 & + & C_{201} & x_1^2 \, x_3 \\ + & C_{120} & x_1 x_2^2 & + & C_{111} & x_1 x_2 x_3 & + & C_{102} & x_1 \, x_3^2 \\ + & C_{030} & x_2^3 & + & C_{021} & x_2^2 x_3 & + & C_{012} & x_2 x_3^2 \\ + & C_{003} & x_3^3 & & & & & & \end{array} \tag{3.100}$$

$$(x = (x_1, x_2, x_3)^T).$$

H_3 has to fulfill the differential equation $\Delta_x H_3(x) = 0, \; x \in \mathbb{R}^3$, i.e.,

$$\begin{aligned} 6\; C_{300}x_1 \;&+\; 2\; C_{210}x_2 + 2\; C_{201}x_3 \\ +2\; C_{120}x_1 \;&+\; 6\; C_{030}x_2 + 2\; C_{021}x_3 \\ +2\; C_{102}x_1 \;&+\; 2\; C_{012}x_2 + 6\; C_{003}x_3 = 0. \end{aligned} \tag{3.101}$$

Since $\Delta_x H_3(x) = 0$ identically for all $x \in \mathbb{R}^3$, we get $\binom{n}{2} = 3$ equations for the coefficients

$$6C_{300} \;+\; 2C_{120} \;+\; 2C_{102} = 0, \tag{3.102}$$

$$2C_{210} \;+\; 6C_{030} \;+\; 2C_{012} = 0, \tag{3.103}$$

$$2C_{201} \;+\; 2C_{021} \;+\; 6C_{003} = 0. \tag{3.104}$$

Using the introduced order for the coefficients $C_\alpha, \; [\alpha] = 3$, the equation $\mathbf{m}\;\mathbf{c} = 0$ reads in matrix notation

$$\begin{pmatrix} 6 & 0 & 0 & \vdots & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \vdots & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 2 & \vdots & 0 & 0 & 0 & 0 & 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} C_{300} \\ C_{210} \\ C_{201} \\ \cdots \\ C_{120} \\ C_{111} \\ C_{102} \\ C_{030} \\ C_{021} \\ C_{012} \\ C_{003} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.105}$$

where we have marked the partitioning of the matrix $\mathbf{m}$ and the vector (C_α) by dashed lines. If we choose

$$C_{120} = -1, C_{111} = ... = C_{003} \;=\; 0 \tag{3.106}$$

the linear system is uniquely solved by the vector

$$(\frac{1}{3}, 0, 0 \,\vdots\, -1, 0, 0, 0, 0, 0, 0)^T. \tag{3.107}$$

Multiplying this vector by 3, all components become integers

$$(C_\alpha^1) = (1, 0, 0 \,\vdots\, -3, 0, 0, 0, 0, 0, 0)^T. \tag{3.108}$$

In the same way, we generate a set of 7 linearly independent solutions of the above system the components of which are all integers, viz.

$$
\begin{aligned}
(C_\alpha^2) &= (0,0,0 \vdots 0,-1,0,0,0,0,0)^T, && (3.109)\\
(C_\alpha^3) &= (1,0,0 \vdots 0,0,-3,0,0,0,0)^T, && (3.110)\\
(C_\alpha^4) &= (0,3,0 \vdots 0,0,0,-1,0,0,0)^T, && (3.111)\\
(C_\alpha^5) &= (0,0,1 \vdots 0,0,0,0,-1,0,0)^T, && (3.112)\\
(C_\alpha^6) &= (0,1,0 \vdots 0,0,0,0,0,-1,0)^T, && (3.113)\\
(C_\alpha^7) &= (0,0,3 \vdots 0,0,0,0,0,0,-1)^T. && (3.114)
\end{aligned}
$$

Thus a linearly independent system $\{H_{3,j}\}_{j=1,\ldots,7}$ of homogeneous harmonic polynomials of degree 3 is found by the following functions:

$$
\begin{aligned}
H_{3,1}(x) &= 1 \cdot x_1^3 - 3 \cdot x_1 x_2^2, && (3.115)\\
H_{3,2}(x) &= -1 \cdot x_1 x_2 x_3, && (3.116)\\
H_{3,3}(x) &= 1 \cdot x_1^3 - 3 \cdot x_1 x_3^2, && (3.117)\\
H_{3,4}(x) &= 3 \cdot x_1^2 x_2 - 1 \cdot x_2^3, && (3.118)\\
H_{3,5}(x) &= 1 \cdot x_1^2 x_3 - 1 \cdot x_2^2, x_3, && (3.119)\\
H_{3,6}(x) &= 1 \cdot x_1^2 x_2 - 1 \cdot x_2 x_3^2, && (3.120)\\
H_{3,7}(x) &= 3 \cdot x_1^2 x_3 - 1 \cdot x_3^3. && (3.121)
\end{aligned}
$$

Let us summarize the solution process once the linear system is given:

(i) Choosing the lower part of the vector identically 0 besides one component.

(ii) Solving the system by backward substitution.

(iii) Multiplying every resulting vector by an appropriate integer.

It is worth mentioning that, corresponding to the linearly independent system $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ of homogeneous harmonic polynomials of degree n, an orthogonal system, in $\{H^*_{n,j}\}_{j=1,\ldots.,2n+1}$ with respect to both the topology of Hom_n and $\mathrm{L}^2(\Omega)$ can be constructed only by integer operations (according to the well-known Gram-Schmidt process). To this end, the functions $H^*_{n,j}$ are computed recursively. We start from

$$H^*_{n,1} = H_{n,1} \ . \tag{3.122}$$

Then we set

$$H^*_{n,2} = a^n_{2,1} H^*_{n,1} + H_{n,2} \ . \tag{3.123}$$

The coefficient $a^n_{2,1}$ has to be chosen such that $H^*_{n,2}$ is orthogonal to $H^*_{n,1}$:

$$(H^*_{n,2}, H^*_{n,1})_{\mathrm{Hom}_n} = 0 \ . \tag{3.124}$$

It turns out that

$$a^n_{2,1} = -\frac{(H_{n,2}, H^*_{n,1})_{\mathrm{Hom}_n}}{(H^*_{n,1}, H^*_{n,1})_{\mathrm{Hom}_n}} \ . \tag{3.125}$$

It should be noted that numerator and denominator may be determined exactly. Now, let

$$H^*_{n,3} = a^n_{3,1} H^*_{n,1} + a^n_{3,2} H^*_{n,2} + H_{n,3} \ . \tag{3.126}$$

The requirements

$$(H^*_{n,3}, H^*_{n,1})_{\mathrm{Hom}_n} = 0, \tag{3.127}$$
$$(H^*_{n,3}, H^*_{n,2})_{\mathrm{Hom}_n} = 0 \tag{3.128}$$

lead to

$$a^n_{3,1} = -\frac{(H_{n,3}, H^*_{n,1})_{\mathrm{Hom}_n}}{(H^*_{n,1}, H^*_{n,1})_{\mathrm{Hom}_n}} , \tag{3.129}$$
$$a^n_{3,2} = -\frac{(H_{n,3}, H^*_{n,2})_{\mathrm{Hom}_n}}{(H^*_{n,2}, H^*_{n,2})_{\mathrm{Hom}_n}} \ . \tag{3.130}$$

Again, the coefficients can be deduced by integer operations. Analogously we get, in general,

$$H^*_{n,1} = H_{n,1} \ , \tag{3.131}$$

$$H^*_{n,k} = a^n_{k,1} H^*_{n,1} + ... + a^n_{k,k-1} H^*_{n,k-1} + H_{n,k}, \quad k = 2, ..., 2n+1, \tag{3.132}$$

where the coefficients

$$a^n_{k,s} = -\frac{(H_{n,k}, H^*_{n,s})_{\mathrm{Hom}_n}}{(H^*_{n,s}, H^*_{n,s})_{\mathrm{Hom}_n}} \tag{3.133}$$

are computable exactly by integer operations, i.e., $a^n_{k,s}$ is known exactly as a fraction of integers.

According to this well-known orthogonalization scheme, each function $H^*_{n,j}$ is a linear combination of the functions $H_{n,1}, ..., H_{n,2n+1}$. The coefficients of this linear combination can be obtained exactly as rational numbers, too. Thus, there exists a vector (B^j_α) such that

$$H^*_{n,j}(x) = \sum_{[\alpha]=n} B^j_\alpha x^\alpha \ , \ j = 1, ..., 2n+1 \ . \tag{3.134}$$

The vectors (B^j_α), $j = 1, ..., 2n+1$, form a matrix $\mathbf{b}$ whose elements consist of fractions of integers (provided that all numbers in the course of computation have been calculated in such a way that numerator and denominator are known as integers).

Lemma 3.18. *There exists a sequence of homogeneous harmonic polynomials* $\{H^*_{n,j}\}_{j=1,\ldots,2n+1}$ *of degree* n *with*

$$(H^*_{n,j}, H^*_{n,l})_{\mathrm{Hom}_n} = 0, \qquad j \neq l \ ,$$

viz.

$$\begin{aligned} H^*_{n,1} &= H_{n,1} \\ H^*_{n,k} &= a^n_{k,1} H^*_{n,1} + \ldots + a^n_{k,k-1} H^*_{n,k-1} + H_{n,k}, \quad k = 2, \ldots, 2n+1, \end{aligned}$$

where all coefficients $a^n_{k,s}$ *are computable by integer operations.*

Remark 3.19. Provided that the expression $\sqrt{(H^*_{n,j}, H^*_{n,j})_{\mathrm{Hom}_n}}$ has been stored as the radicant of an integer, a Hom_n-orthonormal system of homogeneous harmonic polynomials of degree n can be calculated exactly, i.e., by integer operations.

Lemma 3.20. *The system*

$$\sqrt{(H^*_{n,1}, H^*_{n,1})^{-1}_{\mathrm{Hom}_n}} H^*_{n,1}, \ldots, \ \sqrt{(H^*_{n,2n+1}, H^*_{n,2n+1})^{-1}_{\mathrm{Hom}_n}} H^*_{n,2n+1}$$

is an orthonormal system of homogeneous harmonic polynomials of degree n *with respect to* $(\cdot,\cdot)_{\mathrm{Hom}_n}$, *while*

$$\sqrt{\mu_n (H^*_{n,1}, H^*_{n,1})^{-1}_{\mathrm{Hom}_n}} H^*_{n,1}, \ldots, \ \sqrt{\mu_n (H^*_{n,2n+1}, H^*_{n,2n+1})^{-1}_{\mathrm{Hom}_n}} H^*_{n,2n+1}$$

is an orthonormal system of homogeneous harmonic polynomials of degree n *with respect to* $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$. *The values* $(H^*_{n,j}, H^*_{n,j})_{\mathrm{Hom}_n}$ *can be determined entirely by integer operations.*

Example 3.21. We only deal with the degree $n = 3$ (for a table of higher degrees, see W. Freeden, R. Reuter (1984)). According to our orthonormalization process due to Gram-Schmidt, we are able to deduce from the maximal system of linearly independent homogeneous harmonic polynomials $\{H_{3,j}\}_{j=1,\ldots,7}$ an orthogonal system $\{H^*_{3,j}\}_{j=1,\ldots,7}$. The resulting functions are listed below:

$$\begin{aligned} H^*_{3,1}(x) &= x_1^3 - 3x_1x_2^2, \\ H^*_{3,2}(x) &= x_1x_2x_3, \\ H^*_{3,3}(x) &= x_1^3 + x_1x_2^2 - 4x_1x_3^2, \\ H^*_{3,4}(x) &= 3x_1^2x_2 - x_2^3 - x_2^3, \\ H^*_{3,5}(x) &= x_1^2x_3 - x_2^2x_3, \\ H^*_{3,6}(x) &= x_1^2x_2 + x_2^3 - 4x_2x_3^2, \\ H^*_{3,7}(x) &= 3x_1^2x_3 + 3x_2^2x_3 - 2x_3^3. \end{aligned}$$

That means, all components $B_\alpha^j \neq 0$ are decomposed into an integer times a product of the prime numbers 2, 3. An easy calculation gives

$$\begin{array}{rcrcl}
(H^*_{3,1}, H^*_{3,1})_{\mathrm{Hom}_3} & = & 24 & = & 1 \cdot 2^3 \cdot 3^1, \\
(H^*_{3,2}, H^*_{3,2})_{\mathrm{Hom}_3} & = & 1 & = & 1 \cdot 2^0 \cdot 3^0, \\
(H^*_{3,3}, H^*_{3,3})_{\mathrm{Hom}_3} & = & 40 & = & 5 \cdot 2^3 \cdot 3^0, \\
(H^*_{3,4}, H^*_{3,4})_{\mathrm{Hom}_3} & = & 24 & = & 1 \cdot 2^3 \cdot 3^1, \\
(H^*_{3,5}, H^*_{3,5})_{\mathrm{Hom}_3} & = & 4 & = & 1 \cdot 2^2 \cdot 3^0, \\
(H^*_{3,6}, H^*_{3,6})_{\mathrm{Hom}_3} & = & 40 & = & 5 \cdot 2^3 \cdot 3^0, \\
(H^*_{3,7}, H^*_{3,7})_{\mathrm{Hom}_3} & = & 60 & = & 5 \cdot 2^2 \cdot 3^1.
\end{array} \tag{3.135}$$

Thus, the integers are decomposed into a (positive) integer times a product of prime numbers ≤ 3.

Consequently, the orthonormal system

$$\sqrt{(H^*_{n,j}, H^*_{n,j})^{-1}_{\mathrm{Hom}_3}} H^*_{n,j} \tag{3.136}$$

(with respect to $(\cdot,\cdot)_{\mathrm{Hom}_3}$). corresponding to $\{H^*_{n,j}\}_{j=1,\ldots,7}$ may be listed as follows:

$$\begin{aligned}
&\sqrt{(H^*_{3,1}, H^*_{3,1})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,1}(x) \\
&= (1 \cdot 2^0 \cdot 3^0 \cdot x_1^3 x_2^0 x_3^0 - 1 \cdot 2^0 \cdot 3^1 \cdot x_1^1 x_2^2 x_3^0)/\sqrt{1 \cdot 2^3 \cdot 3^1},
\end{aligned}$$

$$\begin{aligned}
&\sqrt{(H^*_{3,2}, H^*_{3,2})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,2}(x) \\
&= (1 \cdot 2^0 \cdot 3^0 \cdot x_1^1 x_2^1 x_3^1)/\sqrt{1 \cdot 2^0 \cdot 3^0},
\end{aligned}$$

$$\begin{aligned}
&\sqrt{(H^*_{3,3}, H^*_{3,3})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,3}(x) \\
&= (1 \cdot 2^0 \cdot 3^0 \cdot x_1^3 x_2^0 x_3^0 + 1 \cdot 2^0 \cdot 3^0 \cdot x_1^1 x_2^2 x_3^0 \\
&\quad -1 \cdot 2^2 \cdot 3^0 \cdot x_1^1 x_2^0 x_3^2)/\sqrt{5 \cdot 2^3 \cdot 3^0},
\end{aligned}$$

$$\begin{aligned}
&\sqrt{(H^*_{3,4}, H^*_{3,4})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,4}(x) \\
&= (1 \cdot 2^0 \cdot 3^1 \cdot x_1^2 x_2^1 x_3^0 - 1 \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^3 x_3^0)/\sqrt{1 \cdot 2^3 \cdot 3^1},
\end{aligned}$$

$$\begin{aligned}
&\sqrt{(H^*_{3,5}, H^*_{3,5})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,5}(x) \\
&= (1 \cdot 2^0 \cdot 3^0 \cdot x_1^2 x_2^0 x_3^1 - 1 \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^2 x_3^1)/\sqrt{1 \cdot 2^2 \cdot 3^0},
\end{aligned}$$

$$\sqrt{(H^*_{3,6}, H^*_{3,6})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,6}(x)$$
$$= (1 \cdot 2^0 \cdot 3^0 \cdot x_1^2 x_2^1 x_3^0 + 1 \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^3 x_3^0 5 \cdot 2^3 \cdot 3^0 \cdot x_1^0 x_2^1 x_3^2)/\sqrt{1 \cdot 2^2 \cdot 3^0},$$

$$\sqrt{(H^*_{3,7}, H^*_{3,7})^{-1}_{\mathrm{Hom}_3}} H^*_{3,7}(x)$$
$$= (1 \cdot 2^0 \cdot 3^1 \cdot x_1^2 x_2^0 x_3^1 + 1 \cdot 2^0 \cdot 3^1 \cdot x_1^0 x_2^2 x_3^1$$
$$-1 \cdot 2^1 \cdot 3^0 \cdot x_1^0 x_2^0 x_3^3)/\sqrt{5 \cdot 2^2 \cdot 3^1}.$$

Finally, the orthonormal system of homogeneous harmonic polynomials of degree n (with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$) is given as follows

$$\sqrt{\mu_3 (H^*_{3,j}, H^*_{3,j})^{-1}_{\mathrm{Hom}_3}}\; H^*_{3,j}, \quad j = 1 \;\; \ldots, 7 \tag{3.137}$$

with

$$\mu_3 = \frac{105}{4\pi} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4\pi}. \tag{3.138}$$

Our considerations have shown how a basis of Harm_n can be computed entirely by integer operations from $2n+1$ systems of linear equations. The basis functions obtained can be orthonormalized exactly by means of the well-known Gram-Schmidt orthonormalization process. As a result, there are $2n+1$ homogeneous harmonic polynomials available (orthonormalized in the sense of $(\cdot,\cdot)_{\mathrm{Hom}_n}$) . But the disadvantage in that approach is that the linear systems of equations result in basis functions which are all involved in the computational work of the orthonormalization. Later on (in Section 3.14), when Legendre harmonics come into play, an algorithm will be presented which reduces the amount of computational work by a factor less than 4, but which is close to 4 if the degree n becomes large enough.

3.4 Definition of Scalar Spherical Harmonics

We begin by introducing scalar spherical harmonics. Essential tool is the theory of homogeneous harmonic polynomials.

Definition 3.22. Let H_n be a homogeneous harmonic polynomial of degree n in $\mathbb{R}^3$, i.e., $H_n \in \mathrm{Harm}_n(\mathbb{R}^3)$. The restriction

$$Y_n = H_n|\Omega \tag{3.139}$$

is called a spherical harmonic of degree n. The space of all spherical harmonics of degree n, i.e., the set of all restrictions $Y_n = H_n|\Omega$, $H_n \in \mathrm{Harm}_n(\mathbb{R}^3)$, is denoted by $\mathrm{Harm}_n(\Omega)$. More explicitly,

$$\mathrm{Harm}_n(\Omega) = \mathrm{Harm}_n(\mathbb{R}^3)|\Omega. \tag{3.140}$$

Remark 3.23. In what follows, we simply write Harm_n instead of $\mathrm{Harm}_n(\mathbb{R}^3)$ (or $\mathrm{Harm}_n(\Omega)$) if no confusion is likely to arise.

We know already that the linear space Harm_n is of dimension $2n+1$, that is $d(\mathrm{Harm}_n) = 2n+1$. From Theorem 3.15, it follows that spherical harmonics of different orders are orthogonal in the sense of the L^2-inner product

$$(Y_n, Y_m)_{\mathrm{L}^2(\Omega)} = \int_\Omega Y_n(\xi) Y_m(\xi)\, d\omega(\xi) = 0, \quad n \neq m. \tag{3.141}$$

Using the standard method of separation, we have $H_n(x) = r^n Y_n(\xi), x = r\xi, r = |x|, \xi \in \Omega$. Observing the identity

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) r^n = n(n+1) r^{n-2} \tag{3.142}$$

we obtain

$$0 = \Delta_x H_n(x) = r^{n-2} n(n+1) Y_n(\xi) + r^{n-2} \Delta^*_\xi Y_n(\xi). \tag{3.143}$$

Thus, we are able to formulate the following lemma.

Lemma 3.24. *Any spherical harmonic $Y_n, n = 0, 1, \ldots,$ is a twice differentiable eigenfunction of the Beltrami operator corresponding to the eigenvalue $-n(n+1)$. More explicitly,*

$$(\Delta^*_\xi - (\Delta^*)^\wedge(n)) Y_n(\xi) = 0, \quad \xi \in \Omega, \quad Y_n \in \mathrm{Harm}_n,$$

where the 'spherical symbol' $\{(\Delta^*)^\wedge(n)\}_{n=0,1,\ldots}$ of the Beltrami operator Δ^* is given by

$$(\Delta^*)^\wedge(n) = -n(n+1), \quad n = 0, 1, \ldots.$$

Remark 3.25. Throughout the book, for convenience, the capital letter Y followed by double indices, for example, $Y_{n,j}$, denotes a member of degree n and order j within an orthonormal system $\{Y_{n,1}, \ldots, Y_{n,2n+1}\}$ with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$. A special realization of an $\mathrm{L}^2(\Omega)$-orthonormal system is presented in Section 3.12 (where we introduce a system involving associated Legendre functions).

In terms of spherical harmonics, the *addition theorem* allows the following reformulation.

Theorem 3.26. *Let $\{Y_{n,j}\}_{j=1,\ldots,2n+1}$ be an $\mathrm{L}^2(\Omega)$-orthonormal system in Harm_n. Then, for any pair $(\xi, \eta) \in \Omega \times \Omega$,*

$$\sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta).$$

Proof. Theorem 3.26 follows immediately from Theorem 3.16. □

Remark 3.27. The addition theorem can be seen in analogy to its two-dimensional counterpart involving the "circular harmonics" $H_{n,j}(2;\cdot):\mathbb{R}^2\to\mathbb{R}$, $j=1,2$, given by

$$\begin{aligned} H_{n,1}(2;x_{(2)}) = H_{n,1}(2;x_1,x_2) &= \frac{1}{\sqrt{\pi}}\mathrm{Re}(x_2+ix_1)^n \qquad (3.144)\\ &= \frac{1}{\sqrt{\pi}}|x|^n\cos n(\frac{\pi}{2}-\varphi)\\ &= \frac{(-1)^{n+1}}{\sqrt{\pi}}|x|^n\sin(n\varphi),\\ H_{n,2}(2;x_{(2)}) = H_{n,2}(2;x_1,x_2) &= \frac{1}{\sqrt{\pi}}\ \mathrm{Im}\ (x_2+ix_1)^n \qquad (3.145)\\ &= \frac{1}{\sqrt{\pi}}|x|^n\sin n(\frac{\pi}{2}-\varphi)\\ &= \frac{(-1)^{n+1}}{\sqrt{\pi}}|x|^n\cos(n\varphi), \end{aligned}$$

$x_{(2)}\in\mathbb{R}^2$, $x_{(2)}=(x_1,x_2)^T, x_1=r\cos\varphi$, $x_2=r\sin\varphi$, $r=|x_{(2)}|=\sqrt{x_1^2+x_2^2}$, $0\le\varphi<2\pi$. Obviously, we have

$$\int_{|x_{(2)}|=1} H_{n,j}(2;x_{(2)})H_{k,l}(2;x_{(2)})\ d\omega(x_{(2)}) = \delta_{nk}\delta_{jl}. \qquad (3.146)$$

Moreover, for $x_{(2)},y_{(2)}\in\mathbb{R}^2$, $x_1=|x_{(2)}|\cos\varphi$, $x_2=|x_{(2)}|\sin\varphi, y_1=|y_{(2)}|\cos\psi$, $y_2=|y_{(2)}|\sin\psi$, $0\le\varphi,\psi<2\pi$, we have

$$\begin{aligned} &\sum_{j=1}^{2} H_{n,j}(2;x_{(2)})H_{n,j}(2;y_{(2)}) \qquad (3.147)\\ &= \frac{|x_{(2)}|^n|y_{(2)}|^n}{\pi}\Big(\cos n\left(\frac{\pi}{2}-\varphi\right)\cos(n(\frac{\pi}{2}-\psi))+\sin(n(\frac{\pi}{2}-\varphi))\sin(n(\frac{\pi}{2}-\psi))\Big)\\ &= \frac{|x_{(2)}|^n|y_{(2)}|^n}{\pi}\cos(n(\varphi-\psi)). \end{aligned}$$

The two-dimensional counterpart of the Legendre polynomial is the *Chebyshev function*

$$\begin{aligned} L_n(2;x_{(2)}) &= \mathrm{Re}(x_2+ix_1)^n \qquad (3.148)\\ &= |x_{(2)}|^n\cos n(\arccos(\sin\varphi))\\ &= |x_{(2)}|^nT_n(\sin\varphi), \end{aligned}$$

that is symmetric with respect to the $(0,1)$-axis and that takes on the value 1 for $(x_1, x_2) = (0,1)$. Clearly,

$$\begin{aligned} L_n(2; x_{(2)}) &= |x_{(2)}|^n \cos n \left(\arccos(\cos(\frac{\pi}{2} - \varphi)) \right) \\ &= |x_{(2)}|^n T_n(\cos(\frac{\pi}{2} - \varphi)) \\ &= |x_{(2)}|^n \cos \left(n \left(\frac{\pi}{2} - \varphi \right) \right). \end{aligned} \tag{3.149}$$

Thus, we finally obtain as *two-dimensional analogue of the addition theorem*

$$\sum_{j=1}^{2} H_{n,j}(2; x_{(2)}) H_{n,j}(2; y_{(2)}) = \frac{|x_{(2)}|^n |y_{(2)}|^n}{\pi} T_n(\xi_{(2)} \cdot \eta_{(2)}) \tag{3.150}$$

with $\xi_{(2)} = (\cos\varphi, \sin\varphi)^T$, $\eta_{(2)} = (\cos\psi, \sin\psi)^T$. Clearly, we have

$$\xi_{(2)} \cdot \eta_{(2)} = \cos(\varphi - \psi), \tag{3.151}$$

which explains the close similarity to the result known from our (three–dimensional) spherical harmonic theory (for higher dimensional generalizations, see C. Müller (1966, 1998)).

Suppose that $\mathbf{t}$ is an orthogonal transformation. Then $\xi \mapsto Y_{n,j}(\mathbf{t}\xi)$, $\xi \in \Omega$, is a spherical harmonic of degree n. Thus, we are able to write this function as linear combination

$$Y_{n,j}(\mathbf{t}\xi) = \sum_{r=1}^{2n+1} \underbrace{\int_\Omega Y_{n,j}(\mathbf{t}\eta) Y_{n,r}(\eta) \, d\omega(\eta)}_{=c^n_{j,r}} Y_{n,r}(\xi) \tag{3.152}$$

Moreover, the addition theorem tells us that, for $\xi, \eta \in \Omega$,

$$\begin{aligned} P_n(\mathbf{t}\xi \cdot \mathbf{t}\eta) &= \sum_{j=1}^{2n+1} Y_{n,j}(\mathbf{t}\xi) Y_{n,j}(\mathbf{t}\eta) \\ &= \sum_{j=1}^{2n+1} \sum_{r=1}^{2n+1} c^n_{j,r} Y_{n,r}(\xi) \sum_{s=1}^{2n+1} c^n_{j,s} Y_{n,s}(\eta) \\ &= \sum_{r=1}^{2n+1} \sum_{s=1}^{2n+1} Y_{n,r}(\xi) Y_{n,s}(\eta) \sum_{j=1}^{2n+1} c^n_{j,r} c^n_{j,s} \\ &= \sum_{r=1}^{2n+1} Y_{n,r}(\xi) Y_{n,r}(\eta) \\ &= P_n(\xi \cdot \eta) \end{aligned} \tag{3.153}$$

such that

$$\begin{aligned}\delta_{jl} &= \sum_{r=1}^{2n+1} \int_\Omega Y_{n,j}(\mathbf{t}\eta)Y_{n,r}(\eta)\ d\omega(\eta) \int_\Omega Y_{n,l}(\mathbf{t}\eta)Y_{n,r}(\eta)\ d\omega(\eta) \\ &= \int_\Omega Y_{n,j}(\mathbf{t}\eta)Y_{n,l}(\mathbf{t}\eta)\ d\omega(\eta). \end{aligned} \tag{3.154}$$

Lemma 3.28. *If* $\mathbf{t} \in O(3)$, *then the matrix*

$$\left(\int_\Omega Y_{n,j}(\mathbf{t}\eta)Y_{n,r}(\eta)\ d\omega(\eta)\right)_{j,r=1,\ldots 2n+1} \tag{3.155}$$

is orthogonal.

Because of $P_n(1) = 1$, we find

$$\sum_{j=1}^{2n+1} (Y_{n,j}(\xi))^2 = \frac{2n+1}{4\pi}, \quad \xi \in \Omega. \tag{3.156}$$

If we remember that every $Y_n \in \mathrm{Harm}_n$ can be written in the form

$$Y_n = \sum_{j=1}^{2n+1} (Y_n, Y_{n,j})_{\mathrm{L}^2(\Omega)} Y_{n,j}, \tag{3.157}$$

we immediately get the following lemma.

Lemma 3.29. *(Reproducing Kernel in* Harm_n*) For every* $Y_n \in \mathrm{Harm}_n$

$$\frac{2n+1}{4\pi} \int_\Omega Y_n(\eta) P_n(\xi\cdot\eta)\, d\omega(\eta) = Y_n(\xi), \quad \xi \in \Omega,$$

that is

$$(\xi,\eta) \mapsto K_{\mathrm{Harm}_n}(\xi,\eta) = \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad (\xi,\eta) \in \Omega\times\Omega,$$

represents the (uniquely determined) reproducing kernel in Harm_n. *Moreover,*

$$(\Delta^*_\eta - (\Delta^*)^\wedge(n))K_{\mathrm{Harm}_n}(\xi,\eta) = 0, \quad \eta \in \Omega,$$

holds for all $\xi \in \Omega$.

From Lemma 3.29, we easily obtain the following result.

Lemma 3.30. *Let*

$$\mathrm{Harm}_{0,\ldots,m} = \bigoplus_{n=0}^{m} \mathrm{Harm}_n . \tag{3.158}$$

Then

$$(\xi,\eta) \mapsto K_{\mathrm{Harm}_{0,\ldots,m}}(\xi,\eta) = \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi\cdot\eta) \tag{3.159}$$

is the (uniquely) determined reproducing kernel in $\mathrm{Harm}_{0,\ldots,m}$*, i.e.,* $K_{\mathrm{Harm}_{0,\ldots,m}}(\xi,\cdot)$*,* $\xi\in\Omega$*, is a member of* $\mathrm{Harm}_{0,\ldots,m}$ *with*

$$\int_\Omega Y(\eta) K_{\mathrm{Harm}}(\xi,\eta)\ d\omega(\eta) = Y(\xi), \quad \xi\in\Omega \tag{3.160}$$

for all $Y\in \mathrm{Harm}_{0,\ldots,m}$.

Observing that

$$\int_\Omega (Y_n(\xi))^2\ d\omega(\xi) = \sum_{j=1}^{2n+1} (Y_n, Y_{n,j})^2_{\mathrm{L}^2(\Omega)} \tag{3.161}$$

we find in connection with (3.156) and (3.157)

$$\begin{aligned} (Y_n(\xi))^2 &\le \sum_{j=1}^{2n+1} (Y_n, Y_{n,j})^2_{\mathrm{L}^2(\Omega)} \sum_{j=1}^{2n+1} (Y_{n,j}(\xi))^2 \\ &= \frac{2n+1}{4\pi} \sum_{j=1}^{2n+1} (Y_n, Y_{n,j})^2_{\mathrm{L}^2(\Omega)} . \end{aligned} \tag{3.162}$$

This yields the following lemma.

Lemma 3.31. *For every* $Y_n\in \mathrm{Harm}_n$

$$\begin{aligned} \|Y_n\|_{\mathrm{C}(\Omega)} &= \sup_{\xi\in\Omega} |Y_n(\xi)| \le \left(\frac{2n+1}{4\pi}\right)^{\frac12} \|Y_n\|_{\mathrm{L}^2(\Omega)} \\ &= \left(\frac{2n+1}{4\pi}\right)^{\frac12} \left(\int_\Omega (Y_n(\xi))^2\ d\omega(\xi)\right)^{1/2} . \end{aligned} \tag{3.163}$$

In particular,

$$\|Y_{n,j}\|_{\mathrm{C}(\Omega)} = \sup_{\xi\in\Omega} |Y_{n,j}(\xi)| \le \left(\frac{2n+1}{4\pi}\right)^{\frac12}, \tag{3.164}$$

$j=1,\ldots,2n+1$.

3.5 Legendre Polynomials

The function $P_n : [-1,1] \to \mathbb{R},\ n = 0,1,...,$ defined by (3.63)

$$P_n(t) \ = \ \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{(2n-2s)!}{2^n(n-2s)!(n-s)!s!} t^{n-2s}, \quad t \in [-1,1] \tag{3.165}$$

is called the *Legendre polynomial.* P_n is uniquely determined by the properties:

(i) P_n is a polynomial of degree n on the interval $[-1,1]$,

(ii) $\int_{-1}^{1} P_n(t) P_m(t)\, dt = 0$ for $n \neq m$,

(iii) $P_n(1) \ = \ 1.$

This is easily seen from the usual process of orthogonalization.

In particular, we have for $n = 0, \ldots, 4$

$$P_0(t) \ = \ 1, \qquad P_1(t) = t, \qquad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{3.166}$$

$$P_3(t) \ = \ \frac{5}{2}t^3 - \frac{3}{2}t, \quad P_4(t) = \frac{35}{8}t^4 - \frac{15}{4}t^2 + \frac{3}{8}. \tag{3.167}$$

A graphical impression of some Legendre polynomials can be found in Fig. 3.1.

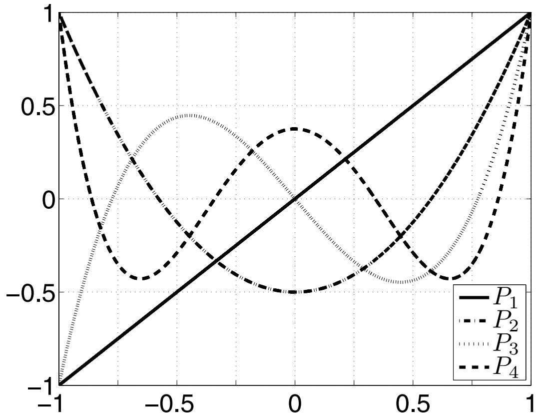


Fig. 3.1: Legendre polynomials $t \mapsto P_n(t)$, $t \in [-1,1]$, $n = 1, \ldots, 4$.

Furthermore,

$$\int_{-1}^{1} P_n(t) P_m(t)\, dt \ = \ \frac{2}{2n+1}\delta_{nm}. \tag{3.168}$$

Definition 3.32. For $\xi \in \Omega$, the function $P_n(\xi\cdot) : \eta \mapsto P_n(\xi \cdot \eta)$, $\eta \in \Omega$, is called the *scalar* ξ*-Legendre kernel* of degree n.

Applying the Cauchy–Schwarz inequality to the addition theorem (Theorem 3.26), we obtain for the scalar ξ-Legendre kernel

$$\begin{aligned}
\frac{2n+1}{4\pi}\left| P_n(\xi\cdot\eta)\right| &= \left|\sum_{j=1}^{2n+1} Y_{n,j}(\xi)Y_{n,j}(\eta)\right| && (3.169)\\
&\leq \sqrt{\sum_{j=1}^{2n+1}\left(Y_{n,j}(\xi)\right)^2}\sqrt{\sum_{j=1}^{2n+1}\left(Y_{n,j}(\eta)\right)^2}\\
&= \frac{2n+1}{4\pi} \mid P_n(1) \mid\\
&= \frac{2n+1}{4\pi} P_n(1). && (3.170)
\end{aligned}$$

Therefore, it follows that

$$|P_n(t)| \leq P_n(1) = 1\ ,\ t \in [-1,1]. \qquad (3.171)$$

Moreover, the Legendre polynomial P_n satisfies the estimate (see, for example, C. Müller (1952))

$$|P_n^{(k)}(t)| \leq |P_n^{(k)}(1)|, \qquad (3.172)$$

where $P_n^{(k)}(1) = O(n^{2k})$. In particular, we have

$$P_n'(1) = \frac{n(n+1)}{2}. \qquad (3.173)$$

Furthermore, for $k = 2, 3, \ldots, n$,

$$P_n^{(k)}(1) = \left(\frac{1}{2}\right)^k \frac{1}{k!}\, n(n+1)\left((n(n+1) - 1\cdot 2)\ldots(n(n+1) - k(k-1))\right).$$

From Lemma 3.24 in combination with Theorem 3.26, it follows that

$$\left((1-t^2)\left(\frac{d}{dt}\right)^2 - 2t\frac{d}{dt} + n(n+1)\right)P_n(t) = 0, \quad t \in [-1,1], \qquad (3.174)$$

where

$$L_t = \left((1-t^2)\left(\frac{d}{dt}\right)^2 - 2t\frac{d}{dt}\right) \qquad (3.175)$$

is the Legendre operator, i.e., the part of the Beltrami operator that depends only on the polar distance t.

We therefore obtain the following lemma.

Lemma 3.33. *The Legendre polynomial P_n is the only twice differentiable eigenfunction of the 'Legendre operator' (3.175) on $[-1,1]$, corresponding to the eigenvalues $-n(n+1)$, $n = 0,1,...$, and bounded on $[-1,1]$ with $P_n(1) = 1$.*

The differential equation (3.174) shows that P_n and P_n' cannot vanish simultaneously such that P_n has no multiple zeros. The orthogonality relation for Legendre polynomials implies that P_n has, at most, k different zeros, $z_1, \ldots z_k$, $k \leq n$, in the interval $(-1,1)$. Letting

$$I_k(t) = (t - z_1)\cdots(t - z_k) \tag{3.176}$$

we get $I_k(1) > 0$ and $P_n = J_{n-k} I_k$. The polynomial J_{n-k} is positive in $[-1,1]$, and we have

$$\int_{-1}^{+1} P_n(t) I_k(t)\ dt = \int_{-1}^{+1} J_{n-k}(t) I_k^2(t)\ dt > 0. \tag{3.177}$$

As P_n is orthogonal to all polynomials of degree $< n$, this is possible only for the case $k = n$. Thus, we can conclude that P_n has n different zeros in the interval $(-1,1)$.

The zeros of the Legendre polynomial for $n = 1, 2, 3, 4$ are listed in Table 3.2.

Table 3.2: Zeros of the Legendre polynomial.

$n = 1$	$z_1 = 0$
$n = 2$	$z_2 = -z_1 = 0.5773502692...$
$n = 3$	$z_3 = -z_1 = 0.7745966692...$
	$z_2 = 0$
$n = 4$	$z_4 = -z_1 = 0.8611363116...$
	$z_3 = -z_2 = 0.3399810436...$

Lemma 3.34. *The Legendre polynomial P_n has n different zeros in the interval $(-1,1)$.*

From the binomial theorem, it follows that

$$(t^2 - 1)^n \ = \ \sum_{s=0}^{n} (-1)^s \frac{n!}{(n-s)!s!} t^{2n-2s}, \quad n = 0, 1, \ldots. \tag{3.178}$$

For all $s \leq [n/2]$ we find

$$\left(\frac{d}{dt}\right)^n t^{2n-2s} \ = \ \frac{(2n-2s)!}{(n-2s)!} t^{n-2s}, \tag{3.179}$$

while for $[n/2] < s \leq n$ we get

$$\left(\frac{d}{dt}\right)^n t^{2n-2s} = 0. \tag{3.180}$$

Therefore, we see that

$$\left(\frac{d}{dt}\right)^n (t^2-1)^n = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{n!}{(n-s)!s!} \frac{(2n-2s)!}{(n-2s)!} t^{n-2s} . \tag{3.181}$$

By comparison with the definition of the Legendre polynomial (3.63), we obtain the *Rodriguez formula.*

Lemma 3.35. *For $n = 0, 1, \ldots,$*

$$P_n(t) = \frac{1}{2^n n!} \left(\frac{d}{dt}\right)^n (t^2-1)^n, \quad t \in [-1,1]. \tag{3.182}$$

Integrating by parts, we obtain the *Rodriguez rule*

$$\int_{-1}^{1} F(t) P_n(t)\, dt = \frac{1}{2^n n!} \int_{-1}^{1} F^{(n)}(t)(1-t^2)^n\, dt \tag{3.183}$$

for every $F \in \mathrm{C}^{(n)}[-1,1]$.

As an application of this formula, we discuss the integrals

$$\int_{-1}^{1} P_k(t)(P_n'(t) - P_{n-2}'(t))\, dt = -\int_{-1}^{1} (P_n(t) - P_{n-2}(t)) P_k'(t)\, dt. \tag{3.184}$$

$P_n' - P_{n-2}'$ is a polynomial of degree $n-1$ on $[-1,1]$. Consequently, the integral (3.184) vanishes for $k \geq n$. On the other hand, P_k' is a polynomial of degree $k-1$. This means that the integral vanishes also for $k < n-1$ so that (3.184) differs from zero only for $k = n-1$. Thus

$$P_n'(t) - P_{n-2}'(t) = c_n P_{n-1}(t), \quad t \in [-1,1]. \tag{3.185}$$

In connection with (3.173), we find

$$P_n'(1) - P_{n-2}'(1) = 2n-1. \tag{3.186}$$

Therefore, it follows that

$$P_n'(t) - P_{n-2}'(t) = (2n-1) P_{n-1}(t) , \quad t \in [-1,1]. \tag{3.187}$$

Equivalently, we have

$$(2n-1) \int_s^1 P_{n-1}(t)\, dt = P_{n-2}(s) - P_n(s), \quad n \geq 2, \tag{3.188}$$

for all $s \in [-1, 1]$. By similar arguments, we are able to show that

$$P'_{n+1}(t) - tP'_n(t) = (n+1)P_n(t), \tag{3.189}$$

$$(t^2 - 1)P'_n(t) = ntP_n(t) - nP_{n-1}(t), \tag{3.190}$$

$$(n+1)P_{n+1}(t) + nP_{n-1}(t) - (2n+1)tP_n(t) = 0. \tag{3.191}$$

The formulas (3.189)–(3.191) are known as *recurrence formulas for the Legendre polynomials.* Moreover, we have the following result.

Lemma 3.36. *For* $n = 1, 2, \ldots,$ $t \in [-1, 1]$,

$$(t^2 - 1)P'_n(t) = \frac{n(n+1)}{2n+1}(P_{n+1}(t) - P_{n-1}(t)). \tag{3.192}$$

Proof. Inserting (3.191) into (3.190) we find

$$\begin{aligned}
(t^2 - 1)P'_n(t) &= n(tP_n(t) - P_{n-1}(t)) \qquad (3.193)\\
&= n\left(\frac{n+1}{2n+1}P_{n+1}(t) + \frac{n}{2n+1}P_{n-1}(t) - P_{n-1}(t)\right)\\
&= n\left(\frac{n+1}{2n+1}P_{n+1}(t) + \frac{n-(2n+1)}{2n+1}P_{n-1}(t)\right)\\
&= \frac{n(n+1)}{2n+1}\left(P_{n+1}(t) - P_{n-1}(t)\right).
\end{aligned}$$

This is the desired result. □

In addition, we mention the following results involving derivatives of the Legendre polynomial.

Lemma 3.37. *The following identities are valid:*

(i) *For* $n = 0, 1, \ldots$

$$P'_{2n+1}(t) = \sum_{k=0}^{n}(4k+1)P_{2k}(t). \tag{3.194}$$

(ii) *For* $n = 1, 2, \ldots$

$$P'_{2n}(t) = \sum_{k=0}^{n-1}(4k+3)P_{2k+1}(t). \tag{3.195}$$

(iii) *For* $n = 1, 2, \ldots$

$$(1+t)P_n'(t) = \sum_{k=0}^{n-1}(2k+1)P_k(t) + nP_n(t). \tag{3.196}$$

Proof. We prove statement (iii) only. It is clear that there exist coefficients $a_{n,0}, \ldots, a_{n,n}$ such that

$$(1+t)P_n'(t) = \sum_{k=0}^{n} a_{n,k}P_k(t). \tag{3.197}$$

Now, by virtue of (3.168),

$$\int_{-1}^{1} P_n'(t)(1+t)P_l(t)\,dt = a_{n,l}\frac{2}{2l+1} \tag{3.198}$$

for $l = 0, \ldots, n$. Integration by parts yields

$$\int_{-1}^{1} P_n'(t)(1+t)P_l(t)\,dt = 2 - \delta_{nl}\frac{2}{2l+1} - \int_{-1}^{1} P_n(t)(1+t)P_l'(t)\,dt. \tag{3.199}$$

For $l = 0, \ldots, n-1$ the last integral on the right-hand side vanishes, since $(1+t)P_l'(t)$ is of degree $\leq n-1$. Thus, it follows that $a_{n,l} = 2l+1$ for $l = 0, \ldots, n-1$. An easy calculation shows that

$$\begin{aligned} \int_{-1}^{1} P_n(t)P_n'(t)(1+t)\,dt &= \frac{1}{2}P_n^2(t)(1+t)\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} P_n^2(t)\,dt \\ &= 1 - \frac{1}{2n+1}, \end{aligned} \tag{3.200}$$

hence, $a_{n,n} = n$. This gives the required result. □

Remark 3.38. The Legendre polynomials satisfy the recurrence relation

$$P_k(t) - \frac{2k-1}{k}P_{k-1}(t) + \frac{k-1}{k}P_{k-2}(t) = 0. \tag{3.201}$$

$k \geq 2$, $t \in [-1,1]$ (remember $P_0(t) = 1, P_1(t) = t$, $t \in [-1,1]$). For every $t \in [-1,1]$ fixed, the sum

$$Q_N(t) = \sum_{k=0}^{N} P_n(t) \tag{3.202}$$

can be calculated by the (stable) algorithm

$$R_{N+1}(t) = R_{N+2}(t) = 0 \tag{3.203}$$

$$R_k(t) = \frac{2k+1}{k+1}R_{k+1}(t) - \frac{k+2}{k+1}R_{k+2}(t) + a_k, \; k = N, \ldots, 1 \tag{3.204}$$

$$Q_N(t) = a_0 - \frac{1}{2}R_2(t) + R_1(t)t. \tag{3.205}$$

The proof follows easily by writing out the above recurrence relation for the Legendre polynomial in matrix form (see P. Deuflhard, A. Hohmann (1991)).

The power series

$$\phi(h) = \sum_{n=0}^{\infty} P_n(t)h^n, \quad t \in [-1,1], \tag{3.206}$$

is absolutely and uniformly convergent for all h with $|h| \leq h_0, h_0 \in [0,1)$. By differentiation with respect to h and comparing coefficients according to (3.206), we find

$$(1 + h^2 - 2ht)\phi'(h) = (t-h)\phi(h). \tag{3.207}$$

This differential equation is uniquely solvable under the initial condition $\phi(0) = 1$. Since it is not hard to show that

$$h \mapsto (1 + h^2 - 2ht)^{-1/2}, \; h \in (-1,1), \tag{3.208}$$

solves this initial value problem, we have the following *generating series expansion of the Legendre polynomials.*

Lemma 3.39. *For* $t \in [-1,1]$ *and all* $h \in (-1,1)$

$$\sum_{n=0}^{\infty} P_n(t)h^n = \frac{1}{\sqrt{1+h^2-2ht}}.$$

Among other areas of application, the subject of potential theory is concerned with forces of attraction due to the presence of a gravitational field. Central to the discussion of gravitational attraction is Newton's law of gravitation for the force field generated by a single particle (cf. Chapter 10): the gravitational force f in free space (i.e., free of point masses) is related to the potential function F according to

$$f(x) = -\nabla_x F(x), \; x \neq y, \tag{3.209}$$

when the potential between a mass point y and a point of free space x has the form

$$F(x) = k|x-y|^{-1}, \ x \neq y \tag{3.210}$$

(k is the gravitational constant). Because of spherical symmetry of the gravitational field, the potential function of a single particle depends only upon the radial distance, i.e., the inner product of the direction vectors of x and y. In order to obtain this result, let us suppose for the sake of definition $x = |x|\xi$, $y = |y|\eta$, $\xi, \eta \in \Omega$, $|x| < |y|$. Then we find

$$\frac{1}{|x-y|} = \frac{1}{|y|}\left(1 + \left(\frac{|x|}{|y|}\right)^2 - 2\frac{|x|}{|y|}\xi \cdot \eta\right)^{-1/2}. \tag{3.211}$$

Returning now to Lemma 3.39 with $t = \xi \cdot \eta$ and $h = |x|/|y|$, we find that the potential function has the series expansion

$$\frac{1}{|x-y|} = \frac{1}{|y|}\sum_{n=0}^{\infty}\left(\frac{|x|}{|y|}\right)^n P_n(\xi \cdot \eta). \tag{3.212}$$

Moreover, our considerations have shown that

$$\frac{1}{|x-y|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}|x|^n(\xi \cdot \nabla_y)^n \frac{1}{|y|}, \tag{3.213}$$

where

$$\frac{(-1)^n}{n!}(\xi \cdot \nabla_y)^n \frac{1}{|y|} = \frac{P_n(\xi \cdot \eta)}{|y|^{n+1}}, \quad n = 0, 1, \ldots \tag{3.214}$$

Identity (3.214) is known as *Maxwell's representation formula.* It shows that the Legendre polynomials may be obtained by repeated differentiations of the '*fundamental solution*' $y \to |y|^{-1}, y \neq 0$, of the Laplace equation in the direction of the unit vector ξ. Thus, the potential on the right-hand side of Maxwell's representation formula may be regarded as the potential of a pole of order n with the axis ξ at the origin.

The power series in Lemma 3.39 can be differentiated for all $h \in (-1, 1)$. Thus, it follows that

$$-\frac{h-t}{(1+h^2-2ht)^{3/2}} = \sum_{n=1}^{\infty} nP_n(t)h^{n-1}. \tag{3.215}$$

Now, it is easy to see that

$$\frac{1}{\sqrt{1+h^2-2ht}} - \frac{2h^2-2ht}{(1+h^2-2ht)^{3/2}} = \frac{1-h^2}{(1+h^2-2ht)^{3/2}}. \tag{3.216}$$

This gives us the following result.

Lemma 3.40. *For all $t \in [-1,1]$ and $h \in (-1,1)$*

$$\frac{1-h^2}{(1+h^2-2ht)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)h^n P_n(t).$$

Lemma 3.39 can be used to prove an integral representation for the Legendre polynomial. To this end, we start from the well known elementary integral

$$\int_0^{\pi} \frac{d\varphi}{1+\gamma\cos\varphi} = \frac{\pi}{\sqrt{1-\gamma^2}} \quad (|\gamma|<1). \tag{3.217}$$

We set

$$\gamma = -\frac{h\sqrt{t^2-1}}{1-ht}. \tag{3.218}$$

On the one hand, it follows that

$$\begin{aligned} \int_0^{\pi} \frac{1}{1+\gamma\cos\varphi}\,d\varphi &= (1-ht)\int_0^{\pi}[1-h(t+\sqrt{t^2-1}\cos\varphi)]^{-1}\,d\varphi \\ &= (1-ht)\int_0^{\pi}\sum_{n=0}^{\infty}(t+\sqrt{t^2-1}\cos\varphi)^n h^n\,d\varphi. \end{aligned} \tag{3.219}$$

On the other hand, we obtain

$$\frac{\pi}{\sqrt{1-\gamma^2}} = \frac{\pi}{\sqrt{1-h^2(t^2-1)/(1-ht)^2}} = \frac{\pi(1-ht)}{\sqrt{1+h^2-2ht}}. \tag{3.220}$$

In connection with Lemma 3.39, this yields

$$\frac{\pi}{\sqrt{1-\gamma^2}} = (1-ht)\pi\sum_{n=0}^{\infty}P_n(t)h^n. \tag{3.221}$$

By comparison, we therefore obtain

$$\sum_{n=0}^{\infty}\int_0^{\pi}(t+\sqrt{t^2-1}\cos\varphi)^n\,d\varphi\ h^n = \pi\sum_{n=0}^{\infty}P_n(t)h^n. \tag{3.222}$$

This gives us the *Laplace representation of Legendre polynomials* .

Lemma 3.41. *For $t \in [-1,1]$ and $n = 0,1,\ldots$*

$$P_n(t) = \frac{1}{\pi}\int_0^{\pi}(t+\sqrt{t^2-1}\cos\varphi)^n\,d\varphi.$$

By the representation ($i = \sqrt{-1}$)

$$P_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \left(t + i\sqrt{1-t^2} \cos\varphi \right)^n \, d\varphi \tag{3.223}$$

we obtain an estimate valid for arbitrary $t \in [-1, 1]$ (cf. C. Müller (1969)):

$$\begin{aligned} |P_n(t)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |t + i\sqrt{1-t^2} \cos\varphi|^n \, d\varphi \\ &\leq (|t| + |\sqrt{1-t^2}|)^n \\ &\leq (|t| + \sqrt{1+|t|^2})^n \\ &\leq 2^n (1+|t|^2)^{n/2}. \end{aligned} \tag{3.224}$$

Moreover, it follows by the substitution $s = \cos\varphi$ that (3.223) is equivalent to

$$P_n(t) = \frac{1}{\pi} \int_{-1}^{1} (t + is\sqrt{1-t^2})^n (1-s^2)^{-1/2} \, ds, \tag{3.225}$$

so that we get from $|t + is\sqrt{1-t^2}| = (1-(1-s^2)(1-t^2))^{1/2}$ the estimate

$$|P_n(t)| \leq \frac{1}{\pi} \int_{-1}^{1} e^{\frac{n}{2}\log(1-(1-t^2)(1-s^2))} (1-s^2)^{-1/2} \, ds. \tag{3.226}$$

Since

$$\log(1-(1-s^2)(1-t^2)) \leq -(1-s^2)(1-t^2) \tag{3.227}$$

we obtain

$$|P_n(t)| \leq \frac{2}{\pi} \int_0^1 e^{-\frac{n}{2}(1-s^2)(1-t^2)} (1-s^2)^{-1/2} \, ds. \tag{3.228}$$

By the substitution $s = 1-u$, we find in connection with $u \leq 1-s^2 \leq 2u$

$$\begin{aligned} |P_n(t)| &\leq \frac{2}{\pi} \int_0^1 e^{-\frac{n}{2}u(1-t^2)} u^{-1/2} \, du \\ &\leq \frac{\sqrt{2}}{\pi} \int_0^\infty e^{-\frac{n}{2}u(1-t^2)} u^{-1/2} \, du \\ &= \frac{2}{\pi} \frac{\sqrt{\pi}}{(n(1-t^2))^{1/2}} \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{4}{n(1-t^2)} \right)^{1/2}. \end{aligned} \tag{3.229}$$

Lemma 3.42. *(Estimate of the Legendre polynomial) For $n = 1, 2, \ldots$ and $t \in (-1, 1)$,*

$$|P_n(t)| \leq \frac{1}{\sqrt{\pi}} \left(\frac{4}{n(1-t^2)} \right)^{1/2}. \tag{3.230}$$

Next, we discuss the *circular average* of a function on the unit sphere Ω. The problem is equivalent to the investigation of the spherical counterpart T_h of the so-called translation operator

$$T_h(F)(\xi) = \frac{1}{2\pi\sqrt{1-h^2}} \int_{\substack{\xi\cdot\eta=h \\ \eta\in\Omega}} F(\eta)\, d\sigma(\eta), \quad F \in \mathrm{L}^2(\Omega), \quad h \in (-1,1), \tag{3.231}$$

where $d\sigma$ is the line element in $\mathbb{R}^3$. T_h is a bounded positive linear operator mapping $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2(\Omega)$ with the following properties:

$$||T_h(F)||_{\mathrm{L}^2(\Omega)} \leq ||F||_{\mathrm{L}^2(\Omega)}, \tag{3.232}$$

$$\int_\Omega T_h(F)(\xi) Y_n(\xi)\, d\omega(\xi) = P_n(h) \int_\Omega F(\xi) Y_n(\xi)\, d\omega(\xi), \tag{3.233}$$

for $n = 0, 1, \ldots, h \in (-1,1), F \in \mathrm{L}^2(\Omega)$. In particular,

$$T_h(Y_n)(\xi) = P_n(h) Y_n(\xi), \quad \xi \in \Omega, Y_n \in \mathrm{Harm}_n. \tag{3.234}$$

Finally,

$$\lim_{\substack{h\to 1 \\ h<1}} ||F - T_h(F)||_{\mathrm{L}^2(\Omega)} = 0. \tag{3.235}$$

In other words, $Y_n \in \mathrm{Harm}_n$ implies $T_h(Y_n) \in \mathrm{Harm}_n$, and $h \mapsto T_h(Y_n)$ is a polynomial of degree $\leq n$. Furthermore, it is not hard to see that for all $G \in \mathrm{L}^2[-1,1]$ and all $F \in \mathrm{L}^2(\Omega)$

$$\int_\Omega G(\xi\cdot\eta) F(\eta) d\omega(\eta) = 2\pi \int_{-1}^{1} G(t) T_t(F)(\xi)\, dt \tag{3.236}$$

almost everywhere.

Clearly, if $P : [-1,1] \to \mathbb{R}$ is a polynomial of degree $\leq m$ and $F \in \mathrm{L}^2(\Omega)$, then

$$\int_\Omega P(\xi\cdot\eta) F(\eta)\, d\omega(\eta) \tag{3.237}$$

is a member of the class of all spherical harmonics of degree $\leq m$, i.e.,

$$\mathrm{Harm}_{0,\ldots,m} = \bigoplus_{n=0}^{m} \mathrm{Harm}_n. \tag{3.238}$$

3.6 Orthogonal (Fourier) Expansions

Let $\{Y_{n,j}\}, n = 0, 1, \ldots, j = 1, \ldots, 2n+1$, be an $\mathrm{L}^2(\Omega)$-orthonormal system of spherical harmonics. Let F be an arbitrary element of class $\mathrm{L}^2(\Omega)$ (or $\mathrm{C}(\Omega)$). The series

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (F, Y_{n,j})_{\mathrm{L}^2(\Omega)} Y_{n,j} \tag{3.239}$$

is the *Fourier series* (or orthogonal expansion) of F in terms of spherical harmonics. The constants

$$F^\wedge(n,j) = (F, Y_{n,j})_{L^2(\Omega)} = \int_\Omega F(\xi) Y_{n,j}(\xi)\, d\omega(\xi) \tag{3.240}$$

are known as the *Fourier coefficients of* F (or orthogonal coefficients of F with respect to $Y_{n,j}$). One frequently writes

$$F \sim \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(n,j) Y_{n,j} \tag{3.241}$$

to indicate that the right-hand sum is associated in a formal way with the left-hand side. In view of the fact that

$$\begin{aligned} \mathrm{Proj}_{Harm_n}(F) &= \sum_{j=1}^{2n+1} F^\wedge(n,j) Y_{n,j} \\ &= \int_\Omega \frac{2n+1}{4\pi} P_n(\cdot\eta) F(\eta)\, d\omega(\eta) \end{aligned} \tag{3.242}$$

we may write

$$F \sim \sum_{n=0}^{\infty} \mathrm{Proj}_{Harm_n}(F), \tag{3.243}$$

hence, the Fourier series (orthogonal expansion in terms of spherical harmonics) of an element F is merely the sum of the projections of the element on the orthonormal system of spherical harmonics. The relation between an element and its Fourier series has been the object of many investigations. Of particular importance for practical purposes are results in the framework of the pre-Hilbert space $(\mathrm{C}(\Omega), (\cdot,\cdot)_{L^2(\Omega)})$ or the Hilbert space $(L^2(\Omega), (\cdot,\cdot)_{L^2(\Omega)})$ which will be established below.

We take the opportunity to base our considerations about Fourier expansion theory in terms of spherical harmonics on two summability methods, namely Bernstein and Abel-Poisson summability.

The point of departure for Bernstein summability is the so-called *Bernstein kernel of degree n*

$$t \mapsto B_n(t) = \frac{n+1}{4\pi} \left(\frac{1+t}{2}\right)^n, \quad t \in [-1,1]. \tag{3.244}$$

Remark 3.43. The name *Bernstein* is motivated by the fact that the kernel (see Fig. 3.2) is proportional to the Bernstein polynomial $B_n^\nu(t) = \binom{n}{\nu} t^\nu (1-t)^{n-\nu}$ scaled to the interval $[-1,1]$ with $\nu = n$ (t is the polar distance between ξ and η, i.e., $t = \xi \cdot \eta$).

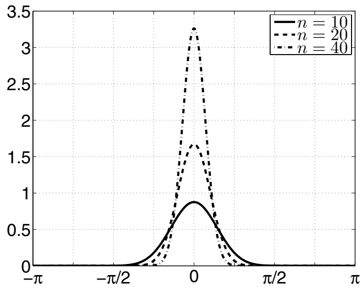


Fig. 3.2: Illustration of the kernel $B_n(\cos\vartheta), \theta \in [-\pi,\pi]$ for the degrees $n = 10$, $n = 20$, $n = 40$.

First, we mention some important properties of the Bernstein kernel.

Lemma 3.44.

(i) *For all* $t \in [-1,1]$ *and* $n = 0, 1, \ldots$ *we have*

$$B_n^\wedge(0) = 2\pi \int_{-1}^{+1} B_n(t)\ dt = 1. \tag{3.245}$$

(ii) *For all* $t \in [-1,1]$

$$B_n(t) \geq 0. \tag{3.246}$$

(iii) *For all* $t \in [-1,1]$

$$\lim_{n\to\infty} B_n(t) = 0. \tag{3.247}$$

(iv) *For* $k = 0, \ldots, n$

$$2\pi \int_{-1}^{1} B_n(t)P_k(t)\ dt = B_n^\wedge(k) = \frac{n!}{(n-k)!}\frac{(n+1)!}{(n+k+1)!} = \frac{\binom{n}{k}}{\binom{n+k+1}{k}}. \tag{3.248}$$

(v) *For* $k \in \mathbb{N}$ *fixed*

$$B_n^\wedge(k) < B_{n+1}^\wedge(k). \tag{3.249}$$

(vi) *For* $k \in \mathbb{N}_0$ *fixed,* $B_n^\wedge(k) \to 1$ *as* $n \to \infty$*, i.e.,*

$$\lim_{n\to\infty} B_n^\wedge(k) = 1. \tag{3.250}$$

Now, suppose that F is continuous on Ω. We use the property (i) to guarantee

$$\begin{aligned} &\int_\Omega B_n(\xi\cdot\eta)F(\eta)\; d\omega(\eta) \\ &\quad = F(\xi) + \int_\Omega B_n(\xi\cdot\eta)\left(F(\eta)-F(\xi)\right)\; d\omega(\eta), \end{aligned} \tag{3.251}$$

$\xi \in \Omega$. We split Ω into two parts, depending on a parameter $\gamma \in (0,1)$:

$$\int_\Omega \cdots = \int_{-1\le\xi\cdot\eta\le 1-\gamma} \cdots + \int_{1-\gamma\le\xi\cdot\eta\le 1} \cdots . \tag{3.252}$$

On the one hand, we find with (ii)

$$\begin{aligned} \left| \int_{-1\le\xi\cdot\eta\le 1-\gamma} B_n(\xi\cdot\eta)F(\eta)\; d\omega(\eta)\right| &\le 4\pi\, \|F\|_{\mathrm{C}(\Omega)} \int_{-1}^{1-\gamma} B_n(t)\; dt \\ &\le 2\, \|F\|_{\mathrm{C}(\Omega)} \left(1-\frac{\gamma}{2}\right)^{n+1} . \end{aligned}$$

On the other hand, F is uniformly continuous on Ω. Thus, there exists a positive function $\mu : \gamma \mapsto \mu(\gamma)$ with

$$\lim_{\substack{\gamma\to 0\\ \gamma>0}} \mu(\gamma) = 0 \tag{3.253}$$

such that

$$|F(\xi)-F(\eta)| \le \mu(\gamma) \tag{3.254}$$

for all $\eta \in \Omega$ with $1-\gamma \le \xi\cdot\eta \le 1$. Thus, it follows that

$$\begin{aligned} \left| \int_{1-\gamma\le\xi\cdot\eta\le 1} B_n(\xi\cdot\eta)(F(\eta)-F(\xi))\; d\omega(\eta)\right| &\le \mu(\gamma)\int_\Omega B_n(\xi\cdot\eta)\; d\omega(\eta) \\ &= \mu(\gamma). \end{aligned} \tag{3.255}$$

Summarizing our results, we obtain the following estimate

$$\begin{aligned}
\sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta)-F(\xi)\right| &= \sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)\left(F(\eta)-F(\xi)\right)\ d\omega(\eta)\right| \\
\leq\ & \sup_{\xi\in\Omega}\left|\int\limits_{\substack{-1\leq\xi\cdot\eta\leq 1-\gamma,\\ |\eta|=1}} B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta)-F(\xi)\int\limits_{\substack{-1\leq\xi\cdot\eta\leq 1-\gamma,\\ |\eta|=1}} B_n(\xi\cdot\eta)\ d\omega(\eta)\right| \\
&+\sup_{\xi\in\Omega}\left|\int\limits_{\substack{1-\gamma\leq\xi\cdot\eta\leq 1,\\ |\eta|=1}} B_n(\xi\cdot\eta)\left(F(\eta)-F(\xi)\right)\ d\omega(\eta)\right| \\
\leq\ & \sup_{\xi\in\Omega}\left|\int\limits_{\substack{-1\leq\xi\cdot\eta\leq 1-\gamma,\\ |\eta|=1}} B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta)\right| \\
&+\|F\|_{\mathrm{C}(\Omega)}\int\limits_{\substack{-1\leq\xi\cdot\eta\leq 1-\gamma,\\ |\eta|=1}} B_n(\xi\cdot\eta)\ d\omega(\eta)+\mu(\gamma) \\
\leq\ & 2\|F\|_{\mathrm{C}(\Omega)}\left(1-\frac{\gamma}{2}\right)^{n+1}+\mu(\gamma).
\end{aligned}$$

This shows us that

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)F(\eta)d\omega(\eta)-F(\xi)\right|\leq\mu(\gamma) \tag{3.256}$$

for every $\gamma\in(0,1)$. Since $\mu(\gamma)\to 0$ as $\gamma\to 0$, we get the following result.

Theorem 3.45. *For* $F\in\mathrm{C}(\Omega)$

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta)-F(\xi)\right|=0.$$

Now, it can be readily seen that

$$B_n(t)=\sum_{k=0}^{n}B_n^\wedge(k)\frac{2k+1}{4\pi}P_k(t) \tag{3.257}$$

for all $t\in[-1,1]$. This shows us that

$$\begin{aligned}
&\int_\Omega B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta) \\
&\quad= \sum_{k=0}^{n}B_n^\wedge(k)\frac{2k+1}{4\pi}\int_\Omega P_k(\xi\cdot\eta)F(\eta)\ d\omega(\eta) \\
&\quad= \sum_{k=0}^{n}B_n^\wedge(k)\ \mathrm{Proj}_{\mathrm{Harm}_k}(F)(\xi).
\end{aligned} \tag{3.258}$$

Thus, we finally have the *'Bernstein summability' of a Fourier series expansion* in terms of spherical harmonics.

Theorem 3.46. *For $F \in \mathrm{C}(\Omega)$,*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \sum_{k=0}^{n} B_n^\wedge(k) \sum_{j=1}^{2k+1} F^\wedge(k,j) Y_{k,j}(\xi) - F(\xi) \right| = 0. \tag{3.259}$$

Theorem 3.46 enables us to prove the *closure of the system of spherical harmonics in the space* $\mathrm{C}(\Omega)$.

Corollary 3.47. *The system $\{Y_{n,j}\}_{n=0,1,\ldots,\ j=1,\ldots,2n+1}$ is closed in $\mathrm{C}(\Omega)$, that is for any given $\varepsilon > 0$ and each $F \in \mathrm{C}(\Omega)$, there exists a linear combination*

$$\sum_{k=0}^{N} \sum_{j=1}^{2k+1} d_{k,j} Y_{k,j}$$

such that

$$\left\| F - \sum_{k=0}^{N} \sum_{j=1}^{2k+1} d_{k,j} Y_{k,j} \right\|_{\mathrm{C}(\Omega)} \leq \varepsilon.$$

Proof. Given $F \in \mathrm{C}(\Omega)$. Then, for any given $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\sup_{\xi\in\Omega} \left| \sum_{k=0}^{N} \sum_{j=1}^{2k+1} \underbrace{B_N^\wedge(k) F^\wedge(k,j)}_{=d_{k,j}} Y_{k,j}(\xi) - F(\xi) \right| \leq \varepsilon, \tag{3.260}$$

which proves Corollary 3.47. □

The point of departure for the Abel-Poisson summability is Lemma 3.40 from which we obtain

$$\int_{-1}^{1} \frac{1-h^2}{(1+h^2-2ht)^{3/2}}\, dt = \sum_{n=0}^{\infty} (2n+1) h^n \int_{-1}^{1} P_n(t)\, dt \tag{3.261}$$

for all $h \in (-1,1)$. Since $\int_{-1}^{1} P_n(t) P_0(t)\, dt = 0$ for $n \geq 1$, we finally have

$$\frac{1}{2} \int_{-1}^{1} \frac{1-h^2}{(1+h^2-2ht)^{3/2}}\, dt = 1. \tag{3.262}$$

Theorem 3.48. *(Abel–Poisson Integral Formula) If F is continuous on Ω, then*

$$\lim_{h\to 1, h<1} \sup_{\xi\in\Omega} \left| \frac{1}{4\pi} \int_{\Omega} \frac{(1-h^2)F(\eta)}{(1+h^2-2h(\xi\cdot\eta))^{3/2}}\, d\omega(\eta) - F(\xi) \right| = 0.$$

Proof. Observing the identity (3.262) we get

$$\begin{aligned} &\frac{1}{4\pi} \int_{\Omega} \frac{(1-h^2)F(\eta)}{(1+h^2-2h(\xi\cdot\eta))^{3/2}}\, d\omega(\eta) - F(\xi) \\ &= \frac{1}{4\pi} \int_{\Omega} \frac{(1-h^2)(F(\eta)-F(\xi))}{(1+h^2-2h(\xi\cdot\eta))^{3/2}}\, d\omega(\eta). \end{aligned} \tag{3.263}$$

For $h \in \left[\frac{1}{2}, 1\right)$ we split the integral into two parts:

$$\int_{\Omega} \cdots \quad = \int\limits_{-1\le\xi\cdot\eta\le 1-\sqrt[3]{1-h}} \cdots \quad + \int\limits_{1-\sqrt[3]{1-h}\le\xi\cdot\eta\le 1} \cdots\,. \tag{3.264}$$

On the one hand, we find

$$1+h^2-2ht = (1-h)^2 + 2h(1-t) \ge 2h\sqrt[3]{1-h} \tag{3.265}$$

and

$$\begin{aligned} \frac{1-h^2}{(1+h^2-2ht)^{3/2}} &\le \frac{1-h^2}{(2h\sqrt[3]{1-h})^{3/2}} \\ &= \frac{1+h}{(2h)^{3/2}} \frac{1-h}{\sqrt{1-h}} \\ &\le 2\sqrt{1-h} \end{aligned} \tag{3.266}$$

provided that $t \in [-1, 1-\sqrt[3]{1-h}]$. This leads us to

$$\begin{aligned} &\left| \int\limits_{-1\le\xi\cdot\eta\le 1-\sqrt[3]{1-h}} \frac{(1-h^2)(F(\xi)-F(\eta))}{(1+h^2-2h(\xi\cdot\eta))^{3/2}}\, d\omega(\eta) \right| \\ &\le 4\pi \|F\|_{C(\Omega)} \int\limits_{-1}^{1-\sqrt[3]{1-h}} \frac{1-h^2}{(1+h^2-2ht)^{3/2}}\, dt \le 16\pi \|F\|_{C(\Omega)} \sqrt{1-h}. \end{aligned} \tag{3.267}$$

On the other hand, F is uniformly continuous on Ω. Thus, there exists a positive function $\mu : h \mapsto \mu(h)$ with

$$\lim_{\substack{h\to 1\\ h<1}} \mu(h) = 0 \tag{3.268}$$

such that

$$|F(\xi) - F(\eta)| \leq \mu(h) \tag{3.269}$$

for all $\eta \in \Omega$ satisfying $1 - \sqrt[3]{1-h} \leq \xi \cdot \eta \leq 1$. Consequently, in connection with (3.262), we are able to deduce that

$$\left| \int\limits_{1-\sqrt[3]{1-h} \leq \xi \cdot \eta \leq 1} \frac{(1-h^2)(F(\xi) - F(\eta))}{(1+h^2-2h(\xi \cdot \eta))^{3/2}} \, d\omega(\eta) \right| \leq \mu(h). \tag{3.270}$$

Letting h tend towards 1, we obtain the desired result. □

As an illustration (see Fig. 3.3), we consider a continuous function F defined on Ω and its Abel–Poisson means

$$\frac{1}{4\pi} \int_\Omega \frac{(1-h^2)F(\eta)}{(1+h^2-2h(\xi \cdot \eta))^{3/2}} \, d\omega(\eta)$$

for the values $h = 0.9, 0.7, 0.4$, respectively. In Fig. 3.3, we show a cut along the equator of the unit sphere.

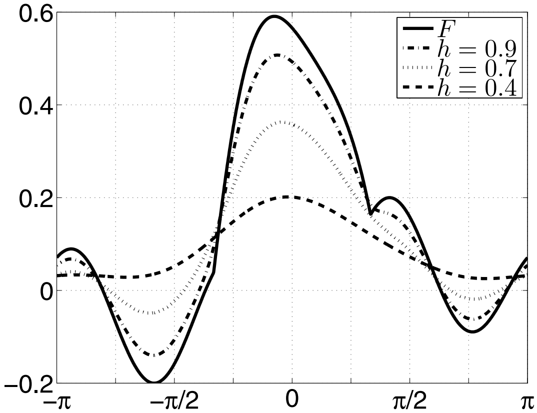


Fig. 3.3: A continuous function F on the sphere and its Abel-Poisson means ($h = 0.9,\ h = 0.7,\ h = 0.4$). The figure shows the profile of the function along the equator of the unit sphere.

Combining Theorem 3.48 and Lemma 3.40, we get the 'Abel-Poisson summability' of a Fourier series expansion.

Theorem 3.49. *Let F be of class $\mathrm{C}(\Omega)$. Then the series*

$$\sum_{n=0}^{\infty} h^n \sum_{j=1}^{2n+1} F^\wedge(n,j) Y_{n,j}(\xi)$$

converges uniformly with respect to all $\xi \in \Omega$ *for fixed* $h \in (0, h_0), h_0 < 1$, *and*

$$\lim_{h \to 1, h<1} \sum_{n=0}^{\infty} h^n \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j}(\xi) \;=\; F(\xi).$$

Again the summability (Theorem 3.49) enables us to prove the *closure of the system of spherical harmonics in the space* $\mathrm{C}(\Omega)$ with respect to $\|\cdot\|_{\mathrm{C}(\Omega)}$.

Corollary 3.50. *The system* $\{Y_{n,j}\}_{n=0,1,\ldots,\ j=1,\ldots,2n+1}$ *is closed in* $\mathrm{C}(\Omega)$, *that is for any given* $\varepsilon > 0$ *and each* $F \in \mathrm{C}(\Omega)$, *there exists a linear combination*

$$\sum_{n=0}^{N} \sum_{j=1}^{2n+1} d_{n,j} Y_{n,j}$$

such that

$$\left\| F - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} d_{n,j} Y_{n,j} \right\|_{\mathrm{C}(\Omega)} \leq \varepsilon.$$

Proof. Given $F \in \mathrm{C}(\Omega)$. Then, on the one hand, for any given $\varepsilon > 0$, there exists a real number $h = h(\varepsilon) < 1$ such that

$$\sup_{\xi \in \Omega} \left| F(\xi) - \sum_{n=0}^{\infty} h^n \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j}(\xi) \right| \leq \frac{\varepsilon}{2}. \tag{3.271}$$

On the other hand, there exists an index $N = N(\varepsilon)$ such that

$$\sup_{\xi \in \Omega} \left| \sum_{n=0}^{\infty} h^n \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j}(\xi) - \sum_{n=0}^{N} h^n \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j}(\xi) \right| \leq \frac{\varepsilon}{2}. \tag{3.272}$$

But this means that

$$\sup_{\xi \in \Omega} \left| F(\xi) - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} h^n F^{\wedge}(n,j) Y_{n,j}(\xi) \right| \leq \varepsilon, \tag{3.273}$$

which proves Corollary 3.50. □

Next, we are interested in closure and completeness in the Hilbert space $(\mathrm{L}^2(\Omega), \|\cdot\|_{\mathrm{L}^2(\Omega)})$.

Theorem 3.51. *The system* $\{Y_{n,j}\}_{n=0,1,\ldots,\ j=1,\ldots,2n+1}$ *is closed in the space* $\mathrm{C}(\Omega)$ *with respect to* $\|\cdot\|_{\mathrm{L}^2(\Omega)}$, *that is for any given* $\varepsilon > 0$ *and any given* $F \in \mathrm{C}(\Omega)$, *there exists a linear combination*

$$\sum_{n=0}^{N} \sum_{j=1}^{2n+1} b_{n,j} Y_{n,j}$$

such that

$$\left\| F - \sum_{n=0}^{N} \sum_{j=1}^{2n+1} b_{n,j} Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon.$$

Proof. Corollary 3.51 follows immediately from Corollary 3.50 by using the norm estimate (2.103). □

Theorem 3.52. *The system* $\{Y_{n,j}\}_{n=0,1,\ldots,\ j=1,\ldots,2n+1}$ *is closed in the space* $\mathrm{L}^2(\Omega)$ *with respect to* $\|\cdot\|_{\mathrm{L}^2(\Omega)}$.

Proof. $\mathrm{C}(\Omega)$ is dense in $\mathrm{L}^2(\Omega)$, that is for every $F \in \mathrm{L}^2(\Omega)$ there exists a function $G \in \mathrm{C}(\Omega)$ with $\|F - G\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon/2$. The function $G \in \mathrm{C}(\Omega)$ admits an arbitrarily close approximation by finite linear combinations of spherical harmonics. Therefore, the proof of the closure is clear. □

Truncated spherical harmonic expansions admit the following *minimum property* which should be mentioned for the convenience of the reader.

Lemma 3.53. *Assume that* $F \in \mathrm{L}^2(\Omega)$. *Then*

$$\left\| F - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} F^\wedge(n,j) Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} = \inf_{Y \in \mathrm{Harm}_{0,\ldots,m}} \|F - Y\|_{\mathrm{L}^2(\Omega)} \ ,$$

i.e., the problem of finding the linear combination in $\mathrm{Harm}_{0,\ldots,m}$ *which is minimal in the* $\mathrm{L}^2(\Omega)$*-norm is solved by the orthogonal projection* $\mathrm{Proj}_{\mathrm{Harm}_{0,\ldots,m}}(F)$ *of* F *onto* $\mathrm{Harm}_{0,\ldots,m}$.

Proof. An easy calculation shows that

$$\left\| F - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} L_{n,j} Y_{n,j} \right\|^2_{\mathrm{L}^2(\Omega)} = (F,F)_{\mathrm{L}^2(\Omega)} - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} | F^\wedge(n,j) |^2 + \sum_{n=0}^{m} \sum_{j=1}^{2n+1} | L_{n,j} - F^\wedge(n,j) |^2 \tag{3.274}$$

for arbitrarily given coefficients $L_{n,j} \in \mathbb{R}$. Therefore, the minimum of the right-hand side of the Eq. (3.274) is achieved if and only if $L_{n,j} = F^{\wedge}(n,j)$ for $n = 0, \ldots, m,\ j = 1, \ldots, 2n+1$. Moreover,

$$\left\| F - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j} \right\|^2_{\mathrm{L}^2(\Omega)} = (F,F)_{\mathrm{L}^2(\Omega)} - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} |F^{\wedge}(n,j)|^2. \tag{3.275}$$

□

We summarize our results in the *fundamental theorem of orthogonal (spherical harmonic) expansions.*

Theorem 3.54. *The closure of the system* $\{Y_{n,j}\}$ *in* $\mathrm{L}^2(\Omega)$ *is equivalent to each of the following statements:*

(i) The orthogonal expansion of any element $H \in \mathrm{L}^2(\Omega)$ *converges in norm to* H*, i.e.,*

$$\lim_{m\to\infty} \left\| H - \sum_{n=0}^{m} \sum_{j=1}^{2n+1} (H, Y_{n,j})_{\mathrm{L}^2(\Omega)} Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} = 0.$$

(ii) Parseval's identity holds. That is, for any $H \in \mathrm{L}^2(\Omega)$*,*

$$\|H\|^2_{L^2(\Omega)} = (H,H)_{\mathrm{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left| (H, Y_{n,j})_{\mathrm{L}^2(\Omega)} \right|^2 .$$

(iii) The extended Parseval identity holds. That is for any $H, K \in \mathrm{L}^2(\Omega)$*,*

$$(H,K)_{L^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (H, Y_{n,j})_{\mathrm{L}^2(\Omega)} (K, Y_{n,j})_{\mathrm{L}^2(\Omega)}.$$

(iv) There is no strictly larger orthonormal system containing the orthonormal system $\{Y_{n,j}\}_{n=0,1,\ldots,\ j=1,\ldots,2n+1}$.

(v) The system $\{Y_{n,j}\}_{n=0,1,\ldots;\ j=1,\ldots,2n+1}$ *has the completeness property. That is,* $H \in \mathrm{L}^2(\Omega)$ *and* $(H, Y_{n,j})_{\mathrm{L}^2(\Omega)} = 0$ *for all* $n = 0, 1, \ldots,\ j = 1, \ldots, 2n+1$*, implies* $H = 0$.

(vi) An element H *of* $\mathrm{L}^2(\Omega)$ *is determined uniquely by its orthogonal coefficients. That is, if* $(H, Y_{n,j})_{\mathrm{L}^2(\Omega)} = (K, Y_{n,j})_{\mathrm{L}^2(\Omega)},\ n = 0, 1, \ldots,\ j = 1, \ldots, 2n+1$*, then* $H = K$.

The proof of Theorem 3.54 is omitted (see, for example, P.J. Davis (1963)). The property (i) is of great importance for practical purposes. In particular, it tells us that any continuous function may be approximated (in the $L^2(\Omega)$-sense) by finite truncations of its Fourier (orthogonal) expansion in terms of any $L^2(\Omega)$-orthonormal system of spherical harmonics $\{Y_{n,j}\}$.

Finally, we are interested in pointwise approximation (see, e.g., C. Müller (1998)).

Theorem 3.55. *Let F be continuous in the point $\xi \in \Omega$. Moreover, assume that F is bounded on Ω. Furthermore, suppose that the sequence $\{S_n(F)\}_{n=0,1,...}$*

$$S_n(F)(\xi) = \sum_{k=0}^{n} \sum_{j=1}^{2k+1} F^\wedge(k,j) Y_{k,j}(\xi)$$

converges in $\xi \in \Omega$. Then

$$F(\xi) = \lim_{n\to\infty} S_n(F)(\xi) = \sum_{k=0}^{\infty} \sum_{j=1}^{2k+1} F^\wedge(k,j) Y_{k,j}(\xi). \tag{3.276}$$

Proof. Observing that

$$S_n(F)(\xi) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} \int_\Omega P_k(\xi\cdot\eta) F(\eta)\ d\omega(\eta) \tag{3.277}$$

we obtain from the Abel–Poisson summability

$$F(\xi) = \lim_{\substack{h\to 1\\ h<1}} \sum_{k=0}^{\infty} h^k \frac{2k+1}{4\pi} \int_\Omega P_k(\xi\cdot\eta) F(\eta) d\omega(\eta). \tag{3.278}$$

The series can be rewritten as follows

$$\begin{aligned} \sum_{k=1}^{\infty} h^k \left(S_k(F)(\xi) - S_{k-1}(F)(\xi)\right) \quad &+ \quad S_0(F)(\xi) \\ &= \quad (1-h) \sum_{k=0}^{\infty} h^k S_k(F)(\xi). \end{aligned} \tag{3.279}$$

It is known from Theorem 3.49 that

$$\lim_{\substack{h\to 1\\ h<1}} (1-h) \sum_{k=0}^{\infty} h^k S_k(F)(\xi) = F(\xi). \tag{3.280}$$

According to our assumption, the sequence $\{S_k(F)(\xi)\}_{k=0,1,\ldots}$ is convergent:

$$\lim_{k\to\infty} S_k(F)(\xi) = S(F)(\xi). \tag{3.281}$$

Consequently, to every $\varepsilon > 0$, there exists an integer $k_0 = k_0(\varepsilon)$ such that

$$S(F)(\xi) - \varepsilon \leq S_k(F)(\xi) \leq S(F)(\xi) + \varepsilon \tag{3.282}$$

holds for all $k \geq k_0(\varepsilon)$. Hence, we are able to show that

$$\frac{h^{k_0}}{1-h}(S(F)(\xi) - \varepsilon) \leq \sum_{k=k_0}^{\infty} h^k S_k(F)(\xi) \leq \frac{h^{k_0}}{1-h}(S(F)(\xi) + \varepsilon). \tag{3.283}$$

The limit $h \to 1$ gives, in connection with (3.280), the estimate

$$S(F)(\xi) - \varepsilon \leq F(\xi) \leq S(F)(\xi) + \varepsilon.$$

This proves the assertion of the theorem. □

Finally, we are concerned with the spherical Fourier transform and its inverse.

Definition 3.56. The *spherical Fourier transform* $FT : F \mapsto (FT)(F), F \in \mathrm{L}^1(\Omega)$, is defined by

$$((FT)(F))(n,j) = F^\wedge(n,j) = \int_\Omega F(\eta) Y_{n,j}(\eta)\, d\omega(\eta).$$

As we have shown above, the restriction of the spherical Fourier transform to $\mathrm{L}^2(\Omega)$ forms a mapping from $\mathrm{L}^2(\Omega)$ into the space

$$\mathrm{L}^2_{FT}(\mathcal{J}) = \left\{ \{H(n,j)\} \left| \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \mid H(n,j) \mid^2 < \infty \right. \right\}, \tag{3.284}$$

where we have used the abbreviation

$$\mathcal{J} = \{(n,j) \mid n = 0,1,\ldots, j = 1,\ldots,2n+1\}. \tag{3.285}$$

From the considerations given above, it is clear that any function $F \in \mathrm{L}^2(\Omega)$ is characterized by its sequence $\{F^\wedge(n,j)\} \in \mathrm{L}^2_{FT}(\mathcal{J})$.

Lemma 3.57. *(Inverse Transform) For $\{H(n,j)\} \in \mathrm{L}^2_{FT}(\mathcal{J})$ define the mapping*

$$(FT)^{-1} : \mathrm{L}^2_{FT}(\mathcal{J}) \to \mathrm{L}^2(\Omega)$$

by

$$(FT)^{-1}(\{H(n,j)\}) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} H(n,j)Y_{n,j}.$$

Then

$$\begin{aligned}(FT)^{-1}(FT) &= I_{\mathrm{L}^2(\Omega)},\\ (FT)(FT)^{-1} &= I_{\mathrm{L}^2_{FT}(\mathcal{J})}.\end{aligned}$$

Moreover it should be noted that, for $F, G \in \mathrm{L}^2(\Omega)$, the relation

$$\lim_{N\to\infty}\left\| G - \sum_{n=0}^{N}\sum_{j=1}^{2n+1} F^{\wedge}(n,j)Y_{n,j}\right\|_{\mathrm{L}^2(\Omega)} = 0 \tag{3.286}$$

implies $F = G$ almost everywhere on Ω. If F is assumed to be Lipschitz continuous, i.e., $F \in \mathrm{Lip}(\Omega)$, T. Gronwall (1914) has shown that F can be recovered by its Fourier expansion in uniform sense:

$$\lim_{N\to\infty}\left\| F - \sum_{n=0}^{N}\sum_{j=1}^{2n+1} F^{\wedge}(n,j)Y_{n,j}\right\|_{\mathrm{C}(\Omega)} = 0. \tag{3.287}$$

3.7 Legendre (Spherical) Harmonics

Next, we are interested in deriving another characterization of the Legendre polynomial, that will be of importance in the representation of spherical harmonics in terms of cartesian coordinates: Consider the function $L_n : \mathbb{R}^3 \to \mathbb{R}, n = 0, 1, \ldots$ defined by ($i = \sqrt{-1}$)

$$L_n(x) = \frac{1}{2\pi}\int_0^{2\pi} (x_3 + \mathrm{i}x_1\cos\alpha + \mathrm{i}x_2\sin\alpha)^n\, d\alpha, \quad x = (x_1, x_2, x_3)^T. \tag{3.288}$$

Clearly, L_n is a homogeneous polynomial of degree n which is symmetric with respect to the x_3-axis. An easy calculation shows that $\Delta_x L_n(x) = 0$ for all $x \in \mathbb{R}^3$, i.e., L_n is harmonic in $\mathbb{R}^3$. Moreover, L_n has the value 1 at ε^3: $L_n(\varepsilon^3) = 1$. Furthermore, by use of the coordinates (2.94) and the Laplace representation Lemma 3.41, we see that

$$\begin{aligned}L_n(x) &= \frac{r^n}{2\pi}\int_0^{2\pi} (t + i\sqrt{1-t^2}\cos(\alpha-\varphi))^n\, d\alpha \qquad (3.289)\\ &= \frac{r^n}{2\pi}\int_0^{2\pi} (t + i\sqrt{1-t^2}\cos\alpha)^n d\alpha\\ &= r^n P_n(t).\end{aligned}$$

This implies that

$$L_n\left(\frac{x}{|x|}\right) = P_n(t), \quad x = |x|\left(t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2)\right), \quad (3.290)$$

for all $x \in \mathbb{R}^3\backslash\{0\}$. This, together with Lemma 2.16 and the addition theorem (Theorem 3.26) hints at the following result.

Theorem 3.58. *Let H_n be a homogeneous, harmonic polynomial of degree n with the following properties:*

(i) $H_n(\mathbf{t}x) = H_n(x)$ *for all orthogonal transformations* $\mathbf{t} \in SO_{\varepsilon^3}(3)$,
(ii) $H_n(\varepsilon^3) = 1$.

Then H_n is uniquely determined, and H_n coincides with the Legendre harmonic L_n of degree n, i.e.,

$$H_n(x) = L_n(x) = r^n P_n(t), \ x = r(t\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2)),$$

where

$$P_n(t) = \sum_{k=0}^{n} C_{\frac{n-k}{2}}(1-t^2)^{\frac{n-k}{2}} t^k \quad (3.291)$$

with

$$C_{\frac{n-k}{2}} = \begin{cases} 0 & , \ n-k \text{ odd} \\ (-\frac{1}{4})^{\frac{n-k}{2}} \frac{n!}{((\frac{n-k}{2})!)^2 k!} & , \ n-k \text{ even.} \end{cases} \quad (3.292)$$

Equivalently,

$$P_n(t) = n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-\frac{1}{4})^l \frac{(1-t^2)^l t^{n-2l}}{(l!)^2(n-2l)!}. \quad (3.293)$$

Proof. We already know that H_n as homogeneous harmonic polynomial of degree n can be written in the form

$$H_n(x) = \sum_{k=0}^{n} A_{n-k}(x_1, x_2) x_3^k, \quad (3.294)$$

where

$$\left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right) A_{n-k}(x_1, x_2) + (k+2)(k+1)A_{n-k-2}(x_1, x_2) = 0 \quad (3.295)$$

for $k = 0, \ldots, n-2$. Therefore, H_n is uniquely determined by the homogeneous polynomials $A_n : (x_1, x_2) \mapsto A_n(x_1, x_2)$ and $A_{n-1} : (x_1, x_2) \mapsto$

$A_{n-1}(x_1, x_2)$. The condition (i) implies that these polynomials depend only on $x_1^2 + x_2^2$. We thus find with a constant $C_{\frac{n-k}{2}}$

$$A_{n-k}(x_1, x_2) = \begin{cases} 0 & , \quad n-k \text{ odd} \\ C_{\frac{n-k}{2}} (x_1^2 + x_2^2)^{\frac{n-k}{2}} & , \quad n-k \text{ even.} \end{cases} \tag{3.296}$$

For $x = \varepsilon^3$ in (3.296), we get $x_1^2 + x_2^2 = 0$ and $x_3 = 1$ such that $C_0 = 1$. In order to determine $C_{\frac{n-k}{2}}$ for even integers $n-k$, we see that

$$\left[\left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 \right] (x_1^2 + x_2^2)^{\frac{n-k}{2}} = 4 (\frac{n-k}{2})^2 (x_1^2 + x_2^2)^{\frac{n-k-2}{2}}. \tag{3.297}$$

In connection with (3.295), we therefore find the recursion relation

$$4C_{\frac{n-k}{2}} (\frac{n-k}{2})^2 (x_1^2 + x_2^2)^{\frac{n-k-2}{2}} = -(k+2)(k+1) C_{\frac{n-k-2}{2}} (x_1^2 + x_2^2)^{\frac{n-k-2}{2}}$$

such that

$$(n-k)^2 C_{\frac{n-k}{2}} + (k+2)(k+1) C_{\frac{n-k-2}{2}} = 0, \tag{3.298}$$

$k = 0, 2, \ldots, n-2$. In other words,

$$C_{\frac{n-k}{2}} = 1 \tag{3.299}$$

for $k = n$, and we have

$$C_{\frac{n-k}{2}} = (-\frac{1}{4})^{\frac{n-k}{2}} \frac{n!}{((\frac{n-k}{2})!)^2 k!}, \quad n-k \text{ even,} \tag{3.300}$$

for $k = 0, \ldots, n-1$. This shows us that H_n is uniquely determined by the conditions (i) and (ii), and we have

$$H_n(x) = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} {}^{\!2} (-\frac{1}{4})^{\frac{n-k}{2}} \frac{n!}{((\frac{n-k}{2})!)^2 k!} (x_1^2 + x_2^2)^{\frac{n-k}{2}} x_3^k, \tag{3.301}$$

where $\sum^2$ means that the sum is extended over all k with $n-k$ even.

By definition, we let $C_{\frac{n-k}{2}} = 0$ for $n-k$ odd. Consequently we are able to write

$$A_{n-k}(x_1, x_2) = C_{\frac{n-k}{2}} (x_1^2 + x_2^2)^{\frac{n-k}{2}} \tag{3.302}$$

with

$$C_{\frac{n-k}{2}} = \begin{cases} 0 & , \quad n-k \text{ odd} \\ (-\frac{1}{4})^{\frac{n-k}{2}} \frac{n!}{((\frac{n-k}{2})!)^2 k!} & , \quad n-k \text{ even} \end{cases} \tag{3.303}$$

Using polar coordinates, we know that H_n depends only on t, as we have

$$x_1^2 + x_2^2 = r^2(1 - t^2). \tag{3.304}$$

Therefore, our considerations have shown that there exists one and only one homogeneous harmonic polynomial H_n (dependent only on t) satisfying the conditions (i) and (ii). More explicitly, $H_n = L_n$, and it follows that

$$P_n(t) = \sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} (-\frac{1}{4})^{\frac{n-k}{2}} \frac{n!}{((\frac{n-k}{2})!)^2 k!} (1 - t^2)^{\frac{n-k}{2}} t^k. \tag{3.305}$$

By letting $n - k = 2l$, $l = 0, \ldots, \lfloor n/2 \rfloor$, we finally find

$$L_n(x) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} C_l (x_1^2 + x_2^2)^l x_3^{n-2l}. \tag{3.306}$$

From (3.298), we are able to deduce that the coefficients C_l satisfy the recursion relation

$$4l^2 C_l + (n - 2l + 2)(n - 2l + 1) C_{l-1} = 0, \tag{3.307}$$

$l = 1, \ldots \lfloor \frac{n}{2} \rfloor$ with $C_0 = 1$. Consequently, in (3.306), we have

$$C_l = (-\frac{1}{4})^l \frac{n!}{(l!)^2 (n - 2l)!}. \tag{3.308}$$

This leads us to the expression

$$L_n(x) = n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-\frac{1}{4})^l \frac{(x_1^2 + x_2^2)^l x_3^{n-2l}}{(l!)^2 (n - 2l)!} \tag{3.309}$$

such that

$$L_n(x) = r^n L_n(\xi) = r^n P_n(t), \tag{3.310}$$

where P_n is given by (3.293). This is the assertion of Theorem 3.58. □

Summarizing our results, we see that the only function $Y_n \in \mathrm{Harm}_n$ satisfying (i) $Y_n(\mathbf{t}\xi) = Y_n(\xi)$, $\xi \in \Omega$, for all orthogonal transformations $\mathbf{t} \in SO_\eta(3)$ (ii) $Y_n(\eta) = 1$, is given by $\xi \mapsto Y_n(\xi) = P_n(\xi \cdot \eta), \xi \in \Omega$. Furthermore, for $n \in \mathbb{N}_0$, the Legendre polynomial P_n of degree n is uniquely represented in the form

$$P_n(t) = \sum_{k=0}^{n} C_{\frac{n-k}{2}} (1 - t^2)^{\frac{n-k}{2}} t^k, \tag{3.311}$$

where, the coefficients $C_{\frac{n-k}{2}}$ are recursively determined by

$$C_{\frac{n-k-2}{2}} = -\frac{(n-k)^2}{(k+2)(k+1)} C_{\frac{n-k}{2}}, \tag{3.312}$$

$k = 0, \ldots, n-2$, with $C_{\frac{n}{2}}, C_{\frac{n-1}{2}}$ given by

$$C_{\frac{n}{2}} = \begin{cases} (-\frac{1}{4})^{\frac{n}{2}} \frac{n!}{((\frac{n}{2})!)^2} & , \quad n \text{ even} \\ 0 & , \quad n \text{ odd} \end{cases} \tag{3.313}$$

and

$$C_{\frac{n-1}{2}} = \begin{cases} 0 & , \quad n \text{ even} \\ (-\frac{1}{4})^{\frac{n-1}{2}} \frac{n!}{((\frac{n-1}{2})!)^2} & , \quad n \text{ odd.} \end{cases} \tag{3.314}$$

Example 3.59. Theorem 3.58 can be used to generate Legendre harmonics recursively. As an example, we discuss the case $n = 3$:

$$L_3(x) = \sum_{k=0}^{3} C_{\frac{3-k}{2}} (x_1^2 + x_2^2)^{\frac{3-k}{2}} x_3^k, \quad x = (x_1, x_2, x_3)^T. \tag{3.315}$$

Equivalently,

$$L_3(x) = C_0 (x_1^2 + x_2^2)^0 x_3^3 + \ C_1 (x_1^2 + x_2^2)^1 x_3^1. \tag{3.316}$$

According to our approach, we get

$$C_{\frac{3}{2}} = 0, \ C_{\frac{2}{2}} = C_1 = -\frac{3}{2}, \tag{3.317}$$

and by recursion

$$C_{\frac{1}{2}} = 0, \quad C_0 = 1. \tag{3.318}$$

Thus we obtain

$$L_3(x) = x_3^3 - \frac{3}{2}(x_1^2 + x_2^2) x_3. \tag{3.319}$$

This gives us the well-known result

$$L_3(x) = L_3(r\xi) = r^3 P_3(t), \ \xi = t\varepsilon^3 + \sqrt{1-t^2}(\varepsilon^1 \cos\varphi + \varepsilon^2 \sin\varphi),$$

where the Legendre polynomial P_3 of degree 3 is given by

$$P_3(t) = t^3 - \frac{3}{2}(1-t^2)t = \frac{5}{2}t^3 - \frac{3}{2}t. \tag{3.320}$$

3.8 Funk–Hecke Formula

An outstanding result in the theory of spherical harmonics is the *Funk–Hecke formula* (cf. H. Funk (1916), E. Hecke (1918), C. Müller (1966, 1998)).

Theorem 3.60. *Suppose that G is absolutely integrable on the interval $[-1,1]$, i.e., $G \in \mathrm{L}^1[-1,1]$. Then, for all $(\xi,\eta) \in \Omega \times \Omega$ and $n = 0,1,\ldots,$*

$$\int_\Omega G(\xi \cdot \zeta) P_n(\eta \cdot \zeta)\, d\omega(\zeta) = G^\wedge(n) P_n(\xi \cdot \eta),$$

where

$$G^\wedge(n) = (G, P_n)_{\mathrm{L}^2[-1,1]} = 2\pi \int_{-1}^{1} G(t) P_n(t)\, dt. \tag{3.321}$$

Proof. For brevity, we set

$$V_n(\xi,\eta) \;=\; \int_\Omega G(\xi \cdot \zeta) P_n(\eta \cdot \zeta)\, d\omega(\zeta). \tag{3.322}$$

Then, with any orthogonal matrix $\mathbf{t}$,

$$\begin{aligned} V_n(\mathbf{t}\xi, \mathbf{t}\eta) &= \int_\Omega G(\mathbf{t}\xi \cdot \zeta) P_n(\mathbf{t}\eta \cdot \zeta)\, d\omega(\zeta) \qquad (3.323)\\ &= (\det\ \mathbf{t}) \int_\Omega G(\mathbf{t}\xi \cdot \mathbf{t}\zeta) P_n(\mathbf{t}\eta \cdot \mathbf{t}\zeta)\, d\omega(\mathbf{t}\zeta). \end{aligned}$$

But this shows us that, on the one hand,

$$V_n(\mathbf{t}\xi, \mathbf{t}\eta) = (\det\ \mathbf{t})^2 V_n(\xi,\eta) \;=\; V_n(\xi,\eta). \tag{3.324}$$

On the other hand, according to (3.322) with ξ fixed, $V_n(\xi,\cdot)$ is a spherical harmonic of degree n which, by virtue of (3.324), is invariant under orthogonal transformations. Therefore, by Theorem 3.58, there exists a constant $G^\wedge(n)$ such that $V_n(\xi,\eta) \;=\; G^\wedge(n) P_n(\xi \cdot \eta)$. In order to determine $G^\wedge(n)$, we set $\xi = \eta$ and find

$$\begin{aligned} G^\wedge(n) &= \int_\Omega G(\xi \cdot \zeta) P_n(\xi \cdot \zeta)\, d\omega(\zeta) \\ &= \int_\Omega G(\varepsilon^3 \cdot \zeta) P_n(\varepsilon^3 \cdot \zeta)\, d\omega(\zeta). \qquad (3.325) \end{aligned}$$

It follows that $G^\wedge(n)$ admits the required representation. □

From Lemma 3.29 and Theorem 3.60, we get the following corollary.

Corollary 3.61. *Assume that* $G \in \mathrm{L}^1[-1,1]$. *Then, for every* $Y_n \in \mathrm{Harm}_n$,

$$\int_\Omega G(\xi \cdot \eta) Y_n(\eta) d\omega(\eta) = G^\wedge(n) Y_n(\xi), \quad \xi \in \Omega,$$

where $G^\wedge(n)$ *is given above, i.e.,*

$$G^\wedge(n) = (G, P_n)_{\mathrm{L}^2[-1,1]} = 2\pi \int_{-1}^{1} G(t) P_n(t)\, dt.$$

The Funk–Hecke formula establishes the close connection between the orthogonal invariance of the sphere and the addition theorem. The principle of the Funk–Hecke formula is that of the integral operator mapping F to the *'convolution of* G *and* F'. The kernel G depends only on the inner product $\xi \cdot \eta$, or equivalently on the distance between ξ and η. The spherical harmonics Y_n are the eigenfunctions of the integral operator corresponding to the eigenvalues $G^\wedge(n)$. Therefore, the Funk–Hecke formula greatly simplifies most manipulations with spherical harmonics.

Definition 3.62. The *Legendre transform* $G \mapsto (LT)(G), G \in \mathrm{L}^1[-1,1]$, is defined by

$$((LT)(G))(n) = G^\wedge(n) = (G, P_n)_{\mathrm{L}^2[-1,1]}.$$

The series

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G^\wedge(n) P_n$$

is called *Legendre expansion* of G (with *Legendre coefficients* $G^\wedge(n), n = 0, 1, \ldots$).

The restriction of the Legendre transform to $\mathrm{L}^2[-1,1]$ is a mapping from $\mathrm{L}^2[-1,1]$ into the space $\mathrm{L}^2_{LT}(\mathbb{N}_0)$ of square-summable sequences $\{A_n\}$:

$$\mathrm{L}^2_{LT}(\mathbb{N}_0) = \left\{ \{A_n\} \,\middle|\, \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} |A_n|^2 < \infty \right\}. \tag{3.326}$$

According to Parseval's identity, we have

$$(G, H)_{\mathrm{L}^2[-1,1]} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} G^\wedge(n) H^\wedge(n) \tag{3.327}$$

for all $G, H \in \mathrm{L}^2[-1,1]$. Moreover

$$\lim_{N \to \infty} \left\| G - \sum_{n=0}^{N} \frac{2n+1}{4\pi} G^\wedge(n) P_n \right\|_{\mathrm{L}^2[-1,1]} = 0 \tag{3.328}$$

for all $G \in \mathrm{L}^2[-1,1]$.

Lemma 3.63. *(Inverse transform) For $\{A_n\} \in \mathrm{L}^2_{LT}(\mathbb{N}_0)$ define the mapping*

$$(LT)^{-1} : \mathrm{L}^2_{LT}(\mathbb{N}_0) \to \mathrm{L}^2[-1,1]$$

by

$$(LT)^{-1}(\{A_n\}) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} A_n P_n.$$

Then

$$\begin{aligned} (LT)^{-1}(LT) &= I_{\mathrm{L}^2[-1,1]}, \\ (LT)(LT)^{-1} &= I_{\mathrm{L}^2_{LT}}(\mathbb{N}_0). \end{aligned}$$

Observe that, for $G \in \mathrm{L}^2[-1,1], G^\wedge(n) = O(n^{-2+\varepsilon}), \varepsilon > 0$, holds for $n \to \infty$. Furthermore, it is not difficult to show that, for $G \in \mathrm{C}^{(2)}[-1,1]$, $(L_t G)^\wedge(n) = -n(n+1)G^\wedge(n)$ so that $G^\wedge(n) = O(n^{-2(k+1)+\varepsilon}), \varepsilon > 0$, is valid for $G \in \mathrm{C}^{(2k)}[-1,1], k \in \mathbb{N}_0$.

Lemma 3.64. *If $G \in \mathrm{C}^{(\infty)}[-1,1]$, then*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (G^\wedge(n))^\alpha < \infty$$

holds for all $\alpha > 0$.

3.9 Eigenfunctions of the Beltrami Operator

Next, we discuss the role played by spherical harmonics as eigenfunctions of the Beltrami operator. From Lemma 3.24, we know that any spherical harmonic of order n is an infinitely often differentiable eigenfunction of the Beltrami operator corresponding to the eigenvalues $(\Delta^*)^\wedge(n) = -n(n+1)$, $n = 0, 1, \ldots$. Furthermore, the completeness of the system of spherical harmonics enables us to show that every eigenfunction $\xi \mapsto Y_n(\xi),\ \xi \in \Omega$, of the Beltrami operator corresponding to the eigenvalue $-n(n+1)$ defines a homogeneous harmonic polynomial (i.e., inner (solid spherical) harmonic)

$$x \mapsto H_n(x) = r^n Y_n(\xi),\ x = r\xi,\ r = |x|,\ \xi \in \Omega, \tag{3.329}$$

of degree n.

We start our considerations with the following lemma.

Lemma 3.65. *Assume that* $\lambda \neq -n(n+1),\ n = 0, 1, \ldots$ *. Let* $K \in \mathrm{C}^{(\infty)}(\Omega)$ *satisfy the differential equation*

$$(\Delta^*_\xi - \lambda)K(\xi) = 0, \quad \xi \in \Omega.$$

Then $K = 0$.

Proof. Let $\{Y_{n,j}\}$ be an $\mathrm{L}^2(\Omega)$-orthonormal system of spherical harmonics. According to the Second Green Surface Theorem, we get

$$\int_\Omega \Big(Y_{n,j}(\xi)\Delta^*_\xi K(\xi) - K(\xi)\Delta^*_\xi Y_{n,j}(\xi) \Big)\, d\omega(\xi) = 0. \tag{3.330}$$

Thus, it follows that

$$\big(\lambda + n(n+1)\big) \int_\Omega K(\xi) Y_{n,j}(\xi)\, d\omega(\xi) = 0 \tag{3.331}$$

for $n = 0, 1, ...;\ j = 1, ..., 2n+1$. Since λ is assumed to be different from $-n(n+1),\ n = 0, 1, ...$, this implies

$$K^\wedge(n,j) = \int_\Omega K(\xi) Y_{n,j}(\xi)\, d\omega(\xi) = 0 \tag{3.332}$$

for $n = 0, 1, ...,\ j = 1, ..., 2n+1$. The completeness property of spherical harmonics, therefore, shows that $K = 0$, as required. □

Lemma 3.66. *Let* $K \in \mathrm{C}^{(\infty)}(\Omega)$ *satisfy the differential equation*

$$\begin{aligned} \big(\Delta^*_\xi + n(n+1)\big)K(\xi) &= (\Delta^*_\xi - (\Delta^*)^\wedge(n))K(\xi) \\ &= 0, \quad \xi \in \Omega. \end{aligned}$$

Then $K \in \mathrm{Harm}_n$.

Proof. By the same arguments as given above, it follows that

$$(n(n+1) - k(k+1)) \int_\Omega K(\xi) Y_{k,j}(\xi)\, d\omega(\xi) = 0 \tag{3.333}$$

for $k = 0, 1, \ldots, j = 1, \ldots, 2k+1$. For all values of k different from n, we thus have $K^\wedge(k,j) = (K, Y_{k,j})_{\mathrm{L}^2(\Omega)} = 0$. But this means that K is a member of the span of $Y_{n,1}, \ldots, Y_{n,2n+1}$, as required. □

Summarizing our results, we therefore obtain the following theorem.

Theorem 3.67. *The spherical harmonics* $Y_n : \xi \mapsto Y_n(\xi)$ *of degree* $n = 0, 1, ...$ *are everywhere on the unit sphere* Ω *infinitely often differentiable eigenfunctions of the Beltrami operator* Δ^* *corresponding to the eigenvalues* $(\Delta^*)^\wedge(n) = -n(n+1),\ n = 0, 1, ...$ *. The 'inner (solid spherical) harmonics'* $x \mapsto H_n(x),\ x \in \mathbb{R}^3$, *with* $H_n(x) = r^n Y_n(\xi),\ x = r\xi, r = |x|$ *are polynomials in cartesian coordinates which satisfy the Laplace equation and are homogeneous of degree* n. *Conversely, every homogeneous harmonic polynomial of degree* n *(i.e., inner harmonic of degree* n*) restricted to the unit sphere* Ω *is a spherical harmonic of degree* n.

3.10 Irreducibility of Scalar Harmonics

For any function $F \in \mathrm{L}^2(\Omega)$, the transformation $\xi \mapsto \mathbf{t}\xi,\ \xi \in \Omega$ produces a change in the functional values of F. Let $F, G \in \mathrm{L}^2(\Omega), \mathbf{t} \in O(3)$. Observing the change of coordinates $\zeta = \mathbf{t}\xi$, we find

$$\int_\Omega F(\mathbf{t}\xi)G(\xi)\, d\omega(\xi) = (\det \mathbf{t})^2 \int_\Omega F(\zeta)G(\mathbf{t}^T\zeta)\, d\omega(\zeta) \tag{3.334}$$

(observe that $d\omega(\mathbf{t}\xi) = (\det \mathbf{t})\, d\omega(\xi)$). Thus, it follows that

$$(R_\mathbf{t} F, G)_{\mathrm{L}^2(\Omega)} = (F, R_{\mathbf{t}^T} G)_{\mathrm{L}^2(\Omega)}. \tag{3.335}$$

As mentioned before, (see Section 2.7), an invariant subspace may itself contain one or more invariant subspaces. If this is the case, $\mathcal{V}$ is said to be *reducible*. If there are no invariant subspaces of $\mathcal{V}$ (other than $\mathcal{V}$ itself), then $\mathcal{V}$ is said to be *irreducible*.

Next, we prove the following theorem.

Theorem 3.68. *The space* Harm_n *of spherical harmonics of order* n *is irreducible.*

Proof. Assume that there exists an invariant subspace $\mathcal{Y}$ of dimension $d(\mathcal{Y}) < d(\mathrm{Harm}_n) = 2n + 1$. Then, we would be able to show that the orthogonal complement $\mathcal{Y}^\perp$ of $\mathcal{Y}$ in Harm_n (with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$) is an invariant subspace (see Lemma 2.14), for

$$\begin{aligned}(F, F^\perp(\mathbf{t}\cdot))_{\mathrm{L}^2(\Omega)} &= \int_\Omega F(\xi)F^\perp(\mathbf{t}\xi) d\omega(\xi) \\ &= (\det \mathbf{t})^2 \int_\Omega F(\mathbf{t}^T\xi)F^\perp(\xi)\, d\omega(\xi) \\ &= 0\end{aligned} \tag{3.336}$$

holds for all $F \in \mathcal{Y}, F^\perp \in \mathcal{Y}^\perp$ (observe that $F(\mathbf{t}\cdot)$ is an element in $\mathcal{Y}$). But this means that $F^\perp(\mathbf{t}\cdot)$ is an element of $\mathcal{Y}^\perp$, i.e., $\mathcal{Y}^\perp$ is an invariant subspace. Now, because of the invariance of $\mathcal{Y}$ and $\mathcal{Y}^\perp$, G and $G^\perp$ being represented in terms of the $\mathrm{L}^2(\Omega)$-orthonormal system $\{Y_{n,j}\}$, respectively,

$$G = \sum_{j=1}^{d(\mathcal{Y})} Y_{n,j}(\varepsilon^3) Y_{n,j} \in \mathcal{Y}, \tag{3.337}$$

$$G^\perp = \sum_{j=d(\mathcal{Y})+1}^{d(\mathrm{Harm}_n)} Y_{n,j}(\varepsilon^3) Y_{n,j} \in \mathcal{Y}^\perp \tag{3.338}$$

satisfy $G(\mathbf{t}\cdot) = G$, $G^\perp(\mathbf{t}\cdot) = G^\perp$ for all $\mathbf{t} \in SO_{\varepsilon^3}(3)$. Moreover, $G, G^\perp$ do not vanish identically (note that there exist elements in $\mathcal{Y}, \mathcal{Y}^\perp$ different from zero at ε^3). Thus, not all values of $Y_{n,j}(\varepsilon^3), j = 1, \ldots, d(\mathcal{Y})$ or $j = d(\mathcal{Y}) + 1, \ldots, d(\mathrm{Harm}_n)$ are zero. From Theorem 3.58, therefore, there exists a constant $a \in \mathbb{R}\backslash\{0\}$ such that $G = aG^\perp$, in contradiction to our assumption. This proves Theorem 3.68. □

The irreducibility of Harm_n leads us to simple representations of spherical harmonics (cf., e.g., W. Freeden et al. (1998)).

Lemma 3.69. *Let Z_n be a member of Harm_n. Then there exist $2n+1$ orthogonal transformations $\mathbf{t}_1, \ldots, \mathbf{t}_{2n+1}$ such that any $Y_n \in \mathrm{Harm}_n$ can be represented with real numbers $a_1, \ldots, a_{2n+1}$ in the form*

$$Y_n = \sum_{j=1}^{2n+1} a_j Z_n(\mathbf{t}_j \cdot).$$

Proof. Consider the set

$$\mathcal{V} = \{Z_n(\mathbf{t}\cdot) \mid \mathbf{t} \in O(3)\}. \tag{3.339}$$

Clearly, there exist $\mathbf{t}_1, \ldots, \mathbf{t}_{2n+1} \in O(3)$ such that

$$\mathcal{V} = \mathrm{span}\{Z_n(\mathbf{t}_1\cdot), \ldots, Z_n(\mathbf{t}_{2n+1}\cdot)\}. \tag{3.340}$$

Therefore, $Z_n \in \mathcal{V}$ implies $Z_n(\mathbf{t}\cdot) \in \mathcal{V}$. From the irreducibility of Harm_n (cf. Theorem 3.68), it follows that $\mathcal{V}$ is an invariant non-void space of dimension $d(\mathcal{V}) = d(\mathrm{Harm}_n)$, and the $2n+1$ linearly independent spherical harmonics $Z_n(\mathbf{t}_1\cdot), \ldots, Z_n(\mathbf{t}_{2n+1}\cdot)$ form a basis. □

Lemma 3.70. *There exist $2n+1$ points $\eta_1, \ldots, \eta_{2n+1}$ of the unit sphere Ω such that any $Y_n \in \mathrm{Harm}_n$ can be represented with real numbers $a_1, \ldots, a_{2n+1}$ in the form*

$$Y_n(\xi) = \sum_{j=1}^{2n+1} a_j P_n(\eta_j \cdot \xi), \quad \xi \in \Omega.$$

Proof. From Lemma 3.69 we know that there exist $2n+1$ orthogonal transformations $\mathbf{t}_1, \ldots, \mathbf{t}_{2n+1}$ such that any $Y_n \in \mathrm{Harm}_n$ admits the form

$$Y_n(\xi) = \sum_{j=1}^{2n+1} a_j P_n(\varepsilon^3 \cdot \mathbf{t}_j \xi), \quad \xi \in \Omega. \tag{3.341}$$

But this means that Lemma 3.70 follows with $\eta_j = \mathbf{t}_j^T \varepsilon^3,\ j = 1, \ldots, 2n+1$. □

In connection with Maxwell's representation formula (3.214), it is possible to write the following lemma.

Lemma 3.71. *For every* $Y_n \in \mathrm{Harm}_n$

$$\frac{Y_n(\xi)}{|x|^{n+1}} = \frac{(-1)^n}{n!} \sum_{j=1}^{2n+1} a_j (\eta_j \cdot \nabla_x)^n \frac{1}{|x|}, \quad x = |x|\xi, \quad x \neq 0. \tag{3.342}$$

In other words, every function

$$H_{-n-1} : x \mapsto H_{-n-1}(x) = |x|^{-(n+1)} Y_n(\xi), \quad x \in \mathbb{R}^3 \backslash \{0\}, \tag{3.343}$$

(i.e., 'outer (solid spherical) harmonic' of degree n) may be regarded as the potential of a linear combination of multipoles with real axes (note that $H_{-n-1}|\Omega = H_n|\Omega = Y_n$). From (3.343), it follows that any outer (solid spherical) harmonic H_{-n-1} can be generated by an 'inner (solid spherical) harmonic' H_n as follows:

$$H_{-n-1}(x) = |x|^{-(2n+1)} H_n(x),\ x \in \mathbb{R}^3 \backslash \{0\}. \tag{3.344}$$

In Lemma 3.71, the system of points $\eta_1, \ldots, \eta_{2n+1}$ corresponds to a system of multipoles in Maxwell's interpretation of spherical harmonics.

Finally, we come to the definition of fundamental systems relative to Harm_n.

Definition 3.72. A system X_{2n+1} of $2n+1$ points $\eta_1, \ldots, \eta_{2n+1}$ of the unit sphere Ω is called a *fundamental system relative to* Harm_n if the matrix

$$\mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1}) = \begin{pmatrix} Y_{n,1}(\eta_1) & \ldots & Y_{n,1}(\eta_{2n+1}) \\ \vdots & & \vdots \\ Y_{n,2n+1}(\eta_1) & \ldots & Y_{n,2n+1}(\eta_{2n+1}) \end{pmatrix} \tag{3.345}$$

is of maximal rank $2n+1$.

From Lemma 3.70, we obtain the following.

Lemma 3.73. *There exists a system* $X_{2n+1} = \{\eta_1, \ldots, \eta_{2n+1}\} \subset \Omega$ *satisfying*

$$\det(\mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1})) \neq 0.$$

Proof. From Lemma 3.70, it follows that there exist $2n+1$ points $\eta_1, \ldots, \eta_{2n+1}$ such that the functions $P_n(\eta_1 \cdot), \ldots, P_n(\eta_{2n+1} \cdot)$ are linearly independent. Therefore the Gram matrix

$$\begin{aligned} &\left(\left(\frac{2n+1}{4\pi}\right)^2 \int_\Omega P_n(\eta_j \cdot \eta) P_n(\eta_k \cdot \eta)\, d\omega(\eta)\right)_{j,k=1,\ldots,2n+1} \\ &= \left(\frac{2n+1}{4\pi} P_n(\eta_j \cdot \eta_k)\right)_{j,k=1,\ldots,2n+1} \end{aligned} \tag{3.346}$$

is of maximal rank, and its determinant is positive. By virtue of the addition theorem, we obtain

$$\begin{aligned} &\left(\mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1})\right)^T \mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1}) \\ &= \left(\frac{2n+1}{4\pi} P_n(\eta_j \cdot \eta_k)\right)_{j,k=1,\ldots,2n+1} \\ &= \frac{2n+1}{4\pi} \mathbf{matr}_{X_{2n+1}}(P_n(\eta_1 \cdot), \ldots, P_n(\eta_{2n+1} \cdot)). \end{aligned} \tag{3.347}$$

Hence,

$$\begin{aligned} & \det\left(\mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1})\right)^2 \\ = \; & \det\left(\frac{2n+1}{4\pi} P_n(\eta_j \cdot \eta_k)\right)_{j,k=1,\ldots,2n+1} \\ > \; & 0. \end{aligned} \tag{3.348}$$

This shows that $\det\left(\mathbf{matr}_{X_{2n+1}}(Y_{n,1}, \ldots, Y_{n,2n+1})\right) \neq 0$, as required. □

3.11 Degree and Order Variances

(Geo-)sciences are much concerned with the space $L^2(\Omega)$ of square-integrable functions on the sphere Ω. The quantity $\|F\|^2_{L^2(\Omega)}$ is called the energy of the 'signal' $F \in L^2(\Omega)$. 'Signals' $F \in L^2(\Omega)$ possess Fourier transforms $F^\wedge(n,k)$ as defined before. From Parseval's identity, we have

$$\|F\|^2_{L^2(\Omega)} = (F, F)_{L^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(F^\wedge(n,k)\right)^2 \; . \tag{3.349}$$

As mentioned already, geoengineers often work more with the '*amplitude spectrum*'

$$\{F^{\wedge}(n,k)\}_{\substack{n=0,1,\ldots;\\ k=1,\ldots,2n+1}} \tag{3.350}$$

than with the 'original signal' $F \in \mathrm{L}^2(\Omega)$. The 'inverse Fourier transform'

$$F = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} F^{\wedge}(n,k) Y_{n,k} \tag{3.351}$$

allows the engineer to consider the potential F as a sum of 'wave functions' $Y_{n,k}$ of different frequencies. Engineers think of their measurements as operating on an 'input signal' F to produce an output signal G,

$$\Lambda F = G, \tag{3.352}$$

where Λ is an operator acting on $\mathrm{L}^2(\Omega)$. Fortunately, it is the case that large portions of interest can be well approximated by operators that are linear and rotation-invariant. If Λ is such an operator on $\mathrm{L}^2(\Omega)$, this means that

$$\Lambda Y_{n,k} = \Lambda^{\wedge}(n) Y_{n,k}, \qquad n = 0, 1, \ldots, k = 1, \ldots, 2n+1, \tag{3.353}$$

where the so-called *symbol* $\{\Lambda^{\wedge}(n)\}_{n\in\mathbb{N}_0}$ is a sequence of real values (independent of k). Thus, we have the fundamental fact that the spherical harmonics are the eigenfunctions of the operator Λ. Different linear operators Λ are characterized by their eigenvalues $\Lambda^{\wedge}(n)$. The 'amplitude spectrum' $\{G^{\wedge}(n,k)\}$ of the response of Λ is described in terms of the amplitude spectrum of functions (signals) by a simple multiplication by the 'transfer' $\Lambda^{\wedge}(n)$.

Physical devices do not transmit spherical harmonics of arbitrarily high frequency without severe attenuation. The 'transfer' $\Lambda^{\wedge}(n)$ usually tends to zero with increasing n. It follows from (3.353) that the amplitude spectra of the responses (observations) to functions (signals) of finite energy also are negligibly small beyond some finite frequency. Thus, both because of the frequency limiting nature of the used devices and because of the nature of the 'transmitted signals', the geoscientist is soon led to consider bandlimited functions. These are the functions $F \in \mathrm{L}^2(\Omega)$ whose 'amplitude spectra' $F^{\wedge}(n,k)$ vanish for all $n \geq N$ ($N \in \mathbb{N}$ fixed). In other words, each bandlimited function $F \in \mathrm{L}^2(\Omega)$ can be written as a finite Fourier transform

$$F = \sum_{n=0}^{N} \sum_{k=1}^{2n+1} F^{\wedge}(n,k) Y_{n,k}. \tag{3.354}$$

A function F of the form (3.354) is said to be *bandlimited with the band* N. In analogous manner, $F \in \mathrm{L}^2(\Omega)$ is said to be *locally supported (space–limited) with spacewidth* ρ around an axis $\eta \in \Omega$, if for some ρ the function

F vanishes on the set of all $\xi \in \Omega$ with $-1 \leq \xi \cdot \eta \leq \rho$. From (3.354), it readily follows that bandlimited functions are infinitely often differentiable everywhere. Moreover, it is clear that F is an analytic function. From the analyticity, it follows immediately that a non-trivial bandlimited function cannot vanish on any (non-degenerate) subset of Ω. The only function that is both bandlimited and space-limited is the trivial function. Now, in addition to bandlimited but non-space-limited functions, numerical analysis would like to deal with space-limited functions. But as we have seen, such a function (signal) of finite (space) support cannot be bandlimited, it must contain spherical harmonics of arbitrarily large frequencies. Thus, there is a dilemma of seeking functions that are somehow concentrated in both space *and* frequency (i.e., (angular) momentum) domain. There is a way of mathematically expressing the impossibility of simultaneous confinement of a function to space and (angular) momentum, namely the *uncertainty principle*. Its mathematical formulation is the content of Section 7.3.

Thus far, only a (deterministic) function model has been discussed. If a comparison of the 'output function' with the actual value were done, discrepancies would be observed. A mathematical description of these discrepancies has to follow the laws of probability theory in a stochastic model. Usually, the observations are looked upon as a function $\tilde{G}$ on the sphere Ω ('$\sim$' for stochastic), i.e.,

$$\tilde{G} = G + \tilde{\varepsilon}, \tag{3.355}$$

where $\tilde{\varepsilon}$ is the *observation noise*. Moreover, in our approach, e.g., motivated by information in satellite technology (see, e.g., R. Rummel (1997) and the references therein), we suppose the *covariance* to be known, i.e.,

$$\mathrm{Cov}\left[\tilde{G}(\xi), \tilde{G}(\eta)\right] \;=\; E\left[\tilde{\varepsilon}(\xi), \tilde{\varepsilon}(\eta)\right] = K(\xi, \eta), \qquad (\xi, \eta) \in \Omega \times \Omega,$$

where the 'covariance kernel' $K : \Omega \times \Omega \to \mathbb{R}$ is of the form

$$K(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} K^\wedge(n,k) Y_{n,k}(\xi) Y_{n,k}(\eta) \tag{3.356}$$

such that its 'symbol' $\{K^\wedge(n,k)\}$ satisfies the conditions

(i) $K^\wedge(n,k) \geq 0, \;\; n = 0, 1, \ldots, k = 1, \ldots, 2n+1,$

(ii) $\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \sup_{k=1,\ldots,2n+1} \left(K^\wedge(n,k)\right)^2 < \infty.$

It is noteworthy that this approach assumes that the first two statistical moments suffice for a complete description, that the error spectrum can be considered invariant over the measurement's period and that one realization in space (or mission time) is enough to deduce the stochastic characteristics. We do not discuss the details of this subject.

Using the fact that any 'output function' (more clearly the output signal, i.e., the observable) can be expanded into an orthogonal series in terms of spherical harmonics

$$\begin{aligned}\tilde{G} = \Lambda\tilde{F} &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \Lambda^{\wedge}(n,k)\tilde{F}^{\wedge}(n,k)Y_{n,k} \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \tilde{G}^{\wedge}(n,k)Y_{n,k} \end{aligned} \tag{3.357}$$

in the sense of $\|\cdot\|_{L^2(\Omega)}$, we get a spectral representation of the form

$$\tilde{G}^{\wedge}(n,k) = (\Lambda\tilde{F})^{\wedge}(n,k) = \Lambda^{\wedge}(n,k)\tilde{F}^{\wedge}(n,k). \tag{3.358}$$

Since this representation clearly distinguishes between the different degrees and orders, one is led to observe the root-mean-square power per spherical harmonic degree and order, respectively per degree, to characterize the signal.

Definition 3.74. Let G be of class $L^2(\Omega)$. Suppose that, for $n = 0, 1, \ldots$, $k = 1, \ldots, 2n+1$, $G^{\wedge}(n,k)$ are the corresponding orthogonal coefficients. Then, the *signal degree and order variances* of G are defined by

$$\begin{aligned}\mathrm{Var}_{n,k}(G) &= \int_{\Omega}\int_{\Omega} G(\xi)G(\eta)Y_{n,k}(\xi)Y_{n,k}(\eta)\, d\omega(\xi)\, d\omega(\eta) \\ &= \left(G^{\wedge}(n,k)\right)^2 . \end{aligned} \tag{3.359}$$

Correspondingly, for $n = 0, 1, \ldots$, the *signal degree variances* of G are defined by

$$\begin{aligned}\mathrm{Var}_{n}(G) &= \frac{2n+1}{4\pi}\int_{\Omega}\int_{\Omega} G(\xi)G(\eta)P_n(\xi\cdot\eta)\, d\omega(\xi)\, d\omega(\eta) \\ &= \sum_{k=1}^{2n+1}\left(G^{\wedge}(n,k)\right)^2 \\ &= \sum_{k=1}^{2n+1} \mathrm{Var}_{n,k}(G) . \end{aligned} \tag{3.360}$$

From Parseval's identity, we get

$$\|G\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} \mathrm{Var}_n(G) = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \mathrm{Var}_{n,k}(G), \tag{3.361}$$

connecting the signal degree and order variances as well as the signal degree variances with the ‘$L^2(\Omega)$-energy’ of the corresponding function.

In order to determine the variances in the case of the ‘output function’ $\tilde{G} = \Lambda \tilde{F}$, we can use the representation (3.358) and end up with

$$\mathrm{Var}_{n,k}\left(\Lambda \tilde{F}\right) = \left(\left(\Lambda \tilde{F}\right)^{\wedge}(n,k)\right)^2 \tag{3.362}$$

and

$$\mathrm{Var}_{n}\left(\Lambda \tilde{F}\right) = \sum_{k=1}^{2n+1} \left(\left(\Lambda \tilde{F}\right)^{\wedge}(n,k)\right)^2 . \tag{3.363}$$

The spectral approach to signal-to-noise thresholding is, in addition to the previously defined degree variances, based on analogous measures calculated from the a priori known covariance kernel of the noise.

Definition 3.75. Let $\{K^{\wedge}(n,k)\}$ be the symbol of a covariance kernel $K : \Omega \times \Omega \to \mathbb{R}$ (as defined above). Then the *degree and order error covariance* of K is defined by

$$\begin{aligned} \mathrm{Cov}_{n,k}(K) &= \int_\Omega \int_\Omega K(\xi,\eta) Y_{n,k}(\xi) Y_{n,k}(\eta)\, d\omega(\xi)\, d\omega(\eta) \\ &= K^{\wedge}(n,k), \quad n = 0,1,\ldots, k = 1,\ldots,2n+1 . \end{aligned} \tag{3.364}$$

For $n = 0, 1, \ldots$, the *spectral degree error covariance* of K is defined by

$$\begin{aligned} \mathrm{Cov}_{n}(K) &= \sum_{k=1}^{2n+1} \int_\Omega \int_\Omega K(\xi,\eta) Y_{n,k}(\xi) Y_{n,k}(\eta)\, d\omega(\xi)\, d\omega(\eta) \\ &= \sum_{k=1}^{2n+1} K^{\wedge}(n,k) \\ &= \sum_{k=1}^{2n+1} \mathrm{Cov}_{n,k}(K). \end{aligned} \tag{3.365}$$

This definition shows that the degree and order error covariance is just given by the orthogonal coefficient of the corresponding covariance kernel K.

In order to make the preceding considerations more concrete, we present two examples of spectral error covariances:

Bandlimited white noise: Suppose that for some $n_K \in \mathbb{N}_0$

$$K^\wedge(n,k) = K^\wedge(n) = \begin{cases} \frac{\sigma^2}{(n_K+1)^2} & , \quad n \le n_K, k = 1,\ldots,2n+1 \\ 0 & , \quad n > n_K, k = 1,\ldots,2n+1, \end{cases} \tag{3.366}$$

where $\tilde{\varepsilon}$ is assumed to be $\mathcal{N}(0,\sigma^2)$-distributed. The associated covariance kernel is isotropic and reads as follows:

$$K(\xi,\eta) = \frac{\sigma^2}{(n_K+1)^2} \sum_{n=0}^{n_K} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta) \ . \tag{3.367}$$

Apart from a multiplicative constant, this kernel can be understood as a truncated Dirac δ-function(al).

Non-bandlimited colored noise: Assume that $K : \Omega \times \Omega \to \mathbb{R}$ is given in such a way that

(i) $K^\wedge(n,k) = K^\wedge(n) > 0$ for an infinite number of pairs (n,k),

(ii) For $\varepsilon > 0$ and for some $\delta \in (1-\varepsilon, 1)$ the integral $\int_{-1}^{\delta} K(t)dt$ is sufficiently small,

(iii) $K(\xi,\xi)$ coincides with σ^2 for all $\xi \in \Omega$.

A concrete realization is given by the kernel

$$K(\xi,\eta) = \frac{\sigma^2}{\exp(-c)} \exp(-c(\xi \cdot \eta)), \tag{3.368}$$

where c is to be understood as the inverse spherical correlation length *(first degree Gauß–Markov model).*

With our definitions at hand, we are now in a position to compare the signal spectrum with that of the noise and thus can decide whether signal or noise is dominant. The next Table 3.3 clarifies the situation (cf. W. Freeden, T. Maier (2002)).

Table 3.3: Spectral signal to noise (hard) thresholding.

Signal and noise spectrum intersect at a *degree and order resolution set* characterized by the following relations:

(i) *Signal dominates noise*

$$\mathrm{Var}_{n,k}(\Lambda\tilde{F}) \ge \mathrm{Cov}_{n,k}(K), \quad n = 0,\ldots,m, \ k = 1,\ldots,2n+1,$$

(ii) *Noise dominates signal*

$$\mathrm{Var}_{n,k}(\Lambda\tilde{F}) < \mathrm{Cov}_{n,k}(K), \quad n = m+1, m+2,\ldots, \ k = 1,\ldots,2n+1 \ .$$

In order to obtain an estimated denoised version $\Lambda\hat{F}$ of the signal $\Lambda\tilde{F}$, the signal must somehow be filtered. Filtering is achieved by convolving a square-summable product kernel $L : \Omega \times \Omega \to \mathbb{R}$ with symbol $\{L^\wedge(n,k)\}$ against $\Lambda\tilde{F}$, i.e.,

$$\Lambda\hat{F} = \int_\Omega L(\cdot,\eta)\Lambda\tilde{F}(\eta)\ d\omega(\eta). \tag{3.369}$$

In spectral language, this reads

$$\begin{aligned} \Lambda\hat{F}(n,k) &= L^\wedge(n,k)\ \Lambda\tilde{F}(n,k), \\ n &= 0,\ldots,m,\ k = 1,\ldots,2n+1. \end{aligned} \tag{3.370}$$

Two important types of filters are well known:

(i) *Spectral thresholding.* This filtering technique is best represented by the filter equation

$$\Lambda\hat{F} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} I_{\mathcal{N}_{\mathrm{res}}}(n,k)L^\wedge(n,k)\left(\Lambda\tilde{F}\right)^\wedge(n,k)Y_{n,k}, \tag{3.371}$$

where I_m denotes the *indicator function of the set* $\{0,\ldots,m\}$.

This approach represents a 'keep or kill' filtering, where the signal dominated coefficients are filtered by a square-summable product kernel, while the noise dominated coefficients are set to zero. This thresholding can be thought of as a non-linear operator on the set of coefficients, resulting in a set of estimated coefficients. As a special realization of this filter, we mention the *ideal low-pass (Shannon) filter*, the kernel of which can best be characterized by its spectral properties:

$$L^\wedge(n,k) = L^\wedge(n) = \begin{cases} 1 & , \quad n = 0,\ldots,m, k = 1,\ldots,2n+1, \\ 0 & , \quad n = m+1, m+2,\ldots, k = 1,\ldots,2n+1. \end{cases} \tag{3.372}$$

In that case, all contributions corresponding to pairs with $n \le m$ are allowed to pass, whereas all other portions of the signal are completely eliminated.

(ii) *Wiener-Kolmogorov filtering.* In the spectral domain, this filter is given by

$$L^\wedge(n) = \frac{\mathrm{Var}_n(\Lambda\tilde{F})}{\mathrm{Var}_n(\Lambda\tilde{F}) + \mathrm{Cov}_n(K)}, \quad n = 0,1,\ldots\ . \tag{3.373}$$

Assuming complete independence of signal and noise, this filter produces an optimal weighting between signal and noise. Note that the Wiener-Kolmogorov filter bears a close resemblance to the Tikhonov kernel used for the regularization of ill-posed inverse problems.

3.12 Associated Legendre Polynomials

By a straightforward calculation, we obtain from the Laplace representation of the Legendre polynomials (see Lemma 3.41)

$$\begin{aligned} & (1-t^2)^{m/2} \left(\frac{d}{dt}\right)^m P_n(t) \\ & = \frac{(n+m)!}{n!} \frac{i^m}{\pi} \int_0^\pi (t + i\sqrt{1-t^2}\cos\varphi)^n \cos(m\varphi)\, d\varphi, \end{aligned} \tag{3.374}$$

$n = 0, 1, \ldots, m = 0, \ldots, n$.

Definition 3.76. For $n = 0, 1, \ldots, m = 0, \ldots, n$, $P_{n,m} : [-1,1] \to \mathbb{R}$ given by

$$P_{n,m}(t) = \frac{(n+m)!}{n!} \frac{i^m}{\pi} \int_0^\pi (t + i\sqrt{1-t^2}\cos\varphi)^n \cos(m\varphi)\, d\varphi \tag{3.375}$$

is called the *associated Legendre function of degree n and order m.*

Thus, it is trivial to see that

$$P_{n,m}(t) = (1-t^2)^{m/2} (\frac{d}{dt})^m P_n(t), \quad t \in [-1,1]. \tag{3.376}$$

Observe that, in the sense of (3.376), $P_{n,m}(t) = 0$ for $m > n$.

By use of the associated Legendre functions, we are immediately able to determine the expansion coefficients of the Fourier series of $(t+\sqrt{t^2-1}\cos\varphi)^n$, $t \in [-1,1]$,

$$(t + \sqrt{t^2-1}\cos\varphi)^n = \frac{a_0}{2} + \sum_{m=1}^n a_m \cos(m\varphi), \tag{3.377}$$

where

$$a_m = \frac{2}{\pi} \int_0^\pi (t + \sqrt{t^2-1}\cos\varphi)^n \cos(m\varphi)\, d\varphi. \tag{3.378}$$

In other words, from Definition 3.76, we obtain

$$a_m = \frac{(2n)!}{(n+m)!} i^m P_{n,m}(t). \tag{3.379}$$

Lemma 3.77. *For $n = 0, 1, \ldots, m = 0, \ldots, n$*

$$\begin{aligned} (t + \sqrt{t^2-1}\cos\varphi)^n &= P_{n,0}(t) \\ &+ (2n)! \sum_{m=1}^n \frac{i^m}{(n+m)!} P_{n,m}(t) \cos(m\varphi), \end{aligned}$$

$t \in [-1,1], \varphi \in [0, 2\pi)$.

Using the Rodriguez formula (see (3.182)), we are led to the identity

$$P_{n,m}(t) = \frac{1}{2^n n!}(1-t^2)^{m/2}\left(\frac{d}{dt}\right)^{n+m}(t^2-1)^n, \quad t\in[-1,1]. \tag{3.380}$$

Moreover,

$$P_{n,0}(t) = P_n(t), \quad t\in[-1,1]. \tag{3.381}$$

Furthermore, we have

$$P_{n,m}(t) = \frac{1}{2^n}(1-t^2)^{m/2}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{(2n-2k)!}{k!(n-k)!(n-2k)!}\left(\frac{d}{dt}\right)^m t^{n-2k}. \tag{3.382}$$

Note that the m-th derivative of the power t^{n-2k} reads

$$\left(\frac{d}{dt}\right)^m t^{n-2k} = \frac{(n-2k)!}{(n-m-2k)!}t^{n-m-2k}. \tag{3.383}$$

This leads us to the following explicit formula for any Legendre function.

Lemma 3.78. *For $n = 0, 1, \ldots,$ $m = 0, \ldots, n$ and $t \in [-1, 1]$*

$$P_{n,m}(t) = (1-t^2)^{m/2}\sum_{k=0}^{\lfloor\frac{n-m}{2}\rfloor}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-m-2k)!}t^{n-m-2k}. \tag{3.384}$$

We give some explicit representations of $P_{n,m}(t)$, $t \in [-1, 1]$:

$$P_{1,0}(t) = t, \tag{3.385}$$

$$P_{1,1}(t) = \sqrt{1-t^2}, \tag{3.386}$$

$$P_{2,0}(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{3.387}$$

$$P_{2,1}(t) = 3t\sqrt{1-t^2}, \tag{3.388}$$

$$P_{2,2}(t) = 3(1-t^2), \tag{3.389}$$

$$P_{3,0}(t) = \frac{5}{2}t^3 - \frac{3}{2}t, \tag{3.390}$$

$$P_{3,1}(t) = \sqrt{1-t^2}\left(\frac{15}{2}t^2 - \frac{3}{2}\right), \tag{3.391}$$

$$P_{3,2}(t) = 15t(1-t^2), \tag{3.392}$$

$$P_{3,3}(t) = 15(1-t^2)^{3/2}. \tag{3.393}$$

Some graphical impressions of Legendre functions can be found in Figs. 3.4 and 3.5.

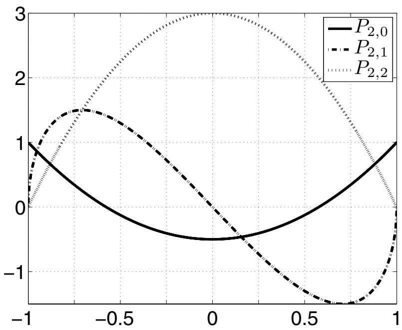


Fig. 3.4: Legendre functions $t \mapsto P_{2,m}(t)$, $t \in [-1, 1]$, $m = 0, 1, 2$.

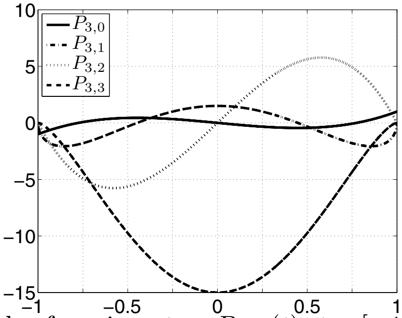


Fig. 3.5: Legendre functions $t \mapsto P_{3,m}(t)$, $t \in [-1, 1]$, $m = 0, \ldots, 3$.

Furthermore, from Theorem 3.288, it follows that

$$P_{n,m}(t) = (1 - t^2)^{m/2} \left(\frac{d}{dt}\right)^m n! \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-\frac{1}{4})^l \frac{(1 - t^2)^l t^{n-2l}}{(l!)^2 (n - 2l)!}. \tag{3.394}$$

By virtue of the product rule for differentiation, elementary calculations yield the following lemma.

Lemma 3.79. *For $n = 0, 1, \ldots, m = 0, \ldots, n,$ and $t \in [-1, 1]$*

$$P_{n,m}(t) = \left(\frac{1}{2}\right)^m (n + m)! \sum_{l=0}^{\lfloor \frac{n-m}{2} \rfloor} \left(-\frac{1}{4}\right)^l \frac{(1 - t^2)^{\frac{m}{2}+l} t^{n-m-2l}}{l!(n - m - 2l)!(l + m)!}. \tag{3.395}$$

From Lemma 3.79, we are able to deduce the following recursion relation.

Lemma 3.80. *For $n = 0, 1, \ldots,$ $m = 0, \ldots, n,$ and $t \in [-1, 1]$*

$$P_{n,m}(t) = \left(\frac{1}{2}\right)^m \frac{(n + m)!}{(n - m)!} \frac{1}{m!} \sum_{l=0}^{\lfloor \frac{n-m}{2} \rfloor} C_l (1 - t^2)^{\frac{m}{2}+l} t^{n-m-2l},$$

where the coefficients C_l, $l = 0, \ldots, \lfloor \frac{n-m}{2} \rfloor - 1$, *are recursively given by*

$$\begin{aligned}(2l+2)(2l+2m+2)C_{l+1} &+ (n-m-2l)(n-m-1-2l)C_l = 0,\\ C_0 &= 1.\end{aligned}$$

Proof. With $C_0 = 1$ and $C_{l+1} = -\frac{(n-m-2l)(n-m-1-2l)}{(2l+2)(2l+2m+2)} C_l$ we get

$$C_1 = -\frac{(n-m)(n-m-1)}{2 \cdot (2m+2)}, \tag{3.396}$$

$$C_2 = \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4 \cdot (2m+2)(2m+4)}. \tag{3.397}$$

By continuing this process, we find

$$\begin{aligned} C_l &= (-1)^l \frac{(n-m)(n-m-1) \cdot \ldots \cdot (n-m-2l+1)}{2 \cdot 4 \cdot \ldots (2l) \cdot (2m+2)(2m+4) \cdot \ldots \cdot (2m+2l)} \\ &= (-1)^l \frac{(n-m)!}{(n-m-2l)!} \frac{1}{2^l l!} \frac{1}{2^l (m+1)(m+2) \cdot \ldots \cdot (m+l)}. \end{aligned} \tag{3.398}$$

Consequently, it follows that

$$C_l = (-\frac{1}{4})^l \frac{(n-m)!}{l!(n-m-2l)!} \frac{m!}{(m+l)!}. \tag{3.399}$$

This is the desired result. □

Lemma 3.80 permits the following reformulation.

Lemma 3.81. *For* $n = 0, 1, \ldots, m = 0, \ldots, n$, *and* $t \in [-1, 1]$

$$P_{n,m}(t) = (\frac{1}{2})^m \frac{(n+m)!}{m!(n-m)!} (1-t^2)^{\frac{m}{2}} \sum_{k=0}^{n-m} C_{\frac{n-m-k}{2}} (1-t^2)^{\frac{n-m-k}{2}} t^k, \tag{3.400}$$

where

$$C_{\frac{n-m-k}{2}} = \begin{cases} (-\frac{1}{4})^{\frac{n-m-k}{2}} \frac{(n-m)!}{k!(\frac{n-m-k}{2})!} \frac{m!}{(\frac{n+m-k}{2})!} & , \ n-m-k \text{ even} \\ 0 & , \ n-m-k \text{ odd.} \end{cases} \tag{3.401}$$

Furthermore, the coefficients $C_{\frac{n-m-k}{2}}$ *are recursively determined by*

$$C_{\frac{n-m-k-2}{2}} = -\frac{(n-m-k)(n+m-k)}{(k+2)(k+1)} C_{\frac{n-m-k}{2}} \tag{3.402}$$

$k = 0, \ldots, n-m-2$, *with* $C_{\frac{n-m}{2}}, C_{\frac{n-m-1}{2}}$ *given by*

$$C_{\frac{n-m}{2}} = \begin{cases} (-\frac{1}{4})^{\frac{n-m}{2}} \frac{1}{(\frac{n-m}{2})!(\frac{n+m}{2})!} & , \ n-m \text{ even} \\ 0 & , \ n-m \text{ odd} \end{cases} \tag{3.403}$$

and

$$C_{\frac{n-m-1}{2}} = \begin{cases} 0 & , \quad n-m \text{ even} \\ (-\frac{1}{4})^{\frac{n-m-1}{2}} \frac{1}{(\frac{n-m-1}{2})!(\frac{n+m-1}{2})!} & , \quad n-m \text{ odd.} \end{cases} \tag{3.404}$$

In connection with $P_{n,m} = 0$ for $m > n$, the preceding lemma (Lemma 3.81) leads us to the following statement.

Lemma 3.82. *For $n = 0, 1, \ldots, m = 0, 1, \ldots,$ and $t \in [-1, 1]$*

$$P_{n,m}(t) = (1-t^2)^{\frac{m}{2}} \sum_{k=0}^{n} C^m_{\frac{n-m-k}{2}} (1-t^2)^{\frac{n-m-k}{2}} t^k, \tag{3.405}$$

where the generating coefficients $C^m_{\frac{n-m-k}{2}}$ of the associated Legendre polynomial of degree n and order m are given by

$$C^m_{\frac{n-m-k}{2}} = \begin{cases} (\frac{1}{2})^m \frac{(n+m)!}{(n-m)!m!} C_{\frac{n-m-k}{2}} & , \quad n-m-k \text{ even}, \ 0 \le k \le n-m \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{3.406}$$

The Legendre polynomials $P_n, n = 0, 1, \ldots,$ are known to satisfy the differential equation

$$\left(\frac{d}{dt}(1-t^2)\frac{d}{dt} + n(n+1) \right) P_n(t) = 0. \tag{3.407}$$

We differentiate this equation m-times with respect to t. In connection with the recursion formulas of the Legendre polynomial, we find

$$\left(\frac{d}{dt}\right)^m \left(-2t\frac{d}{dt}P_n(t) + (1-t^2)\left(\frac{d}{dt}\right)^2 P_n(t) \right) + n(n+1)\left(\frac{d}{dt}\right)^m P_n(t) = 0. \tag{3.408}$$

An elementary calculation starting from (3.408) guarantees the validity of the following differential equation.

Lemma 3.83. *The associated Legendre functions $P_{n,m}$, $n = 0, 1, \ldots,$ $m = 1, \ldots, n$, satisfy the differential equation*

$$(1-t^2)\left(\frac{d}{dt}\right)^2 P_{n,m}(t) - 2t\frac{d}{dt}P_{n,m}(t) + \left(n(n+1) - \frac{m^2}{1-t^2} \right) P_{n,m}(t) = 0.$$

Next, we use the differential equation (Lemma 3.83) to verify the orthogonality of $P_{n,m}$ and $P_{r,m}$ for degrees n, r with $r \neq n$. We notice that the equation

$$\begin{aligned} & P_{r,m}(t)\left(\frac{d}{dt}(1-t^2)\frac{d}{dt}P_{n,m}(t) + \left(n(n+1) - \frac{m^2}{1-t^2}\right)P_{n,m}(t)\right) \\ - \;& P_{n,m}(t)\left(\frac{d}{dt}(1-t^2)P_{r,m}(t) + \left(r(r+1) - \frac{m^2}{1-t^2}\right)P_{r,m}(t)\right) \\ = \;& 0 \end{aligned} \tag{3.409}$$

is equivalent to

$$\begin{aligned} & P_{r,m}(t)\frac{d}{dt}(1-t^2)\frac{d}{dt}P_{n,m}(t) - P_{n,m}(t)\frac{d}{dt}(1-t^2)\frac{d}{dt}P_{r,m}(t) \\ + \;& P_{n,m}(t)P_{r,m}(t)\left(n(n+1) - r(r+1)\right) = 0. \end{aligned} \tag{3.410}$$

Hence, we find

$$\begin{aligned} & \frac{d}{dt}\left((1-t^2)\left(P_{r,m}(t)\frac{d}{dt}P_{n,m}(t) - P_{n,m}(t)\frac{d}{dt}P_{r,m}(t)\right)\right) \\ + \;& P_{n,m}(t)P_{r,m}(t)\left(n(n+1) - r(r+1)\right) = 0. \end{aligned} \tag{3.411}$$

Integration with respect to t over the interval $[-1, 1]$ yields

$$(n-r)(n+r+1)\int_{-1}^{1} P_{n,m}(t)P_{r,m}(t)dt = 0. \tag{3.412}$$

As $n+r+1 > 0$ and $n \neq r$, this leads to the following orthogonality relation.

Lemma 3.84. *For all n, r with $n \neq r$*

$$\int_{-1}^{1} P_{n,m}(t)P_{r,m}(t)\; dt = 0.$$

It is not difficult to see that

$$\begin{aligned} P_{n,m+1}(t) &= (1-t^2)^{\frac{m}{2}+\frac{1}{2}}\left(\frac{d}{dt}\right)^{m+1} P_n(t) \qquad (3.413) \\ &= (1-t^2)^{\frac{m+1}{2}}\frac{d}{dt}\left((1-t^2)^{-\frac{m}{2}}P_{n,m}(t)\right) \\ &= (1-t^2)^{1/2}\frac{d}{dt}P_{n,m}(t) + mt(1-t^2)^{-1/2}P_{n,m}(t). \end{aligned}$$

Thus we arrive at the following *recurrence formula*

$$P_{n,m+1}(t) = (1-t^2)^{1/2}\frac{d}{dt}P_{n,m}(t) + mt(1-t^2)^{-1/2}P_{n,m}(t). \tag{3.414}$$

There is a large palette of three-term recursion relations, including

$$(n-m+1)P_{n+1,m}(t) - (2n+1)tP_{n,m}(t) + (n+m)P_{n-1,m}(t) = 0, \quad (3.415)$$

$$\begin{aligned} &(1-t^2)^{1/2}P_{n,m+1}(t) - 2mtP_{n,m}(t) \\ &+ \quad (n+m)(n-m+1)(1-t^2)^{1/2}P_{n,m-1}(t) = 0, \end{aligned} \quad (3.416)$$

$$P_{n+1,m}(t) - tP_{n,m}(t) - (n+m)(1-t^2)^{1/2}P_{n,m-1}(t) = 0, \quad (3.417)$$

$$tP_{n,m}(t) - P_{n-1,m}(t) - (n-m+1)(1-t^2)^{1/2}P_{n,m-1} = 0, \quad (3.418)$$

$$(n-m+1)P_{n+1,m}(t) + (1-t^2)^{1/2}P_{n,m+1}(t) - (n+m+1)tP_{n,m}(t) = 0, \quad (3.419)$$

$$(n-m)tP_{n,m}(t) - (n+m)P_{n-1,m}(t) + (1-t^2)^{1/2}P_{n,m+1}(t) = 0. \quad (3.420)$$

The derivative of $P_{n,m}(t)$ is given by any of the equivalent formulas

$$\begin{aligned} (1-t^2)\frac{dP_{n,m}(t)}{dt} &= (1-t^2)^{1/2}P_{n,m+1}(t) - mtP_{n,m}(t) \quad (3.421) \\ &= mtP_{n,m}(t) - (n+m)(n-m+1)(1-t^2)^{1/2}P_{n,m-1}(t) \\ &= (n+1)tP_{n,m}(t) - (n-m+1)P_{n+1,m}(t) \\ &= (n+m)P_{n-1,m}(t) - ntP_{n,m}(t). \end{aligned}$$

The effect of a change in the sign of the order or the argument is

$$P_{n,-m}(t) = (-1)^m \frac{(n-m)!}{(n+m)!} P_{n,m}(t), \quad (3.422)$$

$$P_{n,m}(-t) = (-1)^{n+m} P_{n,m}(t). \quad (3.423)$$

The identity (3.414) and the differential equation of the associated Legendre function can be used to verify the following result.

Lemma 3.85. *For* $n = 0, 1, \ldots, m = 1, \ldots, n$

$$\int_{-1}^{1} (P_{n,m}(t))^2 \ dt = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}. \quad (3.424)$$

Proof. By virtue of the recurrence relation (3.414), we obtain

$$\begin{aligned} \int_{-1}^{1} (P_{n,m+1}(t))^2 \ dt &= \int_{-1}^{1} (1-t^2) \left(\frac{d}{dt} P_{n,m}(t) \right)^2 \ dt \quad (3.425) \\ &+ \int_{-1}^{1} m^2 t^2 \frac{1}{1-t^2} (P_{n,m}(t))^2 \ dt \\ &+ 2m \int_{-1}^{1} tP_{n,m}(t) \left(\frac{d}{dt} P_{n,m}(t) \right) \ dt. \end{aligned}$$

Integration by parts yields

$$\begin{aligned}\int_{-1}^{1}(P_{n,m+1}(t))^2\ dt \ &= -\int_{-1}^{1} P_{n,m}(t)\frac{d}{dt}\left\{(1-t^2)\frac{dP_{n,m}}{dt}(t)\right\}\ dt \\ &\quad + P_{n,m}(t)(1-t^2)\frac{dP_{n,m}}{dt}(t)\Big|_{-1}^{1} \\ &\quad +m\int_{-1}^{1} tP_{n,m}(t)\frac{dP_{n,m}}{dt}(t)\ dt \\ &\quad +m\ t(P_{n,m}(t))^2\Big|_{-1}^{1} - m\int_{-1}^{1}\left(\frac{d}{dt}(tP_{n,m}(t))\right)P_{n,m}(t)\ dt \\ &\quad +\int_{-1}^{1}\frac{m^2t^2}{1-t^2}(P_{n,m}(t))^2\ dt.\end{aligned} \tag{3.426}$$

The boundary terms vanish, because $(1-t^2)\Big|_{-1}^{1} = 0$ and $P_{n,m}(\pm 1) = 0$. Hence, we find

$$\begin{aligned}\int_{-1}^{1}(P_{n,m+1}(t))^2\ dt \ &= \ -\int_{-1}^{1} P_{n,m}(t)\left(\frac{d}{dt}(1-t^2)\frac{d}{dt}P_{n,m}(t)\right)\ dt \\ &\quad -m\int_{-1}^{1}(P_{n,m}(t))^2\ dt + \int_{-1}^{1}\frac{m^2t^2}{1-t^2}(P_{n,m}(t))^2\ dt.\end{aligned} \tag{3.427}$$

Consequently we get, in connection with Lemma 3.83,

$$\begin{aligned}&\int_{-1}^{1}(P_{n,m+1}(t))^2\ dt \\ &= \int_{-1}^{1}(P_{n,m}(t))^2\left(n(n+1) - \frac{m^2}{1-t^2} - m + \frac{m^2t^2}{1-t^2}\right)\ dt \\ &= \int_{-1}^{1}(P_{n,m}(t))^2\left(n(n+1) - m(m+1)\right)\ dt.\end{aligned} \tag{3.428}$$

This shows us that

$$\int_{-1}^{1}(P_{n,m+1}(t))^2\ dt = (n-m)(n+m+1)\int_{-1}^{1}(P_{n,m}(t))^2\ dt. \tag{3.429}$$

Once again, note that P_n can be written as $P_{n,0}$, extending the definition of the associated Legendre functions to the case $n = 0, 1, \ldots,\ m = 0, 1, \ldots, n$.

Repeated application of (3.429), therefore, yields

$$\int_{-1}^{1}(P_{n,m}(t))^2\ dt = \frac{n!(n+m)!}{(n-m)!n!}\int_{-1}^{1}(P_n(t))^2\ dt = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}. \tag{3.430}$$

□

Next, we come to the following statement.

Theorem 3.86. *For every fixed* $m = 1, 2, \ldots$, *the system*

$$\left\{ \left(\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \right)^{1/2} P_{n,m} \right\}_{n=m,m+1,\ldots}$$

is a complete orthonormal system in $\mathrm{L}^2[-1,1]$.

Proof. The orthonormality immediately follows from the aforementioned results. To see that the system is complete, let $G \in \mathrm{L}^2[-1,1]$ satisfy

$$\int_{1}^{1} P_{n,m}(t) G(t)\, dt = 0, \ n = m, m+1, \ldots . \tag{3.431}$$

Thus it follows

$$\begin{aligned} 0 &= \int_{-1}^{1} P_{n,m}(t) G(t)\ dt \qquad (3.432) \\ &= \int_{-1}^{1} (1-t^2)^{m/2} \left(\frac{d^m}{dt^m} P_n(t) \right) G(t)\ dt \\ &= \int_{-1}^{1} \left(\frac{d^m}{dt^m} P_n(t) \right) \left((1-t^2)^{m/2} G(t) \right)\ dt \end{aligned}$$

for all $n = m, m+1, \ldots$. Since the system $\left\{ \frac{d^m}{dt^m} P_n \right\}_{n=m,m+1,\ldots}$ is dense in $\mathrm{L}^2[-1,1]$, it follows $(1-t^2)^{m/2} G(t) = 0$, $t \in [-1,1]$, so that $G = 0$ in $\mathrm{L}^2[-1,1]$–sense. □

Finally, we mention that, for $n, l = 0, 1, \ldots$ and $m = 0, \ldots, n$, $k = 0, \ldots, l$, we have

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} P_{n,m}(\cos\vartheta) P_{l,k}(\cos\vartheta) \cos(m\varphi) \cos(k\varphi) \sin\vartheta\ d\vartheta\ d\varphi \qquad (3.433) \\ &= \int_0^{2\pi} \cos(m\varphi)\cos(k\varphi)\ d\varphi \int_0^{\pi} P_{n,m}(\cos\vartheta) P_{l,k}(\cos\vartheta) \sin\vartheta\ d\vartheta \\ &= \delta_{km}\delta_{ln} \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!}, \end{aligned}$$

where, for $m, k \ \varepsilon\ \mathbb{N}$, we have observed the identity

$$\int_0^{2\pi} \cos(m\varphi)\cos(k\varphi)\ d\varphi = \pi\delta_{km}. \tag{3.434}$$

In the same way, we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} P_{n,m}(\cos\vartheta)P_{l,k}(\cos\vartheta)\sin(m\varphi)\sin(l\varphi)\sin\vartheta\, d\vartheta\, d\varphi \\
&= \int_0^{2\pi}\sin(m\varphi)\sin(k\varphi)\,d\varphi \int_0^{\pi} P_{n,m}(\cos\vartheta)P_{l,k}(\cos\vartheta)\sin\vartheta d\vartheta \\
&= \delta_{km}\delta_{ln}\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}. \qquad (3.435)
\end{aligned}$$

3.13 Associated Legendre (Spherical) Harmonics

The functions G, H defined by

$$G : t \mapsto G(t) = P_{n,j}(t), \qquad t \in (-1,1), \qquad (3.436)$$

$$H : \varphi \mapsto H(\varphi) = \begin{cases} \cos(j\varphi) \\ \sin(j\varphi) \end{cases}, \ \varphi \in [0,2\pi), \qquad (3.437)$$

respectively, satisfy the differential equations

$$(1-t^2)G''(t) - 2tG'(t) + \left(n(n+1) - \frac{j^2}{1-t^2}\right)G(t) = 0, \quad (3.438)$$

$$H''(\varphi) + j^2 H(\varphi) = 0. \quad (3.439)$$

Therefore, the functions $L_{n,1},\ldots,L_{n,2n+1} \in \mathrm{C}^{(\infty)}(\Omega)$ given by

$$L_{n,(n+1)+j}(\xi) = \begin{cases} P_{n,|j|}(t)\cos(j\varphi) & , \ j = -n,\ldots,0 \\ P_{n,j}(t)\ \ \sin(j\varphi) & , \ j = \ \ 1,\ldots,n \end{cases} \qquad (3.440)$$

satisfy the differential equation

$$(\Delta^*_\xi + n(n+1))L_{n,(n+1)+j}(\xi) = 0, \quad \xi \in \Omega, \qquad (3.441)$$

$j = -n,\ldots,n$ (note that, as always, $\xi = \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2) + t\varepsilon^3$). In addition, the functions $L_{n,(n+1)+j} \in \mathrm{C}^{(\infty)}(\mathbb{R}^3), j = -n,\ldots,n$, given by

$$L_{n,(n+1)+j}(x) = |x|^n L_{n,(n+1)+j}(\xi), \ x = |x|\xi, \xi \in \Omega, \qquad (3.442)$$

form homogeneous harmonic polynomials of degree n in $\mathbb{R}^3$, i.e., they are members of $\mathrm{Harm}_n(\mathbb{R}^3)$.

Definition 3.87. Let $L_{n,(n+1)+j}, j = -n,\ldots,n$, be defined by (3.440). Then $L_{n,(n+1)+j}$ is called *associated Legendre (spherical) harmonic of degree n and order j*. Correspondingly, the system $\{L^*_{n,(n+1)+j}\}_{j=-n,\ldots,n}$ given by

$$L^*_{n,(n+1)+j} = C_{n,j}L_{n,(n+1)+j}, \quad j = -n,\ldots,n, \qquad (3.443)$$

with

$$C_{n,j} = \sqrt{(2-\delta_{j0})\frac{2n+1}{4\pi}\frac{(n-|j|)!}{(n+|j|)!}} \tag{3.444}$$

is called *(fully)* $\mathrm{L}^2(\Omega)$*-orthonormal system of associated Legendre (spherical) harmonics in* $\mathrm{Harm}_n(\Omega)$.

In terms of associated Legendre harmonics, the addition theorem (cf. Theorem 3.26) allows us the following reformulation (that is standard in all geosciences).

Remark 3.88. (Addition theorem for the system $\{L^*_{n,r}\}$ of *associated Legendre (spherical) harmonics*) Suppose that $\xi, \eta \in \Omega$ are given by

$$\begin{aligned} \xi &= \sqrt{1-t_\xi^2}\cos\varphi_\xi\ \varepsilon^1 + \sqrt{1-t_\xi^2}\sin\varphi_\xi\ \varepsilon^2 + t_\xi\ \varepsilon^3 \\ & \quad -1 \le t_\xi \le 1,\ 0 \le \varphi_\xi < 2\pi, \end{aligned} \tag{3.445}$$

$$\begin{aligned} \eta &= \sqrt{1-t_\eta^2}\cos\varphi_\eta\ \varepsilon^1 + \sqrt{1-t_\eta^2}\sin\varphi_\eta\ \varepsilon^2 + t_\eta \varepsilon^3, \\ & \quad -1 \le t_\eta \le 1,\ 0 \le \varphi_\eta < 2\pi. \end{aligned} \tag{3.446}$$

respectively, so that

$$\begin{aligned} \xi\cdot\eta &= t_\xi t_\eta + \sqrt{1-t_\xi^2}\sqrt{1-t_\eta^2}(\cos\varphi_\xi\cos\varphi_\eta + \sin\varphi_\xi\sin\varphi_\eta) \\ &= t_\xi t_\eta + \sqrt{1-t_\xi^2}\sqrt{1-t_\eta^2}\cos(\varphi_\xi - \varphi_\eta). \end{aligned} \tag{3.447}$$

Then

$$\begin{aligned} & \frac{2n+1}{4\pi} P_n\left(t_\xi t_\eta + \sqrt{1-t_\xi^2}\sqrt{1-t_\eta^2}\cos(\varphi_\xi - \varphi_\eta)\right) \\ &= \frac{1}{4\pi} P_n(t_\xi) P_n(t_\eta) \\ &+ \frac{2n+1}{2\pi} \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{n,m}(t_\xi) P_{n,m}(t_\eta) \cos(m(\varphi_\xi - \varphi_\eta)) \\ &= \sum_{j=-n}^{n} L^*_{n,(n+1)+j}(\xi) L^*_{n,(n+1)+j}(\eta) \\ &= \sum_{r=1}^{2n+1} L^*_{n,r}(\xi) L^*_{n,r}(\eta). \end{aligned} \tag{3.448}$$

Equivalently, we have

$$\begin{aligned}
&\frac{2n+1}{4\pi} P_n(t_\xi t_\eta + \sqrt{1-t_\xi^2}\sqrt{1-t_\eta^2}\cos(\varphi_\xi - \varphi_\eta)) \qquad (3.449)\\
&= \sum_{r=1}^{n+1} C_{n,(n+1)-r} P_{n,(n+1)-r}(t_\xi) P_{n,(n+1)-r}(t_\eta)\\
&\qquad \times \ (\cos(((n+1)-r)\varphi_\xi))(\cos(((n+1)-r)\varphi_\eta))\\
&+ \sum_{r=n+2}^{2n+1} C_{n,(n+1)-r} P_{n,r-(n+1)}(t_\xi) P_{n,r-(n+1)}(t_\eta)\\
&\qquad \times \ (\sin((r-(n+1))\varphi_\xi)\sin((r-(n+1))\varphi_\eta)).
\end{aligned}$$

In other words, summing up all spherical harmonics involving associated Legendre functions via the addition theorem leads (apart from a multiplicative factor) to the orthogonal invariant Legendre (kernel) functions.

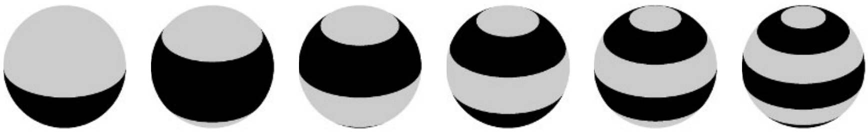


Fig. 3.6: Zonal ($j = 0$) spherical harmonics $L^*_{n,(n+1)+j}$ of different degrees $1, \ldots, 6$ (from left to right). The black and white color indicate the zones of different signs of the function, respectively.

The geometrical interpretation of the spherical harmonics defined via associated Legendre functions is particularly useful. The harmonics with $j = 0$ - that is, the Legendre polynomials - are polynomials of degree n. They have n zeros. These n zeros are all real, different, and situated in the interval $(-1, 1)$. In consequence, the spherical harmonics with $j = 0$ change their sign n times in this interval; furthermore, they do not depend on the variable φ. Since they divide the sphere into zones, they are also called *zonal harmonics* (see Fig. 3.6).

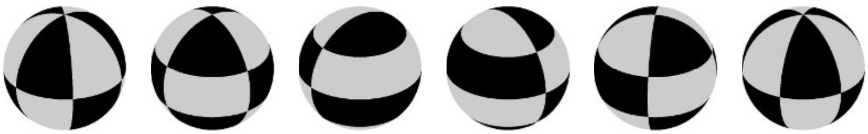


Fig. 3.7: Tesseral ($j \neq \pm n$) spherical harmonics $L^*_{n,(n+1)+j}$ of degree $n = 4$, i.e., $L^*_{4,2}$, $L^*_{4,3}$, $L^*_{4,4}$, $L^*_{4,6}$, $L^*_{4,7}$, $L^*_{4,8}$ (from left to right). The black and white colors indicate the zones of different signs of the function, respectively.

The associated Legendre functions $\vartheta \mapsto P_{n,|j|}(\cos\vartheta)$, $\vartheta \in [0,\pi]$, change their sign $n-|j|$ times in the interval $[0,\pi)$. The trigonometric functions $\varphi \mapsto \cos(j\varphi), j = -n,\ldots,0$, have $2|j|$ zeros in the interval $[0,2\pi)$, the functions $\varphi \mapsto \sin(j\varphi), j = 1,\ldots,n$, have $2j$ zeros in the interval $[0,2\pi)$. Consequently, the geometrical representation of the harmonics for the case $|j| \neq n$ is similar to that of Fig. 3.7. They divide the sphere into compartments in which they are alternately positive and negative, and are called *tesseral harmonics*. 'Tesseral' means a square or rectangle. In particular, for $|j| = n$, they degenerate into functions that divide the sphere into positive and negative sectors, in which case they are called *sectorial harmonics*, see Fig. 3.8.

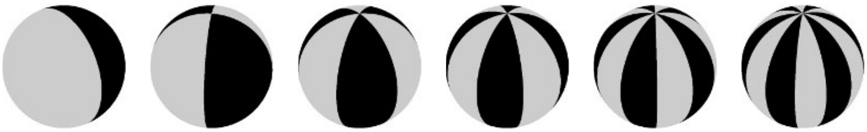


Fig. 3.8: Sectorial ($j = \pm n$) spherical harmonics $L^*_{n,(n+1)+j}$ of different degrees: $L^*_{1,3}$, $L^*_{2,5}$, $L^*_{3,7}$, $L^*_{4,9}$, $L^*_{5,11}$, $L^*_{6,13}$ (from left to right). The black and white colors indicate the zones of different signs of the function, respectively.

Next, we are interested in describing angular derivatives of associated Legendre (spherical) harmonics. For that purpose, we start with the following characterization of the operators ∇^*, L^* in terms of the polar coordinates (2.94).

$$\begin{aligned}\sqrt{1-(\xi\cdot\varepsilon^3)^2}\,\nabla^*_\xi &= \left(-\sin\varphi\varepsilon^1+\cos\varphi\varepsilon^2\right)\frac{\partial}{\partial\varphi} \\ &+ (1-t^2)\left(-t\cos\varphi\varepsilon^1 - t\sin\varphi\varepsilon^2+\sqrt{1-t^2}\varepsilon^3\right)\frac{\partial}{\partial t}\end{aligned} \tag{3.450}$$

and

$$\begin{aligned}\sqrt{1-(\xi\cdot\varepsilon^3)^2}\,\mathrm{L}^*_\xi &= (1-t^2)\left(\sin\varphi\varepsilon^1-\cos\varphi\varepsilon^2\right)\frac{\partial}{\partial t} \\ &+ \left(-t\cos\varphi\varepsilon^1 - t\sin\varphi\varepsilon^2+\sqrt{1-t^2}\varepsilon^3\right)\frac{\partial}{\partial\varphi}.\end{aligned} \tag{3.451}$$

We want to derive explicit representations of both derivatives $\sqrt{1-(\xi\cdot\varepsilon^3)^2}\ \nabla^*_\xi L^*_{n,(n+1)+j}(\xi)$ and $\sqrt{1-(\xi\cdot\varepsilon^3)^2}\ \mathrm{L}^*_\xi L^*_{n,(n+1)+j}(\xi)$, $\xi\in\Omega$, (see, e.g., D. Michel (2007)).

For that purpose, we have to calculate the *angular derivatives* $\frac{\partial}{\partial\varphi}\,L^*_{n,(n+1)+j}$ (*derivative of the latitude*) and $(1-t^2)\frac{\partial}{\partial t}L^*_{n,(n+1)+j}$, (*derivative of the longitude*), respectively.

Lemma 3.89. *For* $n = 1, 2, \ldots, j = -n, \ldots, n$

$$\frac{\partial}{\partial \varphi} L^*_{n,(n+1)+j}(\varphi, t) = j L^*_{n,(n+1)-j}(\varphi, t). \tag{3.452}$$

Proof. The application of the '*operator of the latitude*' $\frac{\partial}{\partial\varphi}$ yields

$$\begin{aligned}
\frac{\partial}{\partial \varphi} L^*_{n,(n+1)+j}(\varphi, t) &= \frac{\partial}{\partial \varphi} C_{n,j} P_{n,|j|}(t) \begin{cases} \cos(j\varphi), & j = -n, \ldots, 0 \\ \sin(j\varphi), & j = \ \ 1, \ldots, n \end{cases}, \\
&= C_{n,j} P_{n,|j|}(t) \begin{cases} -j\sin(j\varphi), & j = -n, \ldots, 0 \\ j\cos(j\varphi), & j = \ \ 1, \ldots, n \end{cases} \\
&= j C_{n,-j} P_{n,|j|}(t) \begin{cases} \sin(-j\varphi), & j = -n, \ldots, 1 \\ \cos(-j\varphi), & j = \ \ 0, \ldots, n \end{cases} \\
&= j L^*_{n,(n+1)-j}(\varphi, t),
\end{aligned} \tag{3.453}$$

where $C_{n,j}$ is given by (3.444)

$$C_{n,j} = \sqrt{(2 - \delta_{|j|0}) \frac{2n+1}{4\pi} \frac{(n-|j|)!}{(n+|j|)!}}. \tag{3.454}$$

(such that $C_{n,j} = C_{n,-j}$). This proves Lemma 3.89 □

Lemma 3.89 tells us that $\frac{\partial}{\partial\varphi}$ applied to spherical harmonics admits a simple structure. But it also informs us that the scalar (Legendre) spherical harmonics are not eigenfunctions of this operator.

Next, we turn to the '*operator of the longitude*' $(1-t^2)\frac{\partial}{\partial t}$.

Lemma 3.90. *For* $n = 1, 2, \ldots, j = -n, \ldots, n,$

$$\begin{aligned}
&(1-t^2) \frac{\partial}{\partial t} L^*_{n,(n+1)+j}(\varphi, t) \\
&\quad = -n D_{n+1,j} L^*_{n+1,(n+1)+j}(\varphi, t) + (n+1) D_{n,j} L^*_{n-1,(n+1)+j}(\varphi, t),
\end{aligned}$$

where

$$D_{n,j} = \sqrt{\frac{(n-|j|)(n+|j|)}{(2n-1)(2n+1)}}. \tag{3.455}$$

Proof. The application of the operator $(1-t^2)\frac{\partial}{\partial t}$ yields

$$
\begin{aligned}
&(1-t^2)\frac{\partial}{\partial t}L^*_{n,(n+1)+j}(\varphi,t) \\
&= C_{n,j}(1-t^2)\frac{d}{dt}P_{n,|j|}(t)\begin{cases}\cos(j\varphi), & j=-n,\ldots,0\\ \sin(j\varphi), & j=\ 1,\ldots,n\end{cases} \\
&= C_{n,j}\left(-\frac{n(n+1)-|j|}{2n+1}P_{n+1,|j|}(t)+\frac{(n+|j|)(n+1)}{2n+1}P_{n-1,|j|}(t)\right) \\
&\qquad\qquad \times\begin{cases}\cos(j\varphi), j=-n,\ldots,0\\ \sin(j\varphi), j=1,\ldots,n\end{cases} \\
&= -\frac{n(n+1)-|j|}{2n+1}\frac{C_{n,j}}{C_{n+1,j}}L^*_{n+1,(n+1)+j}(\varphi,t) \\
&\quad + \frac{(n+|j|)(n+1)}{2n+1}\frac{C_{n,j}}{C_{n-1,j}}L^*_{n-1,(n+1)+j}(\varphi,t).
\end{aligned} \tag{3.456}
$$

Therefore, we obtain

$$
\begin{aligned}
&(1-t^2)\frac{\partial}{\partial t}L^*_{n,(n+1)+j}(\varphi,t) \\
&= -\frac{n(n+1-|j|)}{2n+1}\sqrt{\frac{2n+1}{2n+3}\frac{n+1+|j|}{n+1-|j|}}L^*_{n+1,(n+1)+j}(\varphi,t) \\
&\quad + \frac{(n+|j|)(n+1)}{2n+1}\sqrt{\frac{2n+1}{2n-1}\frac{n-|j|}{n+|j|}}L^*_{n-1,(n+1)+j}(\varphi,t) \\
&= -n\sqrt{\frac{(n+1-|j|)(n+1+|j|)}{(2n+1)(2n+3)}}L^*_{n+1,(n+1)+j}(\varphi,t) \\
&\quad + (n+1)\sqrt{\frac{(n-|j|)(n+|j|)}{(2n-1)(2n+1)}}L^*_{n-1,(n+1)+j}(\varphi,t) \\
&= -nD_{n+1,j}L^*_{n,(n+1)+j}(\varphi,t)+(n+1)D_{n,j}L^*_{n-1,(n+1)+j}(\varphi,t).
\end{aligned} \tag{3.457}
$$

This shows Lemma 3.90. □

Our considerations presented in Lemma 3.90 show us (see (2.138), (2.139) for the explicit expressions of ∇^* and L^*) that

$$
\begin{aligned}
&\sqrt{1-t^2}\ \nabla^* L^*_{n,(n+1)+j}(\varphi,t) \\
&= (-\sin\varphi\varepsilon^1+\cos\varphi\varepsilon^2)jL^*_{n,(n+1)-j}(\varphi,t) \\
&\quad + (-t\cos\varphi\varepsilon^1 - t\sin\varphi\varepsilon^2+\sqrt{1-t^2}\varepsilon^3)\ (-nD_{n+1,j}L^*_{n,(n+1)+j}(\varphi,t) \\
&\quad +(n+1)D_{n,j}L^*_{n-1,(n+1)+j}(\varphi,t))
\end{aligned} \tag{3.458}
$$

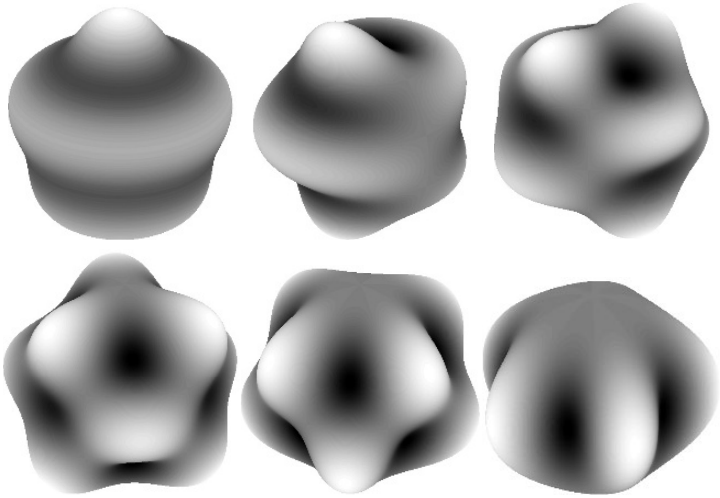


Fig. 3.9: Associated Legendre spherical harmonics $L^*_{n,(n+1)+j}$ of degree $n = 5$ and $j = 0, 1, 2$ (*first row*) and $j = 3, 4, 5$ (*second row*) in three-dimensional view, i.e., functional values are represented by their deviation from the unit sphere.

and

$$\begin{aligned}
&\sqrt{1-t^2}\,\mathrm{L}^* L^*_{n,(n+1)+j}(\varphi, t) \qquad (3.459)\\
&= (\sin\varphi\varepsilon^1 - \cos\varphi\varepsilon^2)(-nD_{n+1,j}L^*_{n,(n+1)-j}(\varphi, t)\\
&\quad +(n+1)D_{n,j}L^*_{n-1,(n+1)+j}(\varphi, t))\\
&\quad + (-t\cos\varphi\varepsilon^1 - t\sin\varphi\varepsilon^2 + \sqrt{1-t^2}\varepsilon^3)jL^*_{n,(n+1)-j}(\varphi, t).
\end{aligned}$$

In other words, the scalar (associated Legendre) harmonics $\{L^*_{n,l}\}_{l=1,\ldots,2n+1}$ are not eigenfunctions to the angular derivatives. On the one hand, the surface gradient as well as the surface curl gradient of spherical harmonics can be expressed using the system $\{L^*_{n,l}\}_{l=1,\ldots,2n+1}$ in elementary form. On the other hand, the representations are lengthy and (at least in the case of the operator of the longitude) rather complicated to handle. Even more, singularities at the poles cannot be avoided for both surface gradient and the surface curl gradient, i.e., ∇^* and L^*, when polar coordinates come into play on the (global) unit sphere Ω.

Remark 3.91. The complete orthonormal system $\{L^*_{n,(n+1)+j}\}_{j=-n,\ldots,n}$ (see Fig. 3.9) is that one used for the representations of the Earth's disturbing potential (see Fig. 3.10)(for example, of the model EGM96 (cf. F.G. Lemoine et al. (1998)), the Earth's magnetic potential IGRF10 (cf. S. Macmillan et al. (2003)), and many others.

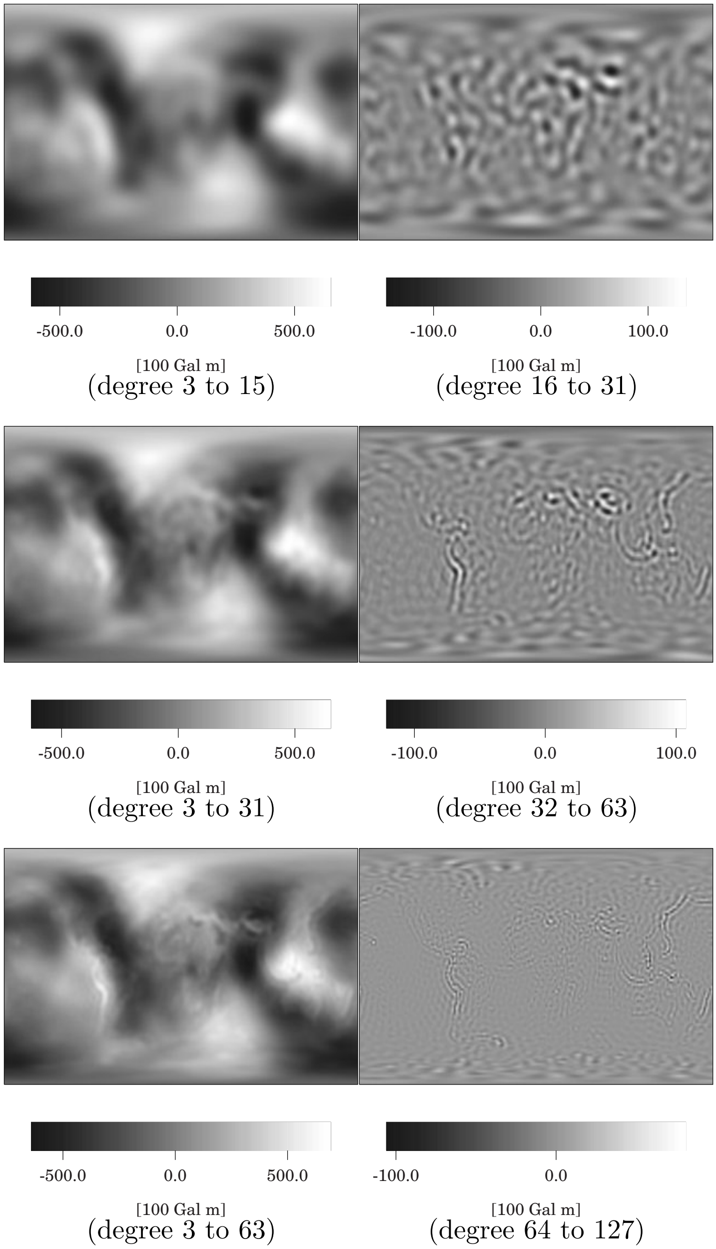


Fig. 3.10: EGM96 spherical harmonic expansion in terms of associated Legendre (spherical) harmonics (from W. Freeden (1999)).

Spherical harmonics involving associated Legendre polynomials, i.e., associated Legendre (spherical) harmonics, share important properties with the 'circle functions', i.e., the trigonometric functions. We already know from Lemma 3.81 that, for $j = 0, \ldots, n$,

$$P_{n,j}(\cos\vartheta) = \left(\frac{1}{2}\right)^j \frac{(n+j)!}{(n-j)!j!} \sin^j\vartheta \sum_{k=0}^{n-j} C_{\frac{n-j-k}{2}} \sin^{n-j-k}(\vartheta)\cos^k(\vartheta), \tag{3.460}$$

where the coefficients $C_{\frac{n-j-k}{2}}$, are recursively determined by (3.402). Furthermore, it is well-known that

$$\cos(j\varphi) = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^i \binom{j}{2i} \cos^{j-2i}(\varphi)\sin^{2i}(\varphi), \tag{3.461}$$

$$\sin(j\varphi) = \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i+1} \binom{j}{2i-1} \cos^{j-2i+1}(\varphi)\sin^{2i-1}(\varphi), \tag{3.462}$$

where, as usual, $\lceil \frac{n}{2} \rceil = \min\{k \in \mathbb{Z} \mid k \geq \frac{n}{2}\}$ and $\lfloor \frac{n}{2} \rfloor = \max\{k \in \mathbb{Z} \mid k \leq \frac{n}{2}\}$.

In terms of spherical coordinates $x = r\xi$, $r = |x|, \xi \in \Omega$, such that $x_i = r\xi_i$, $i = 1, 2, 3$, and $\xi_1 = \sin\vartheta\sin\varphi, \xi_2 = \sin\vartheta\cos\varphi, \xi_3 = \cos\vartheta$, $\varphi \in [0, 2\pi), \vartheta \in [0, \pi]$, this gives us after a simple calculation

$$\begin{aligned} & r^j(\sin\vartheta)^j\cos(j\varphi) \\ = \; & r^j(\sin\vartheta)^j \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^i \binom{j}{2i} \cos^{j-2i}(\varphi)\sin^{2i}(\varphi) \\ = \; & \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^i \binom{j}{2i} \xi_1^{j-2i}\xi_2^{2i} \end{aligned} \tag{3.463}$$

and

$$\begin{aligned} & r^j(\sin\vartheta)^j\sin(j\varphi) \\ = \; & r^j(\sin\vartheta)^j \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i+1} \binom{j}{2i-1} \cos^{j-2i+1}(\varphi)\sin^{2i-1}(\varphi) \\ = \; & \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} \xi_1^{j-2i+1}\xi_2^{2i-1}. \end{aligned} \tag{3.464}$$

Collecting our results, we therefore obtain the following representation in terms of cartesian coordinates.

Lemma 3.92. *For every* $n \in \mathbb{N}_0$, $\{L_{n,(n+1)+j}\}_{j=-n,\ldots,n}$ *with*

$$\begin{aligned} L_{n,(n+1)+j}(x) &= r^n P_{n,|j|}(\cos\vartheta) \begin{cases} \cos(j\varphi) & , \quad j=-n,\ldots,0 \\ \sin(j\varphi) & , \quad j= \quad 1,\ldots,n \end{cases} \\ &= \left(\frac{1}{2}\right)^{|j|} \frac{(n+|j|)!}{(n-|j|)!|j|!} \sum_{k=0}^{n-|j|} C_{\frac{n-|j|-k}{2}} (x_1^2+x_2^2)^{\frac{n-|j|-k}{2}} x_3^k \\ &\quad \times \begin{cases} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{|j|-2i} x_2^{2i} & , \quad j=-n,\ldots,0 \\ \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{j-2i+1} x_2^{2i-1} & , \quad j= \quad 1,\ldots,n \end{cases} \end{aligned} \tag{3.465}$$

forms a maximal $(\cdot,\cdot)_{\mathrm{Hom}_n}$*-orthogonal system and, consequently, a maximal* $\mathrm{L}^2(\Omega)$*–orthogonal system in* $\mathrm{Harm}_n(\mathbb{R}^3)$.

From Lemma 3.92, we are able to deduce the following theorem immediately.

Theorem 3.93. *Let* $A^j_{n-k} : (x_1,x_2) \mapsto A^j_{n-k}(x_1,x_2), k=0,\ldots,n$, $(x_1,x_2)^T \in \mathbb{R}^2$*, be defined by*

$$\begin{aligned} &A^j_{n-k}(x_1,x_2) \\ &= C^{|j|}_{\frac{n-|j|-k}{2}} (x_1^2+x_2^2)^{\frac{n-|j|-k}{2}} \\ &\quad \times \begin{cases} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{|j|-2i} x_2^{2i} & , \quad j=-n,\ldots,0 \\ \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{j-2i+1} x_2^{2i-1} & , \quad j= \quad 1,\ldots,n \end{cases} \end{aligned} \tag{3.466}$$

(with $C^{|j|}_{\frac{n-|j|-k}{2}}$ *given by (3.406)). Then, for* $n=0,1,\ldots,j=-n,\ldots,n$*, the associated Legendre (spherical) harmonic* $L_{n,(n+1)+j} : \mathbb{R}^3 \to \mathbb{R}$ *is representable in the form*

$$L_{n,(n+1)+j}(x) = \sum_{k=0}^{n} A^j_{n-k}(x_1,x_2) x_3^k, \quad x=(x_1,x_2,x_3)^T \in \mathbb{R}^3, \tag{3.467}$$

where

$$\left[\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right] A^j_{n-k}(x_1,x_2) + (k+1)(k+2) A^j_{n-k-2}(x_1,x_2) = 0 \tag{3.468}$$

and, for $k = 0, \ldots, n$,

$$\begin{aligned}
&\left[\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2\right] A^j_{n-k}(x_1, x_2) \\
&\quad = C^{|j|}_{\frac{n-|j|-k}{2}} \\
&\qquad \times \left((n-|j|-k)^2 + 2|j|(n-|j|-k)^2\right)(x_1^2 + x_2^2)^{\frac{n-|j|-k}{2}-1} \\
&\qquad \times \begin{cases} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{|j|-2i} x_2^{2i} & , \quad j = -n, \ldots, 0 \\ \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{j-2i+1} x_2^{2i-1} & , \quad j = \ 1, \ldots, n. \end{cases}
\end{aligned} \tag{3.469}$$

The system $\{L^*_{n,(n+1)+j}\}_{\substack{n=0,1,\ldots \\ j=-n,\ldots,n}}$ with $L^*_{n,(n+1)+j}$ defined by

$$\xi \mapsto L^*_{n,(n+1)+j}(\xi) = C_{n,j} L_{n(n+1)+j}(\xi), \quad \xi \in \Omega, \tag{3.470}$$

(and $C_{n,j}$ given by (3.444)) forms a maximal $\mathrm{L}^2(\Omega)$-orthonormal basis system in $\mathrm{L}^2(\Omega)$.

Introducing the abbreviation $(x_1, x_2) \mapsto B_{\frac{n-|j|-k}{2}}(x_1^2 + x_2^2), x_1, x_2 \in \mathbb{R}$ by

$$B_{\frac{n-|j|-k}{2}}(x_1^2 + x_2^2) = C^{|j|}_{\frac{n-|j|-k}{2}}(x_1^2 + x_2^2)^{\frac{n-|j|-k}{2}} \tag{3.471}$$

we are therefore able to formulate the following theorem.

Theorem 3.94. *For* $n = 0, 1, \ldots, j = -n, \ldots, n$, *the associated Legendre (spherical) harmonic* $L_{n,(n+1)+j} : \mathbb{R}^3 \to \mathbb{R}$ *is given by*

$$L_{n,(n+1)+j}(x) = \sum_{k=0}^{n} A^j_{n-k}(x_1, x_2) x_3^k \tag{3.472}$$

with

$$\begin{aligned}
A^j_{n-k}(x_1, x_2) \quad &= \quad B_{\frac{n-|j|-k}{2}}(x_1^2 + x_2^2) \\
&\quad \times \begin{cases} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{|j|-2i} x_2^{2i} & , \quad j = -n, \ldots, 0 \\ \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{j-2i+1} x_2^{2i-1} & , \quad j = \ 1, \ldots, n. \end{cases}
\end{aligned} \tag{3.473}$$

where $B_{\frac{n-|j|-k}{2}}(x_1^2+x_2^2), x_1, x_2 \in \mathbb{R}$, *satisfies the recursion relation*

$$\begin{aligned} & B_{\frac{n-|j|-k-2}{2}}(x_1^2+x_2^2) \\ &= -\frac{1}{(k+1)(k+2)}\Big((n-|j|-k)^2+2|j|(n-|j|-k)\Big) \\ &\quad \times \frac{1}{x_1^2+x_2^2} B_{\frac{n-|j|-k}{2}}(x_1^2+x_2^2), \end{aligned} \tag{3.474}$$

$k = 0, \ldots, n-|j|-2$, *starting from*

$$\begin{aligned} & B_{\frac{n-|j|}{2}}(x_1^2+x_2^2) \\ &= \begin{cases} (\frac{1}{2})^{|j|} \frac{(n+|j|)!}{|j|!(n-|j|)!} C_{\frac{n-|j|}{2}} (x_1^2+x_2^2)^{\frac{n-|j|}{2}} & , \quad n-|j| \textit{ even} \\ 0 & , \quad n-|j| \textit{ odd} \end{cases} \end{aligned} \tag{3.475}$$

and

$$\begin{aligned} & B_{\frac{n-|j|-1}{2}}(x_1^2+x_2^2) \\ &= \begin{cases} 0 & , \quad n-|j| \textit{ even} \\ (\frac{1}{2})^{|j|} \frac{(n+|j|-1)!}{|j|!(n-1-|j|)!} C_{\frac{n-|j|-1}{2}} (x_1^2+x_2^2)^{\frac{n-|j|-1}{2}} & , \quad n-|j| \textit{ odd.} \end{cases} \end{aligned} \tag{3.476}$$

In other words, for $n-|j|$ even, $L_{n,(n+1)+j}$ can be calculated recursively by (3.474) and (3.475) only by knowing $B_{\frac{n-|j|}{2}}(x_1^2+x_2^2)$, while, for $n-|j|$ odd, $L_{n(n+1)+j}$ can be determined recursively by (3.474) and (3.476) only by knowing $B_{\frac{n-|j|-1}{2}}(x_1^2+x_2^2), x_1, x_2 \in \mathbb{R}$.

Expressed in terms of the (usual) polar coordinates (2.94), we obtain the following corollary.

Corollary 3.95. *For* $n = 0, 1, \ldots, j = -n, \ldots, n$,

$$\begin{aligned} & L_{n,(n+1)+j}(\xi) \\ &= (1-t^2)^{j/2} \sum_{k=0}^{n} B_{\frac{n-|j|-k}{2}}(1-t^2)\; t^k \begin{cases} \cos(j\varphi) & , \quad j = -n, \ldots, 0 \\ \sin(j\varphi) & , \quad j = \quad 1, \ldots, n \end{cases} \end{aligned} \tag{3.477}$$

where

$$\begin{aligned} & B_{\frac{n-|j|-k-2}{2}}(1-t^2) \\ &= -\frac{1}{(k+1)(k+2)}\left((n-|j|-k)^2+2|j|(n-|j|-k)\right) \\ &\quad \times \frac{1}{1-t^2} B_{\frac{n-|j|-k}{2}}(1-t^2), \end{aligned} \tag{3.478}$$

$k = 0, \ldots, n-|j|-2$, *with* $B_{\frac{n-|j|}{2}}(1-t^2)$, *and* $B_{\frac{n-|j|-1}{2}}(1-t^2)$ *given by (3.475) and (3.476), respectively.*

Applying the binomial theorem, we are able to write the homogeneous polynomials $A^j_{n-k} : \mathbb{R}^2 \to \mathbb{R}$ totally separated in terms of the coordinates x_1, x_2. More explicitly, for $k = 0, \ldots, n$,

$$\begin{aligned} & A^j_{n-k}(x_1, x_2) \\ = \;& C^{|j|}_{\frac{n-|j|-k}{2}} \sum_{l=0}^{\frac{n-|j|-k}{2}} \binom{\frac{n-|j|-k}{2}}{l} (x_1^2)^{\frac{n-|j|-k}{2}-l} (x_2^2)^l \qquad (3.479) \\ & \cdot \begin{cases} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{|j|-2i} x_2^{2i} & , \quad j = -n, \ldots, 0 \\ \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{j-2i+1} x_2^{2i-1} & , \quad j = \;\; 1, \ldots, n. \end{cases} \\ = \;& C^{|j|}_{\frac{n-|j|-k}{2}} \begin{cases} \sum_{l=0}^{\frac{n-|j|-k}{2}} \binom{\frac{n-|j|-k}{2}}{l} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{n-k-2l-2i} x_2^{2l+2i} \\ \qquad j = -n, \ldots, 0 \\ \sum_{l=0}^{\frac{n-|j|-k}{2}} \binom{\frac{n-j-k}{2}}{l} \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{n-k-2l-2i+1} x_2^{2l+2i-1} \\ \qquad j = \;\; 1, \ldots, n. \end{cases} \end{aligned}$$

From (3.468) and (3.469) it follows that

$$\begin{aligned} & A^j_{n-k-2}(x_1, x_2) = -\frac{1}{(k+1)(k+2)} C^{|j|}_{\frac{n-|j|-k}{2}} \\ & \times \begin{cases} \sum_{l=0}^{\frac{n-|j|-k}{2}} \binom{\frac{n-|j|-k}{2}}{l} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} \left[\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 \right] x_1^{n-k-2l-2i} x_2^{2l+2i}, \\ \qquad j = -n, \ldots, 0, \\ \sum_{l=0}^{\frac{n-j-k}{2}} \binom{\frac{n-j-k}{2}}{l} \sum_{i=1}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} \left[\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 \right] x_1^{n-k-2l-2i+1} x_2^{2l+2i-1}, \\ \qquad j = 1, \ldots, n. \end{cases} \end{aligned}$$

This yields

$$\begin{aligned} & A^j_{n-k-2}(x_1, x_2) = -\frac{1}{(k+1)(k+2)} C^{|j|}_{\frac{n-|j|-k}{2}} ((n-|j|-k)^2 + 2|j|(n-|j|-k)) \\ & \times \begin{cases} \sum_{l=0}^{\frac{n-|j|-k}{2}} \binom{\frac{n-|j|-k}{2}}{l} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} (-1)^i \binom{|j|}{2i} x_1^{n-k-2(l+1)-2i} x_2^{2l+2i}, \\ \qquad j = -n, \ldots, 0, \\ \sum_{l=0}^{\frac{n-j-k}{2}} \binom{\frac{n-j-k}{2}}{l} \sum_{i=0}^{\lceil \frac{j}{2} \rceil} (-1)^{i-1} \binom{j}{2i-1} x_1^{n-k-2(l+1)-2i+1} x_2^{2l+2i-1}, \\ \qquad j = 1, \ldots, n. \end{cases} \end{aligned}$$

In other words, applying the two-dimensional Laplacian changes the multi-indices, but does not change the even/odd pattern. Therefore, any Legendre (spherical) harmonic $L_{n,(n+1)+j}$ is uniquely determined by the homogeneous polynomials $A_n^j, A_{n-1}^j : \mathbb{R}^2 \to \mathbb{R}$. This is the reason why their explicit representations are of particular significance. For $j = -n, \ldots, 0$ we obtain

$$A_n^j(x_1, x_2) = \begin{cases} C^{|j|}_{\frac{n-|j|}{2}} \sum_{l=0}^{\frac{n-|j|}{2}} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} \binom{\frac{n-|j|}{2}}{l} (-1)^i \binom{|j|}{2i} x_1^{n-2l-2i} x_2^{2l+2i}, & n-|j| \text{ even} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.480}$$

and

$$A_{n-1}^j(x_1, x_2) = \begin{cases} C^{|j|}_{\frac{n-|j|-1}{2}} \sum_{l=0}^{\frac{n-|j|-1}{2}} \sum_{i=0}^{\lfloor \frac{|j|}{2} \rfloor} \binom{\frac{n-|j|-1}{2}}{l} (-1)^i \binom{|j|}{2i} x_1^{n-2l-2i-1} x_2^{2l+2i}, & n-|j| \text{ odd} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.481}$$

while, for $j = 1, \ldots, n$, we find

$$A_n^j(x_1, x_2) = \begin{cases} C^{|j|}_{\frac{n-j}{2}} \sum_{l=0}^{\frac{n-j}{2}} \sum_{i=1}^{\lceil \frac{j}{2} \rceil} \binom{\frac{n-j}{2}}{l} (-1)^{i-1} \binom{j}{2i-1} x_1^{n-2l-2i+1} x_2^{2l+2i-1} & n-j \text{ even} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.482}$$

and

$$A_{n-1}^j(x_1, x_2) = \begin{cases} C^{|j|}_{\frac{n-j-1}{2}} \sum_{l=0}^{\frac{n-|j|-1}{2}} \sum_{i=1}^{\lceil \frac{|j|}{2} \rceil} \binom{\frac{n-j-1}{2}}{l} (-1)^{i-1} \binom{j}{2i-1} x_1^{n-2l-2i} x_2^{2l+2i-1} & n-j \text{ odd.} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.483}$$

Our considerations demonstrate that, for all integers $n \geq 2$, the basis

$$\mathfrak{B}(n) = \{L_{n,(n+1)+j}\}_{j=-n,\ldots,n} \tag{3.484}$$

of the space $\mathrm{Harm}_n(\mathbb{R}^3)$ divides itself into certain subsets. In a first step, we have two subsets

$$\begin{aligned} \mathfrak{B}^{(0)}(n) &= \left\{ \sum_{k=0}^{n} {}_2\, A_{n-k}^j(x_1, x_2) x_3^k \right\}_{j=-n,\ldots,0}, \\ \mathfrak{B}^{(1)}(n) &= \left\{ \sum_{k=1}^{n} {}_2\, A_{n-k}^j(x_1, x_2) x_3^k \right\}_{j=1,\ldots,n}, \end{aligned}$$

where, $A^j_{n-k} : \mathbb{R}^2 \to \mathbb{R}$ is given by (3.479) and, as usual, $\sum 2$ means that every second summand is omitted in the sum. Clearly, the subspaces $\mathrm{Harm}^{(0)}_n(\mathbb{R}^3)$ and $\mathrm{Harm}^{(1)}_n(\mathbb{R}^3)$ generated by $\mathfrak{B}^{(0)}(n)$ and $\mathfrak{B}^{(1)}(n)$, respectively, are of dimensions $n+1$ and n.

In the second step, the subsets $\mathfrak{B}^{(0)}(n), \mathfrak{B}^{(1)}(n)$ are divided in canonical way into two subsets generating an orthogonal splitting of $\mathfrak{B}(n)$ into four subsystems, namely

$$\mathfrak{B}^{(n,0)}(n) = \left\{ \sum_{k=0}^{n} 2\, A^j_{n-k}(x_1,x_2)x_3^k \right\}_{j\le 0, n-|j| \text{ even}}, \tag{3.485}$$

$$\mathfrak{B}^{(n-1,0)}(n) = \left\{ \sum_{k=0}^{n} 2\, A^j_{n-k}(x_1,x_2)x_3^k \right\}_{j\le 0, n-|j| \text{ odd}}, \tag{3.486}$$

$$\mathfrak{B}^{(n,1)}(n) = \left\{ \sum_{k=0}^{n} 2\, A^j_{n-k}(x_1,x_2)x_3^k \right\}_{j> 0, n-|j| \text{ even}}, \tag{3.487}$$

$$\mathfrak{B}^{(n-1,1)}(n) = \left\{ \sum_{k=0}^{n} 2\, A^j_{n-k}(x_1,x_2)x_3^k \right\}_{j> 0, n-|j| \text{ odd}}, \tag{3.488}$$

such that

$$\mathfrak{B}(n) = \mathfrak{B}^{(n,0)}(n) \cup \mathfrak{B}^{(n-1,0)}(n) \cup \mathfrak{B}^{(n,1)}(n) \cup \mathfrak{B}^{(n-1,1)}(n) \tag{3.489}$$

and

$$\mathfrak{B}^{(k,p)}(n) \perp \mathfrak{B}^{(l,q)}(n), (k,p) \neq (l,q), k,l \in \{n-1,n\}, p,q \in \{0,1\}. \tag{3.490}$$

The dimensions of the four subspaces $\mathrm{Harm}^{(n,0)}_n(\mathbb{R}^3)$, $\mathrm{Harm}^{(n-1,0)}_n(\mathbb{R}^3)$, $\mathrm{Harm}^{(n,1)}_n(\mathbb{R}^3)$ and $\mathrm{Harm}^{(n-1,1)}_n(\mathbb{R}^3)$ of the space $\mathrm{Harm}_n(\mathbb{R}^3)$ spanned by the systems $\mathfrak{B}^{(n,0)}(n)$, $\mathfrak{B}^{(n-1,0)}(n)$, $\mathfrak{B}^{(n,)}(n)$, and $\mathfrak{B}^{(n-1,1)}(n)$, respectively, read as follows:

$$\dim\left(\mathrm{Harm}^{(n,0)}_n(\mathbb{R}^3)\right) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & , \quad n \text{ even} \\ \lceil \frac{n}{2} \rceil & , \quad n \text{ odd}, \end{cases} \tag{3.491}$$

$$\dim\left(\mathrm{Harm}^{(n-1,0)}_n(\mathbb{R}^3)\right) = \begin{cases} \lceil \frac{n}{2} \rceil & , \quad n \text{ even} \\ \lceil \frac{n}{2} \rceil & , \quad n \text{ odd}, \end{cases} \tag{3.492}$$

$$\dim\left(\mathrm{Harm}^{(n,1)}_n(\mathbb{R}^3)\right) = \begin{cases} \lceil \frac{n}{2} \rceil & , \quad n \text{ even} \\ \lceil \frac{n}{2} \rceil & , \quad n \text{ odd}, \end{cases} \tag{3.493}$$

$$\dim\left(\mathrm{Harm}^{(n-1,1)}_n(\mathbb{R}^3)\right) = \begin{cases} \lceil \frac{n}{2} \rceil & , \quad n \text{ even} \\ \lceil \frac{n}{2} \rceil - 1 & , \quad n \text{ odd}, \end{cases} \tag{3.494}$$

Note that as superscripts for the first two subsets, we use binary numbers '0' and '1'. The binary digits reflect the sign of j. The second superscripts 'n'

and '$n-1$' reflect the implication of the degree of the two-dimensional homogeneous polynomial to the choice of $n-|j|$ to be even/odd. More explicitly, from the expression (3.479), it becomes obvious that the superscripts equivalently reflect the even/odd pattern of the two-dimensional multiindices involved in the representations of A^j_{n-1}, A^j_n if we understand 'even' for '0' and 'odd' for 1. In fact, in order to characterize the ingredients in the representations (3.480), (3.481), (3.482), and (3.483), respectively, we are lead to the following observation: '$(n,0)$' is associated to two-dimensional monomials $x_1^{\beta_1}x_2^{\beta_2}$ with $\beta_1,\beta_2 \in \mathbb{N}_0$, $\beta_1+\beta_2 = n$, β_2 even, '$(n-1,0)$' is associated to monomials $x_1^{\beta_1}x_2^{\beta_2}$ with $\beta_1,\beta_2 \in \mathbb{N}_0$, $\beta_1+\beta_2 = n-1$, β_2 even, whereas '$(n,1)$' is associated to monomials $x_1^{\beta_1}x_2^{\beta_2}$ with $\beta_1,\beta_2 \in \mathbb{N}_0$, $\beta_1+\beta_2 = n, \beta_2$ odd, and '$(n-1,1)$' is associated to monomials $x_1^{\beta_1}x_2^{\beta_2}$ with $\beta_1,\beta_2 \in \mathbb{N}_0$, $\beta_1+\beta_2 = n-1$, β_2 odd.

Summarizing our results on the splitting of $\mathrm{Harm}_n(\mathbb{R}^3)$ into 'Legendre subspaces', we finally obtain

$$\begin{aligned}\mathrm{Harm}_n(\mathbb{R}^3) &= \mathrm{Harm}_n^{(n,0)}(\mathbb{R}^3) \cup \mathrm{Harm}_n^{(n-1,0)}(\mathbb{R}^3) \\ &\cup \mathrm{Harm}_n^{(n,1)}(\mathbb{R}^3) \cup \mathrm{Harm}_n^{(n-1,1)}(\mathbb{R}^3)\end{aligned} \tag{3.495}$$

such that

$$\mathrm{Harm}_n^{(k,p)}(\mathbb{R}^3) \perp \mathrm{Harm}_n^{(l,q)}(\mathbb{R}^3) \tag{3.496}$$

provided that

$$(k,p) \neq (l,q), k,l \in \{n-1,n\}, p,q \in \{0,1\}. \tag{3.497}$$

Once more, the reason for the validity of (3.495) (3.496) is that the two-dimensional multi-indices generating the basis functions in one subsystem $\mathfrak{B}^{(k,p)}(n)$ $k \in \{n-1,n\}, p \in \{0,1\}$, only show one specific even/odd pattern, and the pattern is different from all the patterns in the other subsystems.

3.14 Exact Computation of Legendre Basis Systems

Earlier, in Section 3.3, it has been shown that a basis system in $\mathrm{Harm}_n(\mathbb{R}^3)$ can be computed entirely by integer operations from $2n+1$ systems of linear equations. The basis functions obtained can be orthonormalized exactly by means of the well-known Gram-Schmidt orthonormalization process. As a result, there are $2n+1$ homogeneous harmonic polynomials available (orthonormalized in the sense of $(\cdot,\cdot)_{\mathrm{Hom}_n}$ and, subsequently, in $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$). But, the disadvantage in that approach is that the linear systems result in basis functions which are all involved in the computational work of the orthonormalization process.

Our investigations about Legendre harmonics (in Section 3.13) have demonstrated that, for every degree $n \geq 2$, the space $\mathrm{Harm}_n(\mathbb{R}^3)$ can be split canonically into four subspaces that are mutually orthogonal and the intersection between any pair of subspaces is empty.

Combining these observations, we are now interested in the exact computation of a basis in $\mathrm{Harm}_n(\mathbb{R}^3)$ which, in $(\cdot,\cdot)_{\mathrm{Hom}_n}$-orthogonal sense, divides itself in a natural way into the four subsystems known from the theory of associated Legendre (spherical) harmonics. The subsystems, indeed, are bases of the spaces $\mathrm{Harm}_n^{(k,p)}(\mathbb{R}^3)$, $k \in \{n-1, n\}, p \in \{0,1\}$, respectively. In addition, they are computed exclusively by integer operations. In doing so, the calculation of coefficients involving factorials such as $C^{|j|}_{\frac{n-|j|-n}{2}}$ are avoided within the computational process. However, it turns out that the basis established by exact computation is only partially orthogonal in $\mathrm{Harm}_n(\mathbb{R}^3)$, i.e., it is not totally orthogonal in $(\cdot,\cdot)_{\mathrm{Hom}_n}$-sense (as in the case of associated Legendre (spherical) harmonics). Actually, it is a compromise between the two methods presented in the preceding sections. Nevertheless, in comparison to the exact computation explained in Section 3.3, the amount of the computational work for (exact) orthonormalization by the Gram-Schmidt procedure is reduced drastically, because the orthonormalization process can be performed separately for the four individual subsets (whose numbers of elements on average is $(2n+1)/4$).

Our concept of exact generation of $(\cdot,\cdot)_{\mathrm{Hom}_n}$-orthonormalized homogeneous harmonic polynomials in $\mathbb{R}^3$ (cf. W. Freeden, R. Reuter (1990)) is based on the observation that the two-dimensional multi-indices occurring in the sets

$$\mathcal{M}(n) = \{(\beta_1, \beta_2)^T \in {\mathbb{N}_0}^2 | \beta_1 + \beta_2 = n\} \tag{3.498}$$

and

$$\mathcal{M}^{(0)}(n) = \{(\beta_1, \beta_2)^T \in \mathcal{M}(n) | \beta_2 \text{ even }\}, \tag{3.499}$$
$$\mathcal{M}^{(1)}(n) = \{(\beta_1, \beta_2)^T \in \mathcal{M}(n) | \beta_2 \text{ odd }\} \tag{3.500}$$

can be recognized in the framework of associated Legendre (spherical) harmonics. In fact,

$$\mu_q(n) = \sharp \mathcal{M}^{(\mathrm{bin}(q))}(n) = \begin{cases} \frac{n+2-b}{2} &, \quad n+b \text{ even} \\ \frac{n+2-(b+1)}{2} &, \quad n+b \text{ odd}, \end{cases} \tag{3.501}$$

where b denotes the number of digits of $\mathrm{bin}(q)$ which are 1, and $\mathrm{bin}(q)$ denotes the representation of the integer q in its binary form. Consequently, we have

$$\mu_q(n-1) = \sharp\mathcal{M}^{(\mathrm{bin}(q))}(n-1) = \begin{cases} \frac{n+1-b}{2} & , \quad n-1+b \text{ even} \\ \frac{n+1-(b+1)}{2} & , \quad n-1+b \text{ odd}. \end{cases} \tag{3.502}$$

By comparison with the results obtained in Section 3.13, this implies the specification of the dimensions of the subspaces as follows:

$$\mu_0(n) = \dim(\mathrm{Harm}_n^{(n,0)}(\mathbb{R}^3)), \tag{3.503}$$
$$\mu_0(n-1) = \dim(\mathrm{Harm}_n^{(n-1,0)}(\mathbb{R}^3)), \tag{3.504}$$
$$\mu_1(n) = \dim(\mathrm{Harm}_n^{(n,1)}(\mathbb{R}^3)), \tag{3.505}$$
$$\mu_1(n-1) = \dim(\mathrm{Harm}_n^{(n-1,1)}(\mathbb{R}^3)). \tag{3.506}$$

Once again, the point of departure for the exact construction of an alternative (Legendre) basis system is the observation that any homogeneous polynomial ${}^{(b,p)}H_n \in \mathrm{Harm}_n^{(b,p)}(\mathbb{R}^3), b \in \{n-1, n\}, p \in \{0,1\}$, in three variables can be represented in the form (3.27)

$${}^{(b,p)}H_n(x_1, x_2, x_3) = \sum_{k=0}^{n} {}^{(k,p)}A_{n-k}(x_1, x_2)x_3^k, \tag{3.507}$$

where ${}^{(b,p)}A_n : \mathbb{R}^2 \to \mathbb{R}$ and ${}^{(b,p)}A_{n-1} : \mathbb{R}^2 \to \mathbb{R}$, $b \in \{n-1, n\}, p \in \{0,1\}$ are homogeneous polynomials of degree n and degree $n-1$, respectively. Furthermore, for $k = 0, \ldots, n-2$, ${}^{(b,p)}A_{n-k-2}$ is defined recursively according to (3.31) by the relations

$$\begin{aligned} &{}^{(b,q)}A_{n-k-2}(x_1, x_2) \\ &= -\frac{1}{(k+1)(k+2)} \left(\left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 \right) {}^{(b,q)}A_{n-k}(x_1, x_2). \end{aligned} \tag{3.508}$$

The functions ${}^{(b,q)}A_n^l : \mathbb{R}^2 \to \mathbb{R}$, $b \in \{n-1, n\}, p \in \{0,1\}$, $l = 1, \ldots, \mu_p(n)$, $p \in \{0,1\}$, generating the basis in $\mathrm{Harm}_n^{(b,q)}(\mathbb{R}^3)$ via the recursion relation (3.508), therefore, can be chosen as follows:

$${}^{(n,0)}A_n^l(x_1, x_2) = x_1^{n-2(l-1)} x_2^{2(l-1)}, \quad l = 1, \ldots, \mu_0(n), \tag{3.509}$$
$${}^{(n-1,0)}A_{n-1}^l(x_1, x_2) = x_1^{n-1-2(l-1)} x_2^{2(l-1)}, \quad l = 1, \ldots, \mu_0(n-1), \tag{3.510}$$
$${}^{(n,1)}A_n^l(x_1, x_2) = x_1^{n-1-2(l-1)} x_2^{2(l-1)+1}, \quad l = 1, \ldots, \mu_1(n), \tag{3.511}$$
$${}^{(n-1,1)}A_{n-1}^l(x_1, x_2) = x_1^{n-2-2(l-1)} x_2^{2(l-1)+1}, \quad l = 1, \ldots, \mu_1(n-1). \tag{3.512}$$

As requested by the recursion (3.508), the Laplacian $(\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2$ has to be applied repeatedly to these functions generating the whole system as indicated by (3.507), respectively. Altogether, we get a basis of $2n+1$ elements in $\mathrm{Harm}_n(\mathbb{R}^3)$, as required. From the formulas involved in the computational process, it is obvious that the partial orthogonal basis subsystems can be computed in an exact arithmetic. The functions of each subsystems can be orthonormalized, as usual, by the Gram-Schmidt orthonormalizing process which can be performed exactly besides a final division of each polynomials by its norm (square root).

Example 3.96. We discuss the case $n = 5$ and start with the set of two–dimensional multi-indices

$$\begin{aligned}\mathcal{M}(4) &= \{(4,0),(3,1),(2,2),(1,3),(0,4)\},\\ \mathcal{M}(5) &= \{(5,0),(4,1),(3,2),(3,3),(2,4),(0,5)\}.\end{aligned}$$

They are split according to our approach into the sets

$$\mathcal{M}^{(0)}(5) = \{(5,0),(3,2),(1,4)\}, \tag{3.513}$$
$$\mathcal{M}^{(0)}(4) = \{(4,0),(2,2),(0,4)\}, \tag{3.514}$$
$$\mathcal{M}^{(1)}(5) = \{(4,1),(2,3),(0,5)\}, \tag{3.515}$$
$$\mathcal{M}^{(1)}(4) = \{(3,1),(1,3)\}. \tag{3.516}$$

Note that, for $n = 5$, we particularly have

$$\mu_0(n) = 3, \tag{3.517}$$
$$\mu_0(n-1) = 3, \tag{3.518}$$
$$\mu_1(n) = 3, \tag{3.519}$$
$$\mu_1(n-1) = 2. \tag{3.520}$$

From each of these sets, the corresponding homogeneous harmonic polynomials are derived. They are written down in a schematic manner. The first polynomial reads as follows:

$$1 \cdot x_1^4 x_2^0 x_3^1 - 2 \cdot x_1^2 x_2^0 x_3^3 + \frac{1}{5} \cdot x_1^0 x_2^0 x_3^5. \tag{3.521}$$

Additionally, the orthonormalized set of polynomials can be listed in similar fashion. The first of these reads:

$$\left(1 \cdot x_1^4 x_2^0 x_3^1 - 2 \cdot x_1^2 x_2^0 x_3^3 + \frac{1}{5} \cdot x_1^0 x_2^0 x_3^5\right) / \sqrt{\frac{384}{5}}. \tag{3.522}$$

We obtain:

$$\begin{aligned}
\mathcal{M}^{(0)}(5) :& 1x_1^5x_2^0x_3^0 - 10x_1^3x_2^0x_3^2 + 5x_1^1x_2^0x_3^4,\\
& 1x_1^3x_2^2x_3^0 - 3x_1^1x_2^2x_3^2 - 1x_1^3x_2^0x_3^2 + 1x_1^1x_2^0x_3^4,\\
& 1x_1^1x_2^4x_3^0 - 6x_1^1x_2^2x_3^2 + 1x_1^1x_2^0x_3^4,
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(0)}(4) :& 1x_1^4x_2^0x_3^1 - 2x_1^2x_2^0x_3^3 + \tfrac{1}{5}x_1^0x_2^0x_3^5,\\
& 1x_1^2x_2^2x_3^1 - \tfrac{1}{3}x_1^0x_2^2x_3^3 - \tfrac{1}{3}x_1^2x_2^0x_3^3 + \tfrac{1}{15}x_1^0x_2^0x_3^5,\\
& 1x_1^0x_2^4x_3^1 - 2x_1^0x_2^2x_3^3 + \tfrac{1}{5}x_1^0x_2^0x_3^5,
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(1)}(5) :& 1x_1^4x_2^1x_3^0 - 6x_1^2x_2^1x_3^2 + 1x_1^0x_2^1x_3^4,\\
& 1x_1^2x_2^3x_3^0 - 1x_1^0x_2^3x_3^2 - 3x_1^2x_2^1x_3^2 + 1x_1^0x_2^1x_3^4.
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(1)}(4) :& 1x_1^3x_2^1x_3^1 - 1x_1^1x_2^1x_3^3,\\
& 1x_1^1x_2^3x_3^1 - 1x_1^1x_2^1x_3^3\ .
\end{aligned}$$

On this linearly independent set of homogeneous harmonic polynomials of degree 5 in $\mathbb{R}^3$ that is partially orthogonal, we apply the Gram-Schmidt orthonormalization process with respect to the $(\cdot,\cdot)_{\mathrm{Hom}_5}$-topology thereby observing the normalization factors (as indicated in Section 3.3):

$$\begin{aligned}
\mathcal{M}^{(0)}(5) :& \sqrt{1920}(1x_1^5x_2^0x_3^0 - 10x_1^3x_2^0x_3^2 + 5x_1x_2^0x_3^4),\\
& \sqrt{54}(1x_1^3x_2^2x_3^0 - 3x_1^1x_2^2x_3^2 + \tfrac{1}{4}x_1^3x_2^0x_3^2\\
&\qquad + \tfrac{1}{4}x_1^3x_2^0x_3^4 + \tfrac{3}{8}x_1^1x_2^0x_3^4 - 1/8x_1^5x_2^0x_3^0),\\
& \sqrt{63}(1x_1^1x_2^4x_3^0 - \tfrac{3}{2}x_1^1x_2^2x_3^2 + \tfrac{1}{8}x_1^1x_2^0x_3^4\\
&\qquad + \tfrac{1}{8}x_1^5x_2^0x_3^0 + \tfrac{1}{4}x_1^3x_2^0x_3^2 - 3/2x_1^3x_1^2x_3^0),
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(0)}(4) :& \sqrt{\tfrac{384}{5}}(1x_1^4x_2^0x_3^1 - 2x_1^2x_2^0x_3^3 + \tfrac{1}{5}x_1^0x_2^0x_3^5),\\
& \sqrt{6}(1x_1^2x_2^2x_3^1 - \tfrac{1}{3}x_1^0x_2^0x_3^3 - \tfrac{1}{12}x_1^2x_2^0x_3^3 + \tfrac{1}{24}x_1^0x_2^0x_3^5\\
& -\tfrac{1}{8}x_1^4x_2^0x_3^1),\\
& \sqrt{63}(1x_1^0x_2^4x_3^1 - \tfrac{3}{2}x_1^0x_2^2x_3^3 + \tfrac{1}{8}x_1^0x_2^0x_3^5 + \tfrac{1}{8}x_1^4x_2^0x_3^1\\
& +\tfrac{1}{4}x_1^2x_2^0x_3^3 - \tfrac{3}{2}x_1^1x_2^2x_3^1),
\end{aligned}$$

$$\begin{aligned}\mathcal{M}^{(1)}(5):\ &\sqrt{192}(1x_1^4x_2^1x_3^0 - 6x_1^2x_2^1x_3^2 + 1x_1^0x_2^1x_3^4),\\ &\sqrt{36}(1x_1^2x_3^3x_3^0 - 1x_1^0x_2^3x_3^2 + \tfrac{1}{2}x_1^0x_2^1x_3^4),\\ &\sqrt{945}\left(1x_1^0x_2^5x_3^0 - 5x_1^0x_2^3x_3^2 + \tfrac{15}{8}x_2^0x_2^1x_3^4\right.\\ &\quad\left.+\tfrac{15}{8}x_1^4x_2^1x_3^0 + \tfrac{15}{4}x_1^2x_2^1x_3^2 - 5x_1^2x_2^3x_3^0\right),\end{aligned}$$

$$\begin{aligned}\mathcal{M}^{(1)}(4):&\sqrt{12}(1\cdot x_1^3x_2^1x_3^1 - 1x_1^1x_2^1x_3^3),\\ &\sqrt{9}(1x_1^1x_2^3x_3^1 - \tfrac{1}{2}x_1^1x_2^1x_3^3 - \tfrac{1}{2}x_1^3x_2^1x_3^1).\end{aligned}$$

3.15 Bibliographical Notes

For more theoretical details on scalar spherical harmonics, the reader may want to consult some of the following references: A. Wangerin (1921), R. Courant, D. Hilbert (1924) O.D. Kellogg (1929), E.T. Whittaker, G.N. Watson (1996), C. Müller (1952, 1966, 1998), P.M. Morse, H. Feshbach (1953), J. Lense (1954), E.W. Hobson (1955), F. John (1955), I.N. Sneddon (1956), A.R. Edmonds (1957), R.T. Seeley (1966), T. McRobert (1967), E.M. Stein, G. Weiss (1971), H. Hochstadt (1971), N.N. Lebedev (1973), N.J. Vilenkin (1968),W. Freeden (1979a), W. Freeden et al. (1998), and many others.

4 Green's Functions and Integral Formulas on the Sphere

An essential tool of our approach to vector and tensor spherical harmonics is Green's function on the unit sphere Ω with respect to the Beltrami operator Δ^*. It offers the perspective of solving the surface gradient equation, the surface curl gradient equation, and (iterated) Beltrami differential equations. The Beltrami equation is needed to assure the explicit structure of the (Helmholtz) decomposition theorems for vectorial and tensorial fields on the unit sphere Ω, later on.

4.1 Green's Function with Respect to the Beltrami Operator Δ^*

Our point of departure for discussing the Green function is the definition of its constituting properties.

Definition 4.1. $G(\Delta^*;\cdot,\cdot) : (\xi,\eta) \mapsto G(\Delta^*;\xi,\eta), -1 \leq \xi \cdot \eta < 1$, is called Green's function on Ω with respect to the operator Δ^*, if it satisfies the following properties:

(i) (*Differential equation*) for every point $\xi \in \Omega$, $\eta \mapsto G(\Delta^*;\xi,\eta)$ is twice continuously differentiable on the set $\{\eta \in \Omega | -1 \leq \xi \cdot \eta < 1\}$, such that
$$\Delta^*_\eta G(\Delta^*;\xi,\eta) = -\frac{1}{4\pi}, \quad -1 \leq \xi \cdot \eta < 1,$$

(ii) (*Characteristic singularity*) for every $\xi \in \Omega$, the function
$$\eta \mapsto G(\Delta^*;\xi,\eta) - \frac{1}{4\pi} \ln(1 - \xi \cdot \eta)$$
is continuously differentiable on Ω (note that
$$\ln|\xi - \eta| = \frac{1}{2}\ln(2 - 2\,\xi \cdot \eta) = \frac{1}{2}\ln(1 - \xi \cdot \eta) + \frac{1}{2}\ln 2,$$
$-1 \leq \xi \cdot \eta < 1$),

(iii) (*Rotational symmetry*) for all orthogonal transformations $\mathbf{t}$

$$G(\Delta^*; \mathbf{t}\xi, \mathbf{t}\eta) = G(\Delta^*; \xi, \eta),$$

(iv) (*Normalization*) for every $\xi \in \Omega$,

$$\frac{1}{4\pi} \int_\Omega G(\Delta^*; \xi, \eta) \; d\omega(\eta) = 0.$$

We first prove the *uniqueness of Green's function with respect to the Beltrami operator* Δ^*.

Lemma 4.2. $G(\Delta^*; \cdot, \cdot)$ *is uniquely determined by its defining properties* (i)–(iv).

Proof. Denote by $D(\Delta^*; \cdot, \cdot)$ the difference between two Green functions satisfying (i)–(iv). Then, we have the following properties:

(i) $D(\Delta^*; \xi, \cdot)$ is twice continuously differentiable for all $\eta \in \Omega$ satisfying $-1 \leq \xi \cdot \eta < 1$, and we have

$$(\Delta^*)_\eta D(\Delta^*; \xi, \eta) = 0, \tag{4.1}$$

(ii) $D(\Delta^*; \xi, \cdot)$ is continuously differentiable on Ω,

(iii) For all orthogonal transformations $\mathbf{t}$,

$$D(\Delta^*; \mathbf{t}\xi, \mathbf{t}\eta) = D(\Delta^*; \xi, \eta), \tag{4.2}$$

(iv) For all $\xi \in \Omega$,

$$\int_\Omega D(\Delta^*; \xi, \eta) \, d\omega(\eta) = 0. \tag{4.3}$$

The properties (i)–(iii) show that $D(\Delta^*; \xi, \cdot)$ is an everywhere on the unit sphere Ω twice continuously differentiable function satisfying the differential equation (i). Therefore, $D(\Delta^*; \xi, \cdot)$ must be a spherical harmonic of degree 0. $D(\Delta^*; \xi, \eta)$ depends only on the scalar product of ξ and η, i.e.,

$$D(\Delta^*; \xi, \eta) = \alpha_0 P_0(\xi \cdot \eta) = \alpha_0. \tag{4.4}$$

From (iv) we obtain

$$\int_\Omega D(\Delta^*; \xi, \eta) \, d\omega(\eta) = 4\pi\alpha_0 \; = \; 0. \tag{4.5}$$

Hence, $\alpha_0 = 0$. But this means that the Green function $G(\Delta^*; \cdot, \cdot)$ is uniquely determined by the defining properties (i)–(iv). □

An easy calculation shows that

$$(\xi, \eta) \mapsto \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2, \quad -1 \leq \xi \cdot \eta < 1 \tag{4.6}$$

satisfies all the defining properties (i)–(iv) of the Green function with respect to Δ^*. Therefore, we have the following result.

Lemma 4.3. *For $\xi, \eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$*

$$G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2.$$

Remark 4.4. Throughout this chapter, we usually write $G(\Delta^*; \xi \cdot \eta)$ instead of $G(\Delta^*; \xi, \eta)$, $(\xi, \eta) \in \Omega \times \Omega$. This indicates that $G(\Delta^*; \xi \cdot \eta)$ depends only on the scalar product of ξ and η, i.e., $G(\Delta^*; \cdot)$ is a zonal function, hence, it may be understood as a function defined on the (one-dimensional) interval $[-1, 1)$.

Observing the logarithmic singularity of the Green function, we see by applying the Second Green Surface Theorem that the spherical harmonics of degree n, i.e., the eigenfunctions of the Beltrami operator Δ^*, are eigenfunctions of Green's (kernel) function $G(\Delta^*; \cdot)$ in the sense of the integral equation

$$-k(k+1) \int_\Omega G(\Delta^*; \xi \cdot \eta) Y_k(\eta) \, d\omega(\eta) = (1 - \delta_{0k}) Y_k(\xi). \tag{4.7}$$

In terms of a (maximal) $\mathrm{L}^2(\Omega)$-orthonormal system $\{Y_{n,m}\}$ of spherical harmonics of degree n and order m, we thus obtain as spectral representation the bilinear expansion (see W. Freeden (1979a))

$$G(\Delta^*; \xi \cdot \eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{-n(n+1)} Y_{n,m}(\xi) Y_{n,m}(\eta),$$

$-1 \leq \xi \cdot \eta < 1$. Observing the addition theorem of the spherical harmonics, we find the following series representation in terms of Legendre polynomials

$$G(\Delta^*; \xi \cdot \eta) = \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{-n(n+1)} P_n(\xi \cdot \eta). \tag{4.8}$$

4.2 Space Regularized Green Function with Respect to the Beltrami Operator

In what follows we are interested in an appropriate 'space regularization' of the Green function. Our considerations are based on Taylor's formula

$$\ln(1-t) = \ln(1-t_0) - \frac{1}{1-t_0}(t-t_0) - \frac{1}{(1-t_0+\vartheta(t-t_0))^2}\frac{(t-t_0)^2}{2}, \quad (4.9)$$

$t_0 \in (-1,1)$, $\vartheta \in (0,1)$. In other words, by letting $1-t_0 = \rho$, $\rho > 0$, we find

$$\frac{1-t}{4\pi\rho} + \frac{1}{4\pi}(\ln\rho - \ln 2) \quad (4.10)$$

as linearization for

$$\frac{1}{4\pi}\ln(1-t) + \frac{1}{4\pi}(1-\ln 2) \quad (4.11)$$

in the (right) neighborhood of $t_0 = 1-\rho$, $\rho > 0$ (see Fig. 4.1).

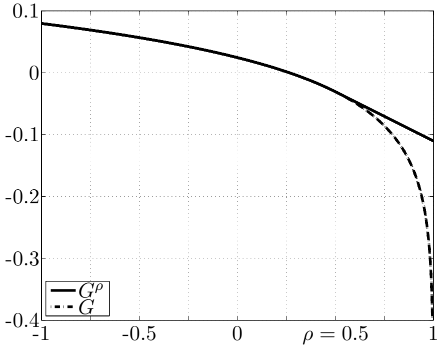


Fig. 4.1: The regularization of G by G^ρ ($\rho = 0.5$)

Keeping the linearization of (4.11) in mind, we next discuss the *(space) ρ-regularized Green function with respect to the Beltrami operator* Δ^*:

$$G^\rho(\Delta^*;\xi\cdot\eta) = \begin{cases} \frac{1}{4\pi}\ln(1-\xi\cdot\eta) + \frac{1}{4\pi}(1-\ln 2), & \xi\cdot\eta < 1-\rho \\ \frac{1-\xi\cdot\eta}{4\pi\rho} + \frac{1}{4\pi}(\ln\rho - \ln 2), & \xi\cdot\eta \geq 1-\rho. \end{cases} \quad (4.12)$$

Obviously, the kernel function $(\xi,\eta) \mapsto G^\rho(\Delta^*;\xi\cdot\eta)$ only depends on the inner product of ξ and η, hence $G^\rho(\Delta^*;\xi\cdot\eta)$ is a zonal function. According to its construction, $G^\rho(\Delta^*;\cdot\eta)$ is a continuously differentiable function on Ω

for every (fixed) $\eta \in \Omega$, $G^\rho(\Delta^*; \xi \cdot)$ is a continuously differentiable function on Ω for every (fixed) $\xi \in \Omega$, and we have

$$t \mapsto G^\rho(\Delta^*; t) = \begin{cases} \frac{1}{4\pi} \ln(1-t) + \frac{1}{4\pi}(1 - \ln 2), & t < 1-\rho \\ \frac{1-t}{4\pi\rho} + \frac{1}{4\pi}(\ln \rho - \ln 2), & t \geq 1-\rho \end{cases} \tag{4.13}$$

is a (one-dimensional) continuously differentiable function on the interval $[-1, 1]$.

The surface gradient of the ρ-regularized Green kernel is given by

$$\nabla^*_\xi G^\rho(\Delta^*; \xi \cdot \eta) = \begin{cases} -\frac{1}{4\pi} \frac{1}{1 - \xi \cdot \eta}(\eta - (\xi \cdot \eta)\xi), & \xi \cdot \eta < 1-\rho \\ -\frac{1}{4\pi\rho}(\eta - (\xi \cdot \eta)\xi), & \xi \cdot \eta \geq 1-\rho, \end{cases} \tag{4.14}$$

while the surface curl gradient of the ρ-regularized Green kernel reads as follows:

$$\mathrm{L}^*_\xi G^\rho(\Delta^*; \xi \cdot \eta) = \begin{cases} -\frac{1}{4\pi} \frac{1}{1 - \xi \cdot \eta}(\xi \wedge \eta), & \xi \cdot \eta < 1-\rho \\ -\frac{1}{4\pi\rho}(\xi \wedge \eta), & \xi \cdot \eta \geq 1-\rho. \end{cases} \tag{4.15}$$

For graphical illustration see Figs. 4.2 and 4.3. In addition, we mention the Beltrami derivative

$$\Delta^*_\xi G^\rho(\Delta^*; \xi \cdot \eta) = \begin{cases} -\frac{1}{4\pi}, & \xi \cdot \eta < 1-\rho \\ \frac{1}{2\pi\rho}(\xi \cdot \eta), & \xi \cdot \eta \geq 1-\rho. \end{cases} \tag{4.16}$$

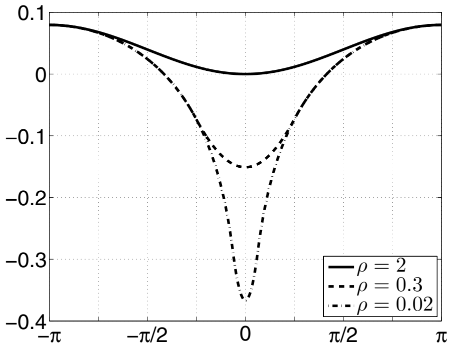


Fig. 4.2: ρ-regularized Green function $\vartheta \mapsto G^\rho(\Delta^*; \cos\ \vartheta)$ for various values ρ.

For $F \in \mathrm{C}(\Omega)$, we consider the potential $P^\rho(F)$ given by

$$P^\rho(F)(\xi) = \int_\Omega G^\rho(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) \tag{4.17}$$

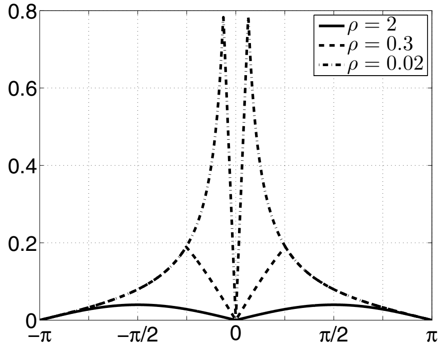


Fig. 4.3: Absolute value of the surface gradient or surface curl gradient of the ρ-regularized Green function $\vartheta \mapsto G^\rho(\Delta^*; \cos\ \vartheta)$ for various values ρ.

as regularized counterpart to the potential $P(F)$ given by

$$P(F)(\xi) = \int_\Omega G(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta), \xi \in \Omega. \tag{4.18}$$

We want to prove the following theorem.

Theorem 4.5. *For (sufficiently small) values $\rho > 0$ and $F \in \mathrm{C}(\Omega)$, the potential $P^\rho(F)$ is of class $\mathrm{C}^{(1)}(\Omega)$, and we have*

$$\lim_{\rho \to 0} \sup_{\xi \in \Omega} \left| \int_\Omega G(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) - \int_\Omega G^\rho(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) \right| = 0 \tag{4.19}$$

and

$$\lim_{\rho \to 0} \sup_{\xi \in \Omega} \left| \int_\Omega \nabla^*_\xi G(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) - \nabla^*_\xi \int_\Omega G^\rho(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) \right| = 0. \tag{4.20}$$

Furthermore,

$$\sup_{\xi \in \Omega} \left| \int_\Omega \nabla^*_\xi G(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) - \nabla^*_\xi \int_\Omega G(\Delta^*; \xi \cdot \eta) F(\eta)\ d\omega(\eta) \right| = 0. \tag{4.21}$$

Proof. First we are concerned with the existence of the occuring integrals. Since F is of class $\mathrm{C}(\Omega)$, we easily see that

$$\left| \int_\Omega \ln(1-\xi\cdot\eta) F(\eta)\, d\omega(\eta) \right| \leq 2\pi \|F\|_{\mathrm{C}(\Omega)} \int_{-1}^{1} \ln(1-t)\, dt < \infty \tag{4.22}$$

and

$$\left| \int_\Omega \nabla^*_\xi \ln(1-\xi\cdot\eta) F(\eta)\, d\omega(\eta) \right| \leq \|F\|_{\mathrm{C}(\Omega)} \int_\Omega \frac{\sqrt{1-(\xi\cdot\eta)^2}}{1-\xi\cdot\eta}\, d\omega(\eta) < \infty. \tag{4.23}$$

For $F \in \mathrm{C}(\Omega)$, $P^\rho(F)$ is of class $\mathrm{C}^{(1)}(\Omega)$. Thus it is clear that

$$\begin{aligned} \nabla^*_\xi P^\rho(F)(\xi) &= \nabla^*_\xi \int_\Omega G^\rho(\Delta^*;\xi\cdot\eta) F(\eta)\, d\omega(\eta) \quad (4.24)\\ &= \int_\Omega \nabla^*_\xi G^\rho(\Delta^*;\xi\cdot\eta) F(\eta)\, d\omega(\eta) \end{aligned}$$

with $G^\rho(\Delta^*;\xi\cdot\eta)$ and $G(\Delta^*;\xi\cdot\eta)$ differing only on the cap with center ξ, more precisely, on the set $\{\eta \in \Omega | 1-\xi\cdot\eta \leq \rho\}$. Thus, we obtain (for sufficiently small values $\rho > 0$)

$$\begin{aligned} &|P^\rho(F)(\xi) - P(F)(\xi)| \quad (4.25)\\ &\leq \frac{1}{4\pi} \|F\|_{\mathrm{C}(\Omega)} \int_{\xi\cdot\eta\geq 1-\rho} \left(|\ln(1-\xi\cdot\eta)| + \ln\rho + 1 + \left| \frac{1-\xi\cdot\eta}{\rho} \right| \right) d\omega(\eta)\\ &\leq \frac{1}{2} \|F\|_{\mathrm{C}(\Omega)} \int_{1-\rho}^{1} (|\ln(1-t)| + 2 + |\ln\rho|)\, dt. \end{aligned} \tag{4.26}$$

Consequently,

$$\sup_{\xi\in\Omega} |P^\rho(F)(\xi) - P(F)(\xi)| = O(\rho \ln\rho). \tag{4.27}$$

In other words, for all $\xi \in \Omega$ and $F \in \mathrm{C}(\Omega)$,

$$|P^\rho(F)(\xi) - P(F)(\xi)| = O(\rho\ln\rho), \quad \rho \to 0. \tag{4.28}$$

This proves the first assertion (4.19) of our theorem. In addition, we are able to verify that

$$\begin{aligned} &\left| \nabla^*_\xi P^\rho(F)(\xi) - \nabla^*_\xi P(F)(\xi) \right| \quad (4.29)\\ &\quad \leq 2\|F\|_{\mathrm{C}(\Omega)} \int_{\substack{\xi\cdot\eta\geq 1-\rho\\ \eta\in\Omega}} \left| \frac{\eta - (\xi\cdot\eta)\xi}{1-\xi\cdot\eta} \right| d\omega(\eta),\\ &\quad = 2\|F\|_{\mathrm{C}(\Omega)} \int_{\substack{\xi\cdot\eta\geq 1-\rho\\ \eta\in\Omega}} \frac{\sqrt{1-(\xi\cdot\eta)^2}}{1-\xi\cdot\eta}\, d\omega(\eta), \end{aligned}$$

i.e.,

$$\left|\nabla_\xi^* P^\rho(F)(\xi) - \nabla_\xi^* P(F)(\xi)\right| = O(\rho^{1/2}), \quad \rho \to 0. \tag{4.30}$$

Altogether, this yields the desired results stated in Theorem 4.5. □

Analogous to Theorem 4.5, we are able to formulate the following statement.

Theorem 4.6. *For (sufficiently small) values $\rho > 0$ and $F \in \mathrm{C}(\Omega)$, the potential $P^\rho(F)$ is of class $\mathrm{C}^{(1)}(\Omega)$, and we have*

$$\lim_{\rho\to 0} \sup_{\xi\in\Omega} \left| \int_\Omega G(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta) - \int_\Omega G^\rho(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta)\right| = 0 \tag{4.31}$$

and

$$\lim_{\rho\to 0} \sup_{\xi\in\Omega} \left| \int_\Omega \mathrm{L}_\xi^* G(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta) - \mathrm{L}_\xi^* \int_\Omega G^\rho(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta)\right| = 0. \tag{4.32}$$

Furthermore,

$$\sup_{\xi\in\Omega} \left| \int_\Omega \mathrm{L}_\xi^* G(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta) - \mathrm{L}_\xi^* \int_\Omega G(\Delta^*;\xi\cdot\eta)F(\eta)\ d\omega(\eta)\right| = 0. \tag{4.33}$$

Next, we determine the Legendre (Fourier) coefficients of the regularized Green function with respect to the Beltrami operator Δ^* (see S. Gramsch (2006)). The bilinear expansion of $G^\rho(\Delta^*;\xi\cdot\eta)$ reads as follows:

$$G^\rho(\Delta^*;\xi\cdot\eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (G^\rho)^\wedge(n) P_n(\xi\cdot\eta), \quad \xi,\eta\in\Omega, \tag{4.34}$$

where

$$(G^\rho)^\wedge(n) = 2\pi \int_{-1}^{1} G^\rho(\Delta^*;t)P_n(t)\ dt. \tag{4.35}$$

In accordance with the definition of the regularized Green function with respect to Δ^* we split the integral into two parts:

$$\begin{aligned}(G^\rho)^\wedge(n) &= 2\pi \int_{-1}^{1-\rho} \left(\frac{1}{4\pi}\ln(1-t) + \frac{1}{4\pi} - \frac{1}{4\pi}\ln 2\right) P_n(t)\ dt \\ &\quad + 2\pi \int_{1-\rho}^{1} \left(\frac{1-t}{4\pi\rho} + \frac{1}{4\pi}\ln\rho - \frac{1}{4\pi}\ln 2\right) P_n(t)\ dt\end{aligned}$$

For $n = 0, 1$ we immediately find

$$(G^\rho)^\wedge(0) = \frac{1}{4}\rho \tag{4.36}$$

and

$$(G^\rho)^\wedge(1) = -\frac{1}{2} + \frac{1}{4}\rho - \frac{1}{24}\rho^2. \tag{4.37}$$

For $n \geq 2$ we make some auxiliary considerations. From (3.188), we know for all $s \in (-1, 1)$ that

$$\int_{-1}^{s} P_n(t)\ dt = \frac{P_{n+1}(s) - P_{n-1}(s)}{2n+1}, \quad n \geq 1. \tag{4.38}$$

For the first derivative of the Legendre polynomials, we get the identity

$$\int_{-1}^{s} P_n'(t)\ dt = P_n(s) - (-1)^n, \quad n \geq 1, \tag{4.39}$$

since we know that $P_n(-1) = (-1)^n$. Furthermore, it follows that for $s \in (-1, 1)$

$$\int_{-1}^{s} tP_n'(t)\ dt = sP_n(s) + (-1)^n + \frac{P_{n-1}(s) - P_{n+1}(s)}{2n+1}, \quad n \geq 1. \tag{4.40}$$

Moreover, we have

$$\int_{-1}^{s} (t+1)P_n'(t)\ dt = (1+s)P_n(s) + \frac{P_{n-1}(s) - P_{n+1}(s)}{2n+1}, \quad n \geq 1. \tag{4.41}$$

Observing these identities, the first integral can be calculated in connection with (3.192) for values $s \in (-1, 1)$ as follows:

$$\begin{aligned}
&\int_{-1}^{s} \ln(1-t)P_n(t)\ dt \qquad (4.42)\\
&= \int_{-1}^{s} \ln(1-t)\frac{1}{2n+1}\left(\frac{d}{dt}\right)\left(P_{n+1}(t) - P_{n-1}(t)\right)\ dt\\
&= \frac{1}{2n+1}\left[\ln(1-t)(P_{n+1}(t) - P_{n-1}(t))\right]\Big|_{-1}^{s}\\
&\quad + \int_{-1}^{s} \frac{1}{1-t}\frac{1}{2n+1}\left(P_{n+1}(t) - P_{n-1}(t)\right)\ dt\\
&= \frac{1}{2n+1}\ln(1-s)\left(P_{n+1}(s) - P_{n-1}(s)\right) - \frac{1}{n(n+1)}\int_{-1}^{s}(t+1)P_n'(t)\ dt.
\end{aligned}$$

Inserting (4.41), we finally obtain for $s \in (-1, 1)$ and $n \geq 1$

$$\begin{aligned} & \int_{-1}^{s} \ln(1-t) P_n(t)\, dt \\ &= \left(\ln(1-s) + \frac{1}{n(n+1)} \right) \frac{P_{n+1}(s) - P_{n-1}(s)}{2n+1} - \frac{(1+s) P_n(s)}{n(n+1)}. \end{aligned} \tag{4.43}$$

For the second integral with $s \in (-1, 1)$, we are led to

$$\begin{aligned} & \int_{s}^{1} \left(\ln\rho - 1 + \frac{1-t}{\rho} \right) P_n(t)\, dt \\ &= (\ln\rho - 1) \int_{s}^{1} P_n(t)\, dt + \frac{1}{\rho} \int_{s}^{1} (1-t) P_n(t)\, dt. \end{aligned} \tag{4.44}$$

Moreover,

$$\int_{s}^{1} t P_n'(t)\, dt = 1 - s P_n(s) - \frac{P_{n-1}(s) - P_{n+1}(s)}{2n+1}, \quad n \geq 1, \tag{4.45}$$

and

$$\begin{aligned} & \int_{s}^{1} t P_n(t)\, dt \\ &= \frac{1}{2n+1} \int_{s}^{1} t \left(\frac{d}{dt} \right) (P_{n+1}(t) - P_{n-1}(t))\, dt \\ &= s \frac{P_{n-1}(s) - P_{n+1}(s)}{2n+1} - \frac{P_n(s) - P_{n+2}(s)}{(2n+1)(2n+3)} + \frac{P_{n-2}(s) - P_n(s)}{(2n+1)(2n-1)}, \end{aligned} \tag{4.46}$$

which shows us that

$$\begin{aligned} & \int_{s}^{1} (1-t) P_n(t)\, dt \\ &= (1-s) \frac{P_{n-1}(s) - P_{n+1}(s)}{2n+1} + \frac{P_n(s) - P_{n+2}(s)}{(2n+1)(2n+3)} - \frac{P_{n-2}(s) - P_n(s)}{(2n+1)(2n-1)}. \end{aligned} \tag{4.47}$$

Summarizing our results, we obtain the Legendre coefficients of the regularized Green function with respect to Δ^* as follows:

Lemma 4.7. *For $\rho \in (0, 2)$, we have*

$$G^{\rho}(\Delta^*; \xi \cdot \eta) = \sum_{k=0}^{\infty} \frac{2n+1}{4\pi} (G^{\rho})^{\wedge}(n) P_n(\xi \cdot \eta), \quad -1 \leq \xi \cdot \eta \leq 1,$$

with

$$\begin{aligned}
(G^\rho)^\wedge(0) &= \frac{1}{4}\rho, \\
(G^\rho)^\wedge(1) &= -\frac{1}{2} + \frac{1}{4}\rho - \frac{1}{24}\rho^2, \\
(G^\rho)^\wedge(n) &= \frac{P_{n+1}(1-\rho) - P_{n-1}(1-\rho)}{2n(n+1)(2n+1)} - \frac{(2-\rho)}{2n(n+1)} P_n(1-\rho) \\
&\quad + \frac{1}{2\rho}\frac{P_n(1-\rho) - P_{n+2}(1-\rho)}{(2n+1)(2n+3)} - \frac{1}{2\rho}\frac{P_{n-2}(1-\rho) - P_n(1-\rho)}{(2n+1)(2n-1)}, \\
&\qquad n = 2, 3, \ldots .
\end{aligned}$$

It should be noted that the Legendre coefficients of the regularized Green function with respect to Δ^* tend to the Legendre coefficients of the Green function with respect to Δ^* as $\rho \to 0$. In detail,

$$\lim_{\rho \to 0} (G^\rho)^\wedge(0) = 0, \tag{4.48}$$

and

$$\lim_{\rho \to 0} (G^\rho)^\wedge(1) = -\frac{1}{2}. \tag{4.49}$$

Observing the identity

$$P_n'(1) = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}_0, \tag{4.50}$$

we find, for $n \geq 2$

$$\lim_{\rho \to 0} \frac{P_{n+1}(1-\rho) - P_{n-1}(1-\rho)}{2n(n+1)(2n+1)} = 0, \tag{4.51}$$

$$\lim_{\rho \to 0} \frac{2-\rho}{2n(n+1)} P_n(1-\rho) = \frac{1}{n(n+1)}, \tag{4.52}$$

$$\lim_{\rho \to 0} \frac{1}{2\rho}\frac{P_n(1-\rho) - P_{n+2}(1-\rho)}{(2n+1)(2n+3)} = 0, \tag{4.53}$$

and

$$\lim_{\rho \to 0} \frac{1}{2\rho}\frac{P_{n-2}(1-\rho) - P_n(1-\rho)}{(2n+1)(2n-1)} = 0, \tag{4.54}$$

which gives us the limit relation

$$\lim_{\rho \to 0} (G^\rho)^\wedge(n) = -\frac{1}{n(n+1)} \tag{4.55}$$

for all integers $n \geq 1$.

4.3 Frequency Regularized Green Function with Respect to the Beltrami Operator

From (3.192), we obtain that for $t \in [-1, 1]$

$$(t^2 - 1)\frac{2k+1}{k(k+1)}P_k'(t) = P_{k+1}(t) - P_{k-1}(t). \tag{4.56}$$

Hence, for $t \neq 1$, a truncated series of the Green function, i.e., a frequency regularization of $G(\Delta^*; t)$, can be expressed as follows:

$$\begin{aligned} &\sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k'(t) \\ &= \frac{1}{4\pi}\frac{1}{1-t^2} \sum_{k=1}^{N} (P_{k+1}(t) - P_{k-1}(t)) \\ &= \frac{1}{4\pi}\frac{1}{1-t^2} \left(\sum_{k=1}^{N-2} P_{k+1}(t) + P_N(t) + P_{N+1}(t) \right) \\ &\quad - \frac{1}{4\pi}\frac{1}{1-t^2} \left(P_0(t) + P_1(t) + \sum_{k=3}^{N} P_{k-1}(t) \right) \\ &= -\frac{1}{4\pi}\frac{1}{1-t^2} (P_0(t) + P_1(t) - P_N(t) - P_{N+1}(t)) \\ &\quad - \frac{1}{4\pi}\frac{1}{1-t^2} \left(\sum_{k=2}^{N-1} P_k(t) - \sum_{k=2}^{N-1} P_k(t) \right). \end{aligned} \tag{4.57}$$

Since $P_0(t) = 1$ and $P_1(t) = t$ we get

$$\begin{aligned} &\sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k'(t) \\ &= -\frac{1}{4\pi}\frac{1}{(1-t)(1+t)} (1 + t - P_N(t) - P_{N+1}(t)) \\ &= -\frac{1}{4\pi}\frac{1}{1-t} + \frac{1}{4\pi}\frac{P_N(t) + P_{N+1}(t)}{1-t^2}. \end{aligned} \tag{4.58}$$

Integrating with respect to t, we find, for $-1 < t_0 \leq t < 1$, t_0 fixed,

$$\begin{aligned} &\sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k(t) - \sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k(t_0) \\ &= \frac{1}{4\pi}\ln(1-t) - \frac{1}{4\pi}\ln(1-t_0) + \frac{1}{4\pi}\int_{t_0}^{t} \frac{P_N(s) + P_{N+1}(s)}{1-s^2}\, ds + C_N(t_0). \end{aligned} \tag{4.59}$$

We choose

$$C_N(t_0) = \sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k(t_0) - \frac{1}{4\pi} \ln(1-t_0) \tag{4.60}$$

such that

$$\begin{aligned}\lim_{N\to\infty} C_N(0) &= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{-k(k+1)} P_k(0) \\ &= \lim_{N\to\infty} \sum_{l=1}^{\lfloor \frac{N}{2} \rfloor} \frac{4l+1}{4\pi} \frac{1}{-2l(2l+1)} \frac{(-1)^l}{4^l} \binom{2l}{l} \\ &= \frac{1}{4\pi}(1-\ln 2)\end{aligned}$$

(note that $P_{2n+1}(0) = 0$ and $P_{2n}(0) = \frac{(-1)^n}{4^n}\binom{2n}{n}$).

Summarizing our results, we obtain the following lemma.

Lemma 4.8. *For all $\xi, \eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$, the N-th frequency regularized Green function with respect to Δ^**

$$G^{(N)}(\Delta^*; \xi \cdot \eta) = -\sum_{k=1}^{N} \frac{2k+1}{4\pi} \frac{1}{k(k+1)} P_k(\xi \cdot \eta) \tag{4.61}$$

satisfies the equation

$$\begin{aligned}G(\Delta^*; \xi \cdot \eta) \quad - \quad & G^{(N)}(\Delta^*; \xi \cdot \eta) \\ = \quad & -\frac{1}{4\pi} \int_0^{\xi\cdot\eta} \frac{P_N(s) + P_{N+1}(s)}{1-s^2} \, ds \\ & + \frac{1}{4\pi}(1-\ln 2) + \sum_{l=1}^{\lfloor \frac{N}{2} \rfloor} \frac{4^l+1}{4\pi} \frac{1}{2l(2l+1)} \frac{(-1)^l}{4l} \binom{2l}{l}.\end{aligned}$$

Figures 4.4 and 4.5 give graphical impressions of the frequency regularized Green function.

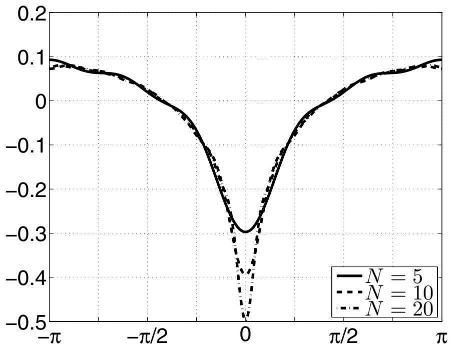


Fig. 4.4: Frequency regularized Green function $\vartheta \mapsto G^{(N)}(\Delta^*; \cos\ \vartheta)$ for various values N.

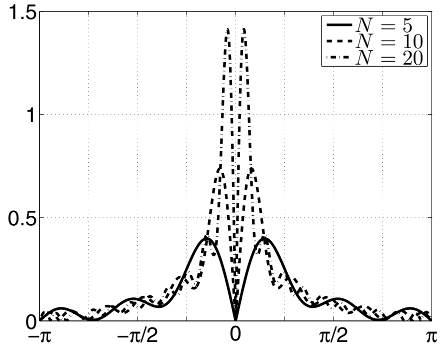


Fig. 4.5: Absolute value of the surface gradient or surface curl gradient of the frequency regularized Green function $\vartheta \mapsto G^{(N)}(\Delta^*; \cos\ \vartheta)$ for various values N.

From Lemma 3.42, we know that for all $s \in [0, t)$, $t < 1$, $|P_N(s)| = O(N^{-1/2})$, hence, it follows that

$$\lim_{N\to\infty} \int_0^{\xi\cdot\eta} \frac{P_N(s) + P_{N+1}(s)}{1 - s^2}\ ds = 0. \tag{4.62}$$

In particular, we are led to the following identities

$$\begin{aligned} \nabla^*_\xi G^{(N)}(\Delta^*; \xi\cdot\eta) &= -\frac{1}{4\pi}\frac{1}{1 - \xi\cdot\eta}(\eta - (\xi\cdot\eta)\xi) \qquad (4.63)\\ &+ \frac{1}{4\pi}\frac{P_N(\xi\cdot\eta) + P_{N+1}(\xi\cdot\eta)}{1 - (\xi\cdot\eta)^2}(\eta - (\xi\cdot\eta)\xi) \end{aligned}$$

and

$$\mathrm{L}^*_\xi G^{(N)}(\Delta^*;\xi\cdot\eta) = -\frac{1}{4\pi}\frac{1}{1-\xi\cdot\eta}(\xi\wedge\eta) + \frac{1}{4\pi}\frac{P_N(\xi\cdot\eta)+P_{N+1}(\xi\cdot\eta)}{1-(\xi\cdot\eta)^2}(\xi\wedge\eta). \tag{4.64}$$

4.4 Green's Functions with Respect to the Beltrami Operators $\partial_n = \Delta^* + n(n+1)$

A fundamental question when approximating a function on the sphere Ω by its truncated Fourier expansion (orthogonal expansion) in terms of spherical harmonics is the existence of a manageable error term. An adequate answer is the construction of integral formulas that will be presented later on. It will be shown that the error term between a function and its truncated Fourier series expansion is explicitly available in integral form provided that sufficient smoothness is imposed on the function under consideration. An essential tool for the integral formulas is the theory of Green's function on the unit sphere Ω with respect to the 'shifted' operators

$$\partial_n = \Delta^* - (\Delta^*)^\wedge(n), \quad (\Delta^*)^\wedge(n) = -n(n+1), \; n = 0,1,2,... \tag{4.65}$$

and their iterations (which later on turn out to play a fundamental role in the (Helmholtz) decomposition of both spherical vector and tensor fields).

We start our considerations with the introduction of Green functions with respect to the operators ∂_n, $n = 0, 1, \ldots$ (cf. W. Freeden (1979a)).

Definition 4.9. $G(\partial_n;\cdot,\cdot):(\xi,\eta)\mapsto G(\partial_n;\xi,\eta)$, $-1\le\xi\cdot\eta<1$, is called Green's function with respect to the operator $\partial_n = \Delta^* - (\Delta^*)^\wedge(n)$ if it satisfies the following properties:

(i) *(Differential equation)* for every point $\xi\in\Omega, \eta\mapsto G(\partial_n;\xi,\eta)$ is twice continuously differentiable on $\{\eta\in\Omega \mid -1\le\xi\cdot\eta<1\}$, and we have

$$(\partial_n)_\eta G(\partial_n;\xi,\eta) = -\frac{2n+1}{4\pi}P_n(\xi\cdot\eta), \quad -1\le\xi\cdot\eta<1,$$

where $(\partial_n)_\eta$ means that the operator ∂_n is applied to the variable η.

(ii) *(Characteristic singularity)* for every $\xi\in\Omega$, the function

$$\eta\mapsto G(\partial_n;\xi,\eta) - \frac{1}{4\pi}P_n(\xi\cdot\eta)\ln(1-\xi\cdot\eta)$$

is continuously differentiable on Ω.

(iii) *(Rotational symmetry)* for all orthogonal transformations $\mathbf{t}$,

$$G(\partial_n; \mathbf{t}\xi, \mathbf{t}\eta) = G(\partial_n; \xi, \eta).$$

(iv) *(Normalization)* for every $\xi \in \Omega$,

$$\int_\Omega G(\partial_n; \xi, \eta) P_n(\xi \cdot \eta) \, d\omega(\eta) = 0.$$

We prove the *uniqueness of Green's function* with respect to the operator ∂_n. The concept closely parallels the proof of Lemma 4.2.

Lemma 4.10. $G(\partial_n; \cdot, \cdot)$ *is uniquely determined by its defining properties* (i)*–*(iv)*.*

Proof. Denote by $D(\partial_n; \cdot, \cdot)$ the difference between two Green functions satisfying (i)–(iv). Then, we have the following properties:

(i) $D(\partial_n; \xi, \cdot)$ is twice continuously differentiable for all points $\eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$, and we have

$$(\partial_n)_\eta D(\partial_n; \xi, \eta) = (\Delta^*_\eta + n(n+1)) D(\partial_n; \xi, \eta) = 0, \tag{4.66}$$

(ii) $D(\partial_n; \xi, \cdot)$ is continuously differentiable for all $\eta \in \Omega$,

(iii) For all orthogonal transformations $\mathbf{t}$,

$$D(\partial_n; \mathbf{t}\xi, \mathbf{t}\eta) = D(\partial_n; \xi, \eta), \tag{4.67}$$

(iv) For all $\xi \in \Omega$

$$\int_\Omega D(\partial_n; \xi, \eta) P_n(\xi \cdot \eta) \, d\omega(\eta) = 0. \tag{4.68}$$

The properties (i)–(iii) show that $D(\partial_n; \xi, \cdot)$ is an everywhere on the unit sphere Ω infinitely often differentiable function satisfying the differential equation (i). Therefore $D(\partial_n; \xi, \cdot)$ must be a spherical harmonic of order n. $D(\partial_n; \xi, \eta)$ depends only on the scalar product of ξ and η, i.e.,

$$D(\partial_n; \xi, \eta) = \alpha_n P_n(\xi \cdot \eta). \tag{4.69}$$

From (iv) we obtain

$$\int_\Omega D(\partial_n; \xi, \eta) P_n(\xi \cdot \eta) \, d\omega(\eta) = \alpha_n \int_\Omega P_n(\xi \cdot \eta) P_n(\xi \cdot \eta) \, d\omega(\eta) = 0. \tag{4.70}$$

Hence, $\alpha_n = 0$. But this means that the Green function $G(\partial_n; \cdot, \cdot)$ is uniquely determined by the defining properties (i)–(iv). □

Remark 4.11. Because of (iii), $G(\partial_n;\cdot)$ is a zonal function.

Graphical impressions of the Green functions $G(\partial_n;\cdot,\cdot)$ can be found in Fig. 4.6.

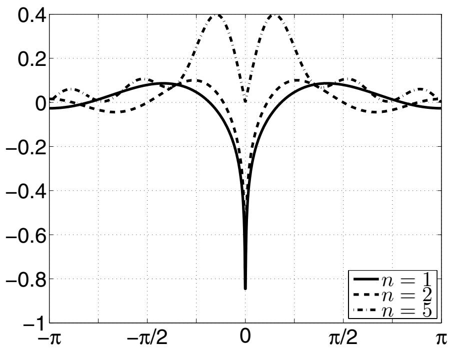


Fig. 4.6: The Green functions $\vartheta \mapsto G(\partial_n;\cos\vartheta)$ for various values n.

Observing the characteristic singularity of the Green function, we see by applying the Second Green Surface Theorem that the spherical harmonics of degree n, i.e., the eigenfunctions Y_n of the Beltrami operator Δ^* with respect to the eigenvalues $(\Delta^*)^\wedge(n),\ n = 0,1,...,$ are eigenfunctions of Green's (kernel) function $G(\partial_n;\cdot,\cdot)$ in the sense of the integral equation

$$\big(n(n+1) - k(k+1)\big)\int_\Omega G(\partial_n;\xi\cdot\eta)Y_k(\eta)\,d\omega(\eta) = (1-\delta_{nk})Y_k(\xi). \quad (4.71)$$

Furthermore, if $\xi,\eta \in \Omega$ with $-1 \leq \xi\cdot\eta < 1$, $G(\partial_n;\xi,\eta)$ allows the bilinear expansion

$$G(\partial_n;\xi\cdot\eta) = \sum_{(\partial_n)^\wedge(k)\neq 0} \frac{1}{(\partial_n)^\wedge(k)} \sum_{j=1}^{2k+1} Y_{k,j}(\xi)Y_{k,j}(\eta), \quad (4.72)$$

where $(\partial_n)^\wedge(k)$ is given by

$$(\partial_n)^\wedge(k) = (\Delta^*)^\wedge(k) - (\Delta^*)^\wedge(n) = -k(k+1) + n(n+1). \quad (4.73)$$

The symbol $\sum_{(\partial_n)^\wedge(k)\neq 0}$ means that the sum is to be extended over all non-negative integers k for which the denominator $(\partial_n)^\wedge(k)$ is different from zero, i.e., $(\Delta^*)^\wedge(k) \neq (\Delta^*)^\wedge(n)$, such that $k \neq n$. Using the addition theorem, we are able to rewrite the bilinear expansion of $G(\partial_n;\xi,\eta)$ in the form

$$G(\partial_n;\xi\cdot\eta) = \sum_{(\partial_n)^\wedge(k)\neq 0} \frac{2k+1}{4\pi}\left((\partial_n)^\wedge(k)\right)^{-1} P_k(\xi\cdot\eta). \quad (4.74)$$

This leads us to the formulation of the following result.

Lemma 4.12. *For* $n = 1, 2, \ldots$ *and* $\xi, \eta \in \Omega$ *with* $-1 \le \xi \cdot \eta < 1$

$$\begin{aligned} G(\partial_n; \xi \cdot \eta) &= \frac{1}{4\pi} P_n(\xi \cdot \eta) \ln(1 - \xi \cdot \eta) \\ &\quad - \frac{1}{2\pi} \sum_{k=0}^{n-1} \frac{2k+1}{(\partial_n)^\wedge(k)} P_k(\xi \cdot \eta) \\ &\quad - \frac{1}{4\pi} \Big(\frac{2n+1}{2} \int_{-1}^{1} P_n^2(t) \ln(1-t)\, dt \Big) P_n(\xi \cdot \eta). \end{aligned}$$

Proof. We have to show that $G(\partial_n; \cdot)$ satisfies the defining properties (i)–(iv). An easy calculation yields

$$\begin{aligned} \left((1-t^2) \left(\frac{d}{dt} \right)^2 - 2t \frac{d}{dt} + n(n+1) \right) P_n(t) \ln(1-t) \qquad (4.75) \\ = -2(1+t) P_n'(t) - P_n(t). \end{aligned}$$

In connection with Lemma 3.37, we therefore obtain

$$\begin{aligned} &\left((1-t^2) \left(\frac{d}{dt} \right)^2 - 2t \frac{d}{dt} + n(n+1) \right) (-P_n(t) \ln(1-t)) \qquad (4.76) \\ &+ \left((1-t^2) (\frac{d}{dt})^2 - 2t \frac{d}{dt} + n(n+1) \right) \left(-2 \sum_{k=0}^{n-1} \frac{2k+1}{n(n+1) - k(k+1)} P_k(t) \right) \\ &= 2(1+t) P_n'(t) + P_n(t) - 2 \sum_{k=0}^{n-1} (2k+1) P_k(t) \\ &= (2n+1) P_n(t). \end{aligned}$$

This shows that condition (i) is valid. Because of $P_n(1) = 1, n = 1, 2, \ldots,$ condition (ii) is certainly satisfied. Finally, it is not difficult to see that

$$\int_\Omega G(\partial_n; \xi \cdot \eta) P_n(\xi \cdot \eta)\, d\omega(\eta) = 0. \qquad (4.77)$$

□

4.5 Integral Formulas Involving Green's Function with Respect to the Beltrami Operator Δ^*

Next, we come to integral formulas on the unit sphere under explicit formulation of the remainder term between function value and integral (involving Green's function with respect to the Beltrami operator Δ^*).

Let ξ be a fixed point of the unit sphere Ω. Assume that F is continuously differentiable on Ω. Then, for each sufficiently small $\rho > 0$, the First Green Surface Theorem gives

$$\begin{aligned}\int_{\xi\cdot\eta\le 1-\rho,|\eta|=1} &\left\{F(\eta)\Delta^*_\eta G(\Delta^*;\xi\cdot\eta) + \nabla^*_\eta F(\eta)\cdot\nabla^*_\eta G(\Delta^*;\xi\cdot\eta)\right\}\, d\omega(\eta) \\ &= \int_{\xi\cdot\eta=1-\rho,|\eta|=1} F(\eta)\frac{\partial}{\partial\nu_\eta}G(\Delta^*;\xi\cdot\eta)\, d\sigma(\eta), \qquad (4.78)\end{aligned}$$

where ν is the unit normal to the circle consisting of all points $\eta \in \Omega$ with $\xi\cdot\eta = 1-\rho$, tangential to Ω, and directed exterior to the set of all points $\eta\in\Omega$ with $\xi\cdot\eta \le 1-\rho$. Explicitly, written out, we have

$$\nu_\eta = -(1-(\xi\cdot\eta)^2)^{-\frac{1}{2}}\eta\wedge(\eta\wedge\xi). \qquad (4.79)$$

In the identity (4.78), we first observe the differential equation of Green's function

$$\int_{\xi\cdot\eta\le 1-\rho,|\eta|=1} F(\eta)\Delta^*_\eta G(\Delta^*;\xi\cdot\eta)\, d\omega(\eta) = -\frac{1}{4\pi}\int_{\xi\cdot\eta\le 1-\rho,|\eta|=1} F(\eta)\, d\omega(\eta). \qquad (4.80)$$

By virtue of the logarithmic singularity of the Green function $G(\Delta^*;\cdot)$, we get in analogy to well known results of potential theory (cf. O.D. Kellogg (1929))

$$\begin{aligned}&\int_{\xi\cdot\eta=1-\rho,|\eta|=1} F(\eta)\frac{\partial}{\partial\nu_\eta}G(\Delta^*;\xi\cdot\eta)d\sigma(\eta) \qquad (4.81)\\ &= \int_{\xi\cdot\eta=1-\rho,|\eta|=1} F(\eta)\frac{\xi-(1-\rho)\eta}{\sqrt{1-(1-\rho)^2}}\cdot\left(-\frac{1}{4\pi\rho}(\xi-(1-\rho)\eta)\right)d\sigma(\eta)\\ &= -\frac{1}{4\pi}\int_{\xi\cdot\eta=1-\rho,|\eta|=1} F(\eta)\frac{\sqrt{1-(1-\rho)^2}}{\rho}d\sigma(\eta).\end{aligned}$$

From the Mean Value Theorem, we are able to deduce that

$$\begin{aligned}&-\frac{1}{4\pi}\frac{\sqrt{1-(1-\rho)^2}}{\rho}\int_{\xi\cdot\eta=1-\rho,|\eta|=1} F(\eta)\, d\sigma(\eta) \qquad (4.82)\\ &= -\frac{1}{4\pi}\frac{\sqrt{1-(1-\rho)^2}}{\rho}2\pi\sqrt{1-(1-\rho)^2}F(\eta_\rho)\\ &= -\frac{1}{2}(2-\rho)F(\eta_\rho)\end{aligned}$$

for some η_ρ lying on the circle $\{\eta\in\Omega \mid 1-\xi\cdot\eta = \rho\}$. The continuity of F yields $F(\eta_\rho)\to F(\xi)$ as $\eta_\rho\to\xi$ for $\rho\to 0$ such that

$$\lim_{\rho\to 0}\int_{1-\xi\cdot\eta\ge\rho,|\eta|=1} F(\eta)\frac{\partial}{\partial\nu_\eta}G(\Delta^*;\xi\cdot\eta)d\sigma(\eta) = -F(\xi). \qquad (4.83)$$

Summarizing our results, we therefore obtain the following result.

Theorem 4.13. *(Third Green Surface Theorem for ∇^*) Let ξ be a fixed point of the unit sphere Ω. Suppose that F is a continuously differentiable function on Ω. Then*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega F(\eta)\, d\omega(\eta) - \int_\Omega \big(\nabla^*_\eta G(\Delta^*; \xi \cdot \eta)\big) \cdot \big(\nabla^*_\eta F(\eta)\big)\, d\omega(\eta).$$

In the same way, we are able to formulate the following corollary.

Corollary 4.14. *(Third Green Surface Theorem for L^*) Under the assumptions of Theorem 4.13*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega F(\eta)\, d\omega(\eta) - \int_\Omega (\mathrm{L}^*_\eta G(\Delta^*; \xi \cdot \eta)) \cdot (\mathrm{L}^*_\eta F(\eta))\, d\omega(\eta).$$

Proof. Applying Green's surface identity for the operator L^*, we get for every (sufficiently small) $\rho > 0$ and $F \in \mathrm{C}^{(1)}(\Omega)$,

$$\begin{aligned} &\int_{\xi\cdot\eta\leq 1-\rho, |\eta|=1} \mathrm{L}^*_\eta F(\eta) \cdot \mathrm{L}^*_\eta G(\Delta^*; \xi \cdot \eta)\ d\omega(\eta) \\ &\qquad + \int_{\xi\cdot\eta\leq 1-\rho, |\eta|=1} F(\eta) \Delta^*_\eta G(\Delta^*; \xi \cdot \eta)\ d\omega(\eta) \\ &= \int_{\xi\cdot\eta = 1-\rho, |\eta|=1} F(\eta) \tau_\eta \cdot \mathrm{L}^*_\eta G(\Delta^*; \xi \cdot \eta)\ d\sigma(\eta), \end{aligned} \tag{4.84}$$

where τ is defined as the (unit) surface vector on Ω tangential to the circle $\{\eta \in \Omega \,|\, 1 - \xi \cdot \eta = \rho\}$. Explicitly, for $\eta \in \Omega$ with $1 - \xi \cdot \eta = \rho$, we have

$$\tau_\eta = (1 - (\xi \cdot \eta)^2)^{-1/2} \xi \wedge \eta. \tag{4.85}$$

Moreover, we know that

$$\mathrm{L}^*_\eta G(\Delta^*; \xi \cdot \eta) = -\frac{1}{4\pi} \frac{\eta \wedge \xi}{1 - \xi \cdot \eta}. \tag{4.86}$$

In other words, the same reasoning as in Theorem 4.13 guarantees Corollary 4.14. □

From Green's second surface identity, we get for each sufficiently small $\rho > 0$

$$\int_{\xi\cdot\eta\le 1-\rho,|\eta|=1} \{G(\Delta^*;\xi\cdot\eta)\Delta^*_\eta F(\eta) - F(\eta)\Delta^*_\eta G(\Delta^*;\xi\cdot\eta)\}\ d\omega(\eta)$$
$$= \int_{\xi\cdot\eta=1-\rho,|\eta|=1} \left\{G(\Delta^*;\xi\cdot\eta)\frac{\partial}{\partial\nu_\eta}F(\eta) - F(\eta)\frac{\partial}{\partial\nu_\eta}G(\Delta^*;\xi\cdot\eta)\right\} d\sigma(\eta), \tag{4.87}$$

provided that F is twice continuously differentiable on Ω. Observing the defining properties of the Green function with respect to Δ^*, we can use the same arguments as known from potential theory (cf. O.D. Kellogg (1929)). In fact, the continuous differentiability of F on Ω leads us to

$$\lim_{\rho\to 0}\int_{\xi\cdot\eta=1-\rho,|\eta|=1} G(\Delta^*;\xi\cdot\eta)\frac{\partial}{\partial\nu_\eta}F(\eta)d\sigma(\eta) = 0. \tag{4.88}$$

Together with (4.83), this shows us the following result (see W. Freeden (1979a)).

Theorem 4.15. *(Third Green Surface Theorem for $\partial_0 = \Delta^*$) Let ξ be a fixed point of the unit sphere Ω. Suppose that F is a twice continuously differentiable function on Ω. Then*

$$F(\xi) = \frac{1}{4\pi}\int_\Omega F(\eta)\,d\omega(\eta) + \int_\Omega G(\Delta^*;\xi\cdot\eta)(\Delta^*_\eta F(\eta))\,d\omega(\eta).$$

In other words, the Green theorems as stated above compare the value of a function at a point $\xi \in \Omega$ with the integral mean of F relative to the unit sphere Ω under explicit representation of the error term in integral form. Essential tool is the Green function with respect to the Beltrami operator Δ^*.

The Third Green Surface Theorem for $\partial_n = \Delta^* + n(n+1)$, can be formulated analogously to the case of the operator $\partial_0 = \Delta^*$.

Theorem 4.16. *(Third Green Surface Theorem for $\partial_n = \Delta^* + n(n+1)$). Let ξ be a fixed point of the unit sphere Ω. Suppose that F is of class $\mathrm{C}^{(2)}(\Omega)$. Then*

$$\begin{aligned} F(\xi) &= \frac{2n+1}{4\pi}\int_\Omega F(\eta)P_n(\xi\cdot\eta)\ d\omega(\eta) \\ &\quad + \int_\Omega G(\Delta^*+n(n+1);\xi\cdot\eta)(\Delta^*_\eta + n(n+1))F(\eta)\ d\omega(\eta). \end{aligned}$$

In order to complete our consideration, we finally mention the Green function with respect to $\Delta^* + \lambda$ for $\lambda \neq n(n+1)$, $n \in \mathbb{N}_0$:

Definition 4.17. For $\lambda \neq -(\Delta^*)^\wedge(n)$, $(\Delta^*)^\wedge(n) = -n(n+1)$, $n \in \mathbb{N}_0$, Green's function

$$G(\Delta^* + \lambda; \cdot, \cdot) : (\xi, \eta) \mapsto G(\Delta^* + \lambda; \xi, \eta),$$

is defined by the following properties:

(i) *(Differential equation)* for every point $\xi \in \Omega$, $n \mapsto G(\Delta^* + \lambda; \xi, \eta)$ is twice continuously differentiable on the set

$$\{\eta \in \Omega | -1 \leq \xi \cdot \eta < 1\},$$

and we have

$$(\Delta^*_\eta + \lambda) G(\Delta^* + \lambda; \xi, \eta) = 0, \qquad -1 \leq \xi \cdot \eta < 1.$$

(ii) *(Characteristic singularity)* for every $\xi \in \Omega$,

$$G(\Delta^* + \lambda; \xi, \eta) = O(\ln(1 - \xi \cdot \eta)).$$

(iii) *(Rotational symmetry)* for all orthogonal transformations $\mathbf{t}$

$$G(\Delta^* + \lambda; \mathbf{t}\xi, \mathbf{t}\eta) = G(\Delta^* + \lambda; \xi, \eta).$$

Without proof, we list the following properties: $G(\Delta^* + \lambda; \cdot, \cdot)$ is uniquely determined by its defining properties. Its bilinear expansion reads as follows

$$\begin{aligned} G(\Delta^* + \lambda; \xi, \eta) &= G(\Delta^* + \lambda; \xi \cdot \eta) \\ &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{2n+1}{\lambda - n(n+1)} P_n(\xi \cdot \eta), \quad -1 \leq \xi \cdot \eta < 1. \end{aligned}$$

Corollary 4.18. *For $F \in \mathrm{C}^{(2)}(\Omega)$ and all $\lambda \in \mathbb{R}$ with $\lambda \neq n(n+1)$, $n \in \mathbb{N}_0$, the integral equation*

$$F(\xi) = \int_\Omega G(\Delta^* + \lambda; \xi \cdot \eta)(\Delta^*_\eta + \lambda) F(\eta) \, d\omega(\eta) \tag{4.89}$$

is valid.

In other words, Corollary (4.18) does not establish a canonical relationship between functional value and integral expression. Nevertheless, Green's theorem shows us (cf. W. Freeden (1981a)) that for all $\lambda \neq n(n+1), n = 0, 1, \ldots,$

$$\begin{aligned} F(\xi) &= \frac{1}{4\pi} \int_\Omega F(\eta)\, d\omega(\eta) \\ &+ \int_\Omega G_0\left(\Delta^* + \lambda; \xi \cdot \eta\right) \left(\Delta^*_\eta + \lambda\right) F(\eta)\, d\omega(\eta), \end{aligned} \tag{4.90}$$

where we have used the abbreviation

$$G_0(\Delta^* + \lambda; \xi \cdot \eta) = \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{\lambda - n(n+1)} P_n(\xi \cdot \eta), \quad -1 \leq \xi \cdot \eta < 1. \tag{4.91}$$

Consequently, for all values $\lambda \in \mathbb{R}$ (even for the case $G(\partial_n; \cdot)$ with $n > 0$), we are able to compare a functional value at $\xi \in \Omega$ and the mean integral value of a twice continuously differentiable function F on Ω under explicit knowledge of the remainder term in integral form.

4.6 Differential Equations Involving Green's Functions with Respect to the Beltrami Operator Δ^*

Combining Theorem 4.13 and observing the surface gradient of the Green function $G(\Delta^*; \cdot, \cdot)$, we obtain the following theorem.

Theorem 4.19. *(Differential Equation for ∇^* on Ω) Let $v : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field on Ω with $\xi \cdot v(\xi) = 0, \mathrm{L}^*_\xi \cdot v(\xi) = 0$, $\xi \in \Omega$. Then*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega \frac{1}{1 - \xi \cdot \eta} (\xi - (\xi \cdot \eta)\eta) \cdot v(\eta)\, d\omega(\eta) \tag{4.92}$$

is the uniquely determined solution of the differential equation

$$\nabla^*_\xi F(\xi) = v(\xi), \quad \xi \in \Omega, \tag{4.93}$$

satisfying

$$\frac{1}{4\pi} \int_\Omega F(\eta)\, d\omega(\eta) = 0. \tag{4.94}$$

Analogously, we get the following result on the differential equation of the surface curl gradient.

Theorem 4.20. *(Differential Equation for* L^* *on* Ω*) Let* $v : \Omega \to \mathbb{R}^3$ *be a continuously differentiable vector field on* Ω *with* $\xi \cdot v(\xi) = 0$ $\nabla^*_\xi \cdot v(\xi) = 0$*,* $\xi \in \Omega$*. Then*

$$F(\xi) = \frac{1}{4\pi} \int_\Omega \frac{1}{1 - \xi \cdot \eta} (\eta \wedge \xi) \cdot v(\eta) \, d\omega(\eta) \tag{4.95}$$

is the uniquely determined solution of the differential equation

$$\mathrm{L}^*_\xi F(\xi) = v(\xi), \quad \xi \in \Omega, \tag{4.96}$$

satisfying

$$\frac{1}{4\pi} \int_\Omega F(\xi) \, d\omega(\eta) = 0. \tag{4.97}$$

From Theorem 4.15, we are able to verify the following result on the Beltrami differential equation.

Theorem 4.21. *(Differential Equations for* Δ^* *on* Ω*) Let* H *be a continuous function on* Ω *with*

$$\frac{1}{4\pi} \int_\Omega H(\xi) \, d\omega(\xi) = 0. \tag{4.98}$$

Let $F \in \mathrm{C}^{(2)}(\Omega)$ *satisfy the Beltrami differential equation*

$$\Delta^* F = H \tag{4.99}$$

such that

$$\frac{1}{4\pi} \int_\Omega F(\xi) \, d\omega(\xi) = 0. \tag{4.100}$$

Then

$$F(\xi) = \int_\Omega G(\Delta^*; \xi \cdot \eta) H(\eta) \, d\omega(\eta), \quad \xi \in \Omega. \tag{4.101}$$

Remark 4.22. The '*surface gradient equation*' $\nabla^* F = v$ is of particular importance in physical geodesy for determining the geoid undulations from deflections of the vertical (see W. Freeden, M. Schreiner (2006), T. Fehlinger et al. (2007b), W. Freeden, K. Wolf (2008), and the references therein), while the 'surface curl gradient equation' $\mathrm{L}^* F = w$ occurs in ocean circulation for characterizing geostrophic flow W. Freeden et al. (2005), T. Fehlinger et al. (2007a). For more details the reader is referred to Chapter 10. The Beltrami differential equation of Theorem 4.21 plays a particular role in the Helmholtz decomposition theorems for spherical vector and tensor fields (see Sections 5.2 and 6.5).

4.7 Approximate Integration and Spline Interpolation

By virtue of the Cauchy–Schwarz inequality, we get from Theorem 4.15

$$\begin{aligned}\left| F(\xi) - \frac{1}{4\pi}\int_\Omega F(\eta)\, d\omega(\eta)\right| & \qquad (4.102)\\ \leq \left(G\left((\Delta^*)^2;\xi\cdot\xi\right)\right)^{1/2}\left(\int_\Omega |F(\eta)|^2\, d\omega(\eta)\right)^{1/2} & \\ = \left(G((\Delta^*)^2;1)\right)^{1/2}\left(\int_\Omega |F(\eta)|^2\, d\omega(\eta)\right)^{1/2} &\end{aligned}$$

for all $\xi \in \Omega$, where $G((\Delta^*)^2;\cdot) : (\xi,\eta) \mapsto G((\Delta^*)^2;\xi\cdot\eta)$, $\xi,\eta\in\Omega$, is defined by convolution

$$G((\Delta^*)^2;\xi\cdot\eta) = \int_\Omega G(\Delta^*;\xi\cdot\zeta)G(\Delta^*;\zeta\cdot\eta)\, d\omega(\zeta). \qquad (4.103)$$

Obviously, the bilinear series reads

$$\begin{aligned} G((\Delta^*)^2;\xi\cdot\eta) & \qquad (4.104)\\ = \sum_{n=1}^\infty\sum_{k=1}^\infty \frac{2n+1}{4\pi}\frac{2k+1}{4\pi}\frac{1}{-k(k+1)}\frac{1}{-n(n+1)}\int_\Omega P_n(\xi\cdot\zeta)P_k(\zeta\cdot\eta)\, d\omega(\zeta) & \\ = \sum_{n=1}^\infty \frac{2n+1}{4\pi}\frac{1}{(-n(n+1))^2}P_n(\xi\cdot\eta). &\end{aligned}$$

Moreover, it follows that, for all $\xi,\eta\in\Omega$ with $-1\leq\xi\cdot\eta<1$

$$\Delta^*_\eta G((\Delta^*)^2;\xi\cdot\eta) = G(\Delta^*;\xi\cdot\eta).$$

Next, we are concerned with the explicit calculation of the iterated Green function $G((\Delta^*)^2;\cdot) : t\mapsto G((\Delta^*)^2;t)$, $t\in[-1,1]$ given by

$$G((\Delta^*)^2;t) = \frac{1}{4\pi}\sum_{k=1}^\infty \frac{2k+1}{(-k(k+1))^2}P_k(t). \qquad (4.105)$$

First, it is not hard to see

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2}. \qquad (4.106)$$

This gives us for all $\xi\in\Omega$

$$\begin{aligned} G((\Delta^*)^2;\xi\cdot\xi) = G((\Delta^*)^2;1) &= \frac{1}{4\pi}\left(\sum_{k=1}^\infty\frac{1}{k^2} - \sum_{k=1}^\infty\frac{1}{k^2} + 1\right) \qquad (4.107)\\ &= \frac{1}{4\pi}. \end{aligned}$$

Moreover, from the well-known result $\sum_{k=1}^{\infty}(-1)^k k^{-2} = -\pi^2/12$, we obtain

$$G((\Delta^*)^2;-1) = \frac{1}{4\pi} - \frac{\pi}{24}. \tag{4.108}$$

For the explicit calculation of $G((\Delta^*)^2;t)$ $t \in (-1,1)$, we use the differential equation

$$\begin{aligned} \frac{d}{dt}(1-t^2)\frac{d}{dt}G((\Delta^*)^2;t) &= G(\Delta^*;t) \\ &= \frac{1}{4\pi}\ln(1-t) + \frac{1}{4\pi}(1-\ln 2). \end{aligned} \tag{4.109}$$

First, for all $t \in (-1,1)$, we get by elementary manipulations

$$\begin{aligned} \int_{-1}^{t} \frac{d}{dt}(1-x^2)\frac{d}{dx}G((\Delta^*)^2;x)\ dx &= (1-t^2)\frac{d}{dt}G((\Delta^*)^2;t) \\ &= -\frac{1}{4\pi}(1-t)\ln(1-t) + \frac{\ln 2}{4\pi}(1-t). \end{aligned} \tag{4.110}$$

Second, for all $t \in (-1,1)$, we find

$$\int_{t}^{1} \frac{d}{dx}G((\Delta^*)^2;x)\ dx = G((\Delta^*)^2;1) - G((\Delta^*)^2;t) \tag{4.111}$$

$$= -\frac{1}{4\pi}\int_{t}^{1}\frac{1}{1+x}\ln(1-x)\ dx + \frac{\ln 2}{4\pi}\int_{t}^{1}\frac{1}{1+x}\ dx.$$

Substituting $1-x=u$ we are led to

$$-\frac{1}{4\pi}\int_{t}^{1}\frac{1}{1+x}\ln(1-x)\ dx = \frac{1}{4\pi}\int_{1-t}^{0}\frac{1}{2-u}\ln u\ du. \tag{4.112}$$

From a table of integrals (see, e.g., W. Gröbner, N. Hofreiter (1975)), we borrow

$$\int \frac{1}{2-u}\ln u\ du = -\ln u \ln\frac{2-u}{2} - \mathcal{L}_2(\frac{u}{2}) \quad (+C), \tag{4.113}$$

where $\mathcal{L}_2$ defines the so-called *dilogarithm*

$$\mathcal{L}_2(u) = -\int_0^u \frac{\ln(1-v)}{v}\ dv = \sum_{k=1}^{\infty}\frac{u^k}{k^2}. \tag{4.114}$$

It follows that the iterated Green function is expressible by means of the dilogarithm

$$\underbrace{G((\Delta^*)^2;1)}_{=\frac{1}{4\pi}} - G((\Delta^*)^2;t) = \frac{1}{4\pi}\left(\ln(1-t)(\ln(1+t)-\ln 2)\right) + \frac{1}{4\pi}\mathcal{L}_2\left(\frac{1-t}{2}\right) + \frac{(\ln 2)^2}{4\pi} - \frac{\ln 2}{4\pi}\ln(1+t). \tag{4.115}$$

Note that

$$\lim_{\substack{t\to 1\\ t<1}} \left(\ln(1-t)\left(\ln(1+t)-\ln 2\right)\right) = 0 \tag{4.116}$$

and

$$\lim_{\substack{t\to -1\\ t>-1}} \left(\ln(1-t)\left(\ln(1+t)-\ln 2\right) - \ln 2\ln(1+t)\right) = -(\ln 2)^2. \tag{4.117}$$

This finally gives us the following representation.

Lemma 4.23. *For $t \in (-1,1)$,*

$$\begin{aligned} G((\Delta^*)^2;t) &= \frac{1}{4\pi} - \frac{1}{4\pi}\ln(1-t)\left(\ln(1+t)-\ln 2\right) \\ &- \frac{1}{4\pi}\mathcal{L}_2\left(\frac{1-t}{2}\right) - \frac{(\ln 2)^2}{4\pi} + \frac{\ln 2}{4\pi}\ln(1+t), \end{aligned} \tag{4.118}$$

where

$$\mathcal{L}_2\left(\frac{1-t}{2}\right) = \sum_{k=1}^{\infty}\left(\frac{1-t}{2}\right)^k \frac{1}{k^2} \tag{4.119}$$

and

$$\lim_{\substack{t\to 1\\ t<1}} G((\Delta^*)^2;t) = G((\Delta^*)^2;1) = \frac{1}{4\pi}, \tag{4.120}$$

$$\lim_{\substack{t\to -1\\ t>-1}} G((\Delta^*)^2;t) = G((\Delta^*)^2;-1) = \frac{1}{4\pi} - \frac{\pi}{24}. \tag{4.121}$$

Summarizing our results, we therefore obtain the following corollary.

Corollary 4.24. *The Green function $G((\Delta^*)^2;\cdot) : (\xi,\eta) \mapsto G((\Delta^*)^2;\xi\cdot\eta)$, $\xi,\eta \in \Omega$, with respect to the operator $(\Delta^*)^2$ is continuous, and we have*

$$G((\Delta^*)^2;\xi\cdot\eta) = \sum_{k=1}^{\infty}\frac{2k+1}{4\pi}\frac{1}{(-k(k+1))^2}P_k(\xi\cdot\eta) \tag{4.122}$$

$$= \begin{cases} \frac{1}{4\pi} & , \quad 1-\xi\cdot\eta = 0 \\ -\frac{1}{4\pi}\ln(1-\xi\cdot\eta)\ln(1+\xi\cdot\eta) & \\ +\frac{\ln 2}{4\pi}\ln(1-(\xi\cdot\eta)^2) - \frac{1}{4\pi}\mathcal{L}_2\left(\frac{1-\xi\cdot\eta}{2}\right) & \\ +\frac{1}{4\pi}(1-(\ln 2)^2) & , \quad 1\pm\xi\cdot\eta \neq 0 \\ \frac{1}{4\pi} - \frac{\pi}{24} & , \quad 1+\xi\cdot\eta = 0. \end{cases}$$

A graphical impression of the iterated Green function is given by Figure 4.7.

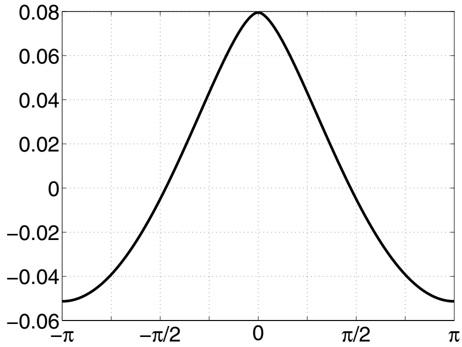


Fig. 4.7: The iterated Green function $\vartheta \mapsto G((\Delta^*)^2;\cos\vartheta)$.

It should be mentioned that the integral formulas with respect to Δ^* can be used for *approximate integration methods* on the sphere Ω. We give a concrete application: If $a_1,\ldots,a_N \in \mathbb{R}$ satisfy $\sum_{k=1}^N a_k = 1$, we have for all nodes $\eta_1,\ldots,\eta_N \in \Omega$,

$$\sum_{k=1}^{N} a_k F(\eta_k) - \frac{1}{4\pi}\int_\Omega F(\eta)\, d\omega(\eta) \tag{4.123}$$

$$= \int_\Omega \sum_{k=1}^{N} a_k G(\Delta^*;\eta_k\cdot\eta)\Delta^*_\eta F(\eta)\, d\omega(\eta),$$

provided that F is twice continuously differentiable on Ω.

Using the Cauchy–Schwarz inequality, we get from (4.123)

$$\left|\frac{1}{4\pi}\int_\Omega F(\eta)\, d\omega(\eta) - \sum_{k=1}^N a_k F(\eta_k)\right| \tag{4.124}$$
$$\leq \left(\sum_{p=1}^N\sum_{q=1}^N a_p a_q G((\Delta^*)^2;\eta_p\cdot\eta_q)\right)^{1/2}\left(\int_\Omega (\Delta^*_\eta F(\eta))^2\, d\omega(\eta)\right)^{1/2}.$$

Thus, the "*best approximation*" formula corresponding to the given nodes $\eta_1,\ldots,\eta_N$

$$\sum_{k=1}^N \mathring{a}_k F(\eta_k) \tag{4.125}$$

to the integral

$$\frac{1}{4\pi}\int_\Omega F(\eta)\, d\omega(\eta) \tag{4.126}$$

is the solution of the quadratic optimization problem:

$$\sum_{p=1}^N\sum_{q=1}^N \mathring{a}_p\mathring{a}_q G\left((\Delta^*)^2;\eta_p\cdot\eta_q\right) \rightarrow \min. \tag{4.127}$$

under the constraints

$$\sum_{k=1}^N \mathring{a}_k = 1. \tag{4.128}$$

Therefore, it is not difficult to show in accordance with Lagrange's method of multipliers, that the solution $(\mathring{a}_1,\ldots,\mathring{a}_N)^T$ of the best approximation formula can be obtained by solving the linear system

$$\begin{array}{ccccccccc}
G((\Delta^*)^2;\eta_1\cdot\eta_1)\mathring{a}_1 & + & \ldots & + & G((\Delta^*)^2;\eta_N\cdot\eta_1)\mathring{a}_N & - & \mathring{\lambda} & = & 0\\
\vdots & & & & \vdots & & \vdots & & \vdots\\
G((\Delta^*)^2;\eta_1\cdot\eta_N)\mathring{a}_1 & + & \ldots & + & G((\Delta^*)^2;\eta_N\cdot\eta_N)\mathring{a}_N & - & \mathring{\lambda} & = & 0\\
\\
\mathring{a}_1 & + & \ldots & + & \mathring{a}_N & & & = & 1
\end{array} \tag{4.129}$$

such that

$$\mathring{\lambda} = \sum_{p=1}^N\sum_{q=1}^N G((\Delta^*)^2,\eta_p\cdot\eta_q)\mathring{a}_p\,\mathring{a}_q. \tag{4.130}$$

In other words, the linear system does not only provide the coefficients $\mathring{a}_1,\ldots,\mathring{a}_N$ of the best approximation, but also the accuracy of the integration formula via the Lagrange multiplier (4.130)

$$\left|\frac{1}{4\pi}\int_\Omega F(\eta)\, d\omega(\eta) - \sum_{k=1}^N \mathring{a}_k F(\eta_k)\right| \leq \mathring{\lambda}^{1/2}\left(\int_\Omega \left(\Delta^*_\eta F(\eta)\right)^2\, d\omega(\eta)\right)^{1/2}. \tag{4.131}$$

Similar integration procedures can be formulated for all differential operators $\Delta^* + \lambda, \lambda \in \mathbb{R}$. This fact can be used to adapt the specific properties of a function under consideration to an operator $\Delta^* + \lambda$ in order to minimize the remainder terms in numerical integration formulas (see, e.g., W. Freeden (1981a), W. Freeden, J. Fleck (1987)).

Next, we deal with *spherical splines*: If $a_1, \ldots, a_N \in \mathbb{R}$ satisfy $\sum_{k=1}^N a_k = 0$ and if $\eta_1, \ldots, \eta_N$ are prescribed nodes on Ω, then

$$\int_\Omega \Delta^*_\eta S(\eta)\ \Delta^*_\eta F(\eta)\ d\omega(\eta) = \sum_{k=1}^N a_k F(\eta_k), \tag{4.132}$$

where $S : \Omega \to \mathbb{R}$ is given by

$$S(\eta) = C_{0,1} Y_{0,1}(\xi) + \sum_{k=1}^N a_k G((\Delta^*)^2; \eta_k, \xi). \tag{4.133}$$

Moreover, we have

$$\int_\Omega \Delta^*_\eta S(\eta)\ \Delta^*_\eta S(\eta)\ d\omega(\eta) = \sum_{k=1}^N a_k S(\eta_k). \tag{4.134}$$

Let $\alpha_1, \ldots, \alpha_N$ be given real values. Then, there exists one and only one function S of type (4.133) satisfying $\Sigma_{k=1}^N a_k = 0$ such that $S(\eta_i) = \alpha_i$, $i = 1, \ldots, N$. We denote this function by S_N.

Now, for all $F \in \mathrm{C}^{(2)}(\Omega)$ with $F(\eta_i) = \alpha_i$, $i = 1, \ldots, N$, we find

$$\begin{aligned} &\int_\Omega \Delta^*_\eta (S_N(\eta) - F(\eta)) \Delta^*_\eta (S_N(\eta) - F(\eta))\ d\omega(\eta) \\ &= \sum_{k=1}^N a_k \alpha_k - 2 \sum_{k=1}^N a_k \alpha_k + \int_\Omega (\Delta^*_\eta F(\eta))^2\ d\omega(\eta). \end{aligned} \tag{4.135}$$

Therefore we find

$$\int_\Omega (\Delta^*_\eta F(\eta))^2 d\omega(\eta) = \int_\Omega (\Delta^*_\eta S_N(\eta))^2 d\omega(\eta) + \int_\Omega (\Delta^*_\xi (S_N(\eta) - F(\eta)))^2 d\omega(\eta). \tag{4.136}$$

Expressed in terms of the 'bending energy'

$$\mathcal{E}(F) = \int_\Omega (\Delta^*_\xi F(\eta))^2\ d\omega(\eta) \tag{4.137}$$

we obtain

$$\mathcal{E}(S_N) \le \mathcal{E}(F) \tag{4.138}$$

for all $F \in \mathrm{C}^{(2)}(\Omega)$ satisfying $F(\eta_i) = \alpha_i$, $i = 1, \ldots, N$. In other words, S_N is the *interpolating spline* (to the data points $(\eta_k, \alpha_k), k = 1, \ldots, N$) with smallest 'energy' (relative to Δ^*).

The integral (4.137) may be physically interpreted (at least in linearized sense under some simplifying assumptions) as the bending energy of a (thin) membrane spanned wholly over the (unit) sphere, F denotes the deflection normal to the rest position supposed, of course, to be spherical. This physical model is suggested by the classical interpretation of the one-dimensional integral $\int_a^b |F''(x)|^2 \, dx$ as the potential energy of a statically deflected thin beam which indeed is proportional to the integral taken over the square of the (linearized) curvature of the elastica of the beam.

Next, we explain the intimate relationship between best approximate and spline integration. In fact, if $\mathring{a}_1, \ldots, \mathring{a}_N$ solve the linear system (4.129), we see that

$$\begin{aligned} \frac{1}{4\pi} \int_\Omega S(\xi) \, d\omega(\xi) &= \frac{1}{\sqrt{4\pi}} \sum_{k=1}^N \mathring{a}_k C_{0,1} \qquad (4.139) \\ &= \sum_{k=1}^N \mathring{a}_k (C_{0,1} Y_{0,1}(\eta_k) + \sum_{r=1}^N a_r G((\Delta^*)^2; \eta_r, \eta_k)) \\ &= \sum_{k=1}^N \mathring{a}_k S(\eta_k) \end{aligned}$$

holds for all splines S of the form (4.133). In other words, the best approximation to the integral is precisely the unique approximation that is exact for spline functions.

4.8 Integral Formulas with Respect to Iterated Beltrami Operators $\partial_{0,\ldots,m} = \partial_0 \ldots \partial_m$

According to the classical Fredholm–Hilbert theory of linear integral equations (see, e.g., R. Courant, D. Hilbert (1924)), we inductively define the Green functions with respect to iterated operators

$$\partial_{0,\ldots,m} = \partial_0 \ldots \partial_m, \qquad (4.140)$$

where

$$\partial_n = \Delta^* - (\Delta^*)^\wedge(n), \quad n = 0, \ldots, m, \qquad (4.141)$$

by the following convolutions

$$G(\partial_{0,\ldots,m};\xi\cdot\eta) = \int_\Omega G(\partial_{0,\ldots,m-1};\xi\cdot\zeta)G(\partial_m;\zeta\cdot\eta)\,d\omega(\zeta),\ m=1,2\ldots, \quad (4.142)$$

$$G(\partial_0;\xi\cdot\eta) = G(\Delta^*;\xi\cdot\eta). \quad (4.143)$$

$G(\partial_{0,\ldots,m};\cdot,\cdot) : \Omega\times\Omega \to \mathbb{R}$ is called *Green's function with respect to the operator* $\partial_{0,\ldots,m}$. An illustration of $G(\partial_{0,\ldots,m},\cdot,\cdot)$ is given in Fig. 4.8

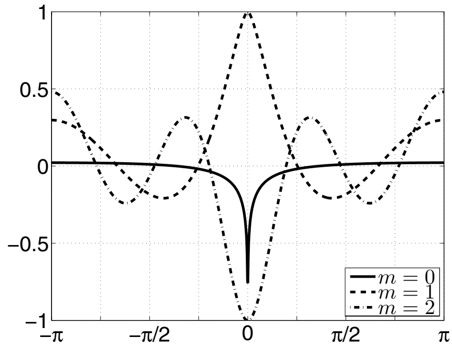


Fig. 4.8: Green's functions $\vartheta\mapsto G(\partial_{0,\ldots,m};\cos\vartheta)$ for $m=0,\ 1,\ 2$ (normalized).

For later use (more precisely, for the Helmholtz theorem involving spherical tensor fields) we are interested in deriving the Green function with respect to $\partial_{0,\ldots,1}=\partial_0\partial_1=\Delta^*(\Delta^*+2)$. First we have

$$\begin{aligned} G(\partial_0\partial_1;\xi\cdot\eta) &= \sum_{k=2}^{\infty}\frac{2k+1}{4\pi}\frac{1}{-k(k+1)(-k(k+1)+2)}P_k(\xi\cdot\eta) \\ &= \frac{1}{4\pi}\sum_{k=2}^{\infty}\frac{2k+1}{k(k+1)(k-1)(k+2)}P_k(\xi\cdot\eta). \end{aligned} \quad (4.144)$$

Considering the derivatives of the sum

$$G(t) = \sum_{k=2}^{\infty}\frac{2k+1}{(k-1)k(k+1)(k+2)}P_k(t) \quad (4.145)$$

we obtain in a first step in connection with Lemma 3.199

$$\begin{aligned} G'(t) &= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k'(t) \\ &= \frac{1}{t^2-1} \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} \frac{k(k+1)}{2k+1} \left(P_{k+1}(t) - P_{k-1}(t)\right) \\ &= \frac{1}{t^2-1} \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+2)} \left(P_{k+1}(t) - P_{k-1}(t)\right). \end{aligned} \tag{4.146}$$

By index shifts, we are able to see that

$$\begin{aligned} G'(t) &= \frac{1}{t^2-1} \left(\sum_{k=3}^{\infty} \frac{1}{(k-2)(k+1)} P_k(t) - \sum_{k=1}^{\infty} \frac{1}{k(k+3)} P_k(t) \right) \\ &= \frac{1}{t^2-1} \left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k(t) - \frac{1}{4} t - \frac{1}{10} \left(\frac{3}{2} t^2 - \frac{1}{2} \right) \right). \end{aligned} \tag{4.147}$$

Another differentiation yields the expression

$$\begin{aligned} G''(t) &= \frac{-2t}{(t^2-1)^2} \left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k(t) - \frac{1}{4} t - \frac{1}{10} \left(\frac{3}{2} t^2 - \frac{1}{2} \right) \right) \\ &\quad + \frac{1}{t^2-1} \left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k'(t) - \frac{1}{4} - \frac{3}{10} t \right). \end{aligned} \tag{4.148}$$

The second sum can be transformed by use of the recurrence relation (Lemma 3.199) as follows

$$\begin{aligned} &\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k'(t) \\ &= \frac{1}{t^2-1} \sum_{k=3}^{\infty} \frac{2}{(k-2)(k+3)} \left(P_{k+1}(t) - P_{k-1}(t)\right) \\ &= \frac{1}{t^2-1} \left(\sum_{k=4}^{\infty} \frac{4(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t) - \frac{1}{3} P_2(t) - \frac{1}{7} P_3(t) \right). \end{aligned} \tag{4.149}$$

This provides us with the following representation of the second derivative

$$\begin{aligned} G''(t) &= \frac{1}{(t^2-1)^2} \left(\sum_{k=3}^{\infty} \frac{-4(2k+1)}{(k-2)k(k+1)(k+3)} t P_k(t) + \frac{1}{2} t^2 + \frac{1}{5} t P_2(t) \right) \\ &\quad + \frac{1}{(t^2-1)^2} \left(\sum_{k=4}^{\infty} \frac{4(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t) - \frac{1}{3} P_2(t) \right. \\ &\qquad \left. - \frac{1}{7} P_3(t) - \frac{1}{4}(t^2-1) - \frac{3}{10} t (t^2-1) \right). \end{aligned} \tag{4.150}$$

Now, in the first sum, the three-term-recurrence relation $(2k+1)tP_k(t) = (k+1)P_{k+1}(t) + kP_{k-1}(t)$ of the Legendre polynomials P_n can be applied leading to the series

$$\begin{aligned}
&\sum_{k=3}^{\infty} \frac{2k+1}{(k-2)k(k+1)(k+3)} tP_k(t) \qquad (4.151)\\
&\quad = \sum_{k=3}^{\infty} \frac{1}{(k-2)k(k+1)(k+3)} \left((k+1)P_{k+1}(t) + kP_{k-1}(t)\right)\\
&\quad = \sum_{k=4}^{\infty} \frac{2k+1}{(k-3)(k-1)(k+2)(k+4)} P_k(t) + \frac{1}{24}P_2(t) + \frac{1}{70}P_3(t).
\end{aligned}$$

Summarizing our results, we therefore have the second derivative given by

$$\begin{aligned}
&G''(t) \qquad (4.152)\\
&= \frac{1}{(t^2-1)^2}\Bigg(-4\sum_{k=4}^{\infty} \frac{2k+1}{(k-3)(k-1)(k+2)(k+4)} P_k(t)\\
&\qquad -\frac{1}{6}P_2(t) - \frac{2}{35}P_3(t) + \frac{1}{2}t^2 + \frac{1}{5}tP_2(t)\\
&\qquad +4\sum_{k=4}^{\infty} \frac{(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t)\\
&\qquad -\frac{1}{3}P_2(t) - \frac{1}{7}P_3(t) - \frac{1}{4}(t^2-1) - \frac{3}{10}t(t^2-1)\Bigg)\\
&= \frac{1}{(t^2-1)^2}\left(-\frac{1}{2}(t-1)(t+1)^2\right)\\
&= \frac{1}{2}\frac{1}{1-t}.
\end{aligned}$$

This enables us to establish an elementary representation of $G(t)$ by integration. To this end, we need certain values of $G(t)$ to determine the constants of integration. In fact, we have

$$\begin{aligned}
G(1) &= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k(-1) \qquad (4.153)\\
&= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} (-1)^k = \frac{1}{6}
\end{aligned}$$

and

$$G'(1) = \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k'(-1) \tag{4.154}$$
$$= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} \frac{k(k+1)}{2} (-1)^{k+1}$$
$$= \sum_{k=2}^{\infty} \frac{2k+1}{2(k-1)(k+2)} (-1)^{k+1}$$
$$= -\frac{5}{12}.$$

Thus ordinary integration shows us that

$$G'(t) = -\frac{1}{2} \ln(1-t) + \frac{1}{2} \ln(2) - \frac{5}{12} \tag{4.155}$$

such that

$$G(t) = \frac{1}{2} \ln(1-t)(1-t) - \frac{1}{2} + \frac{1}{2} t + \frac{1}{2} \ln(2) t - \frac{5}{12} t \tag{4.156}$$
$$- \frac{1}{2} \ln(2) + 1 - \frac{5}{12} + \frac{1}{6}.$$

Altogether we get the following result.

Lemma 4.25. *The Green function $G(\partial_0 \partial_1; \cdot)$ with respect to the operator $\partial_0 \partial_1$ is continuous for all $(\xi, \eta) \in \Omega^2$. Its explicit representation reads*

$$G(\partial_0 \partial_1; \xi \cdot \eta) = \frac{1}{8\pi}(1 - \xi \cdot \eta) \ln(1 - \xi \cdot \eta) + \left(\frac{1}{12} + \frac{\ln(2)}{2} \right) \frac{\xi \cdot \eta}{4\pi} + \frac{1}{4\pi} \left(\frac{1}{4} - \frac{\ln(2)}{2} \right).$$

For positive integers m, $G(\partial_{0,\ldots,m}; \xi \cdot \eta)$ is continuous on the whole sphere Ω as a function of η with ξ fixed, or as a function of ξ with η fixed. On the other hand, the bilinear expansion of $G(\partial_{0,\ldots,m}; \xi \cdot \eta), m \geq 1$,

$$\sum_{k=m+1}^{\infty} \sum_{l=1}^{2k+1} \frac{1}{(\partial_{0,\ldots,m})^\wedge(k)} Y_{k,l}(\xi) Y_{k,l}(\eta) = \frac{1}{4\pi} \sum_{k=m+1}^{\infty} \frac{2k+1}{(\partial_{0,\ldots,m})^\wedge(k)} P_k(\xi \cdot \eta) \tag{4.157}$$

with

$$(\partial_{0,\ldots,m})^\wedge(k) = (\partial_0)^\wedge(k) \cdot \ldots \cdot (\partial_m)^\wedge(k) \tag{4.158}$$

is absolutely and uniformly convergent both in ξ and η respectively and uniformly in ξ and η together. Hence, the representation theorem of the theory of orthogonal expansions (see Theorem 3.55) yields

$$G(\partial_{0,\ldots,m}; \xi \cdot \eta) = \frac{1}{4\pi} \sum_{k=m+1}^{\infty} \frac{2k+1}{(\partial_{0,\ldots,m})^\wedge(k)} P_k(\xi \cdot \eta). \tag{4.159}$$

Let m be an integer with $m \geq 2$. Then the derivative

$$(\partial_2 \cdot \ldots \cdot \partial_m)_\eta \; G(\partial_{0,\ldots,m}; \xi \cdot \eta) = G(\partial_0 \partial_1; \xi \cdot \eta), \quad -1 \leq \xi \cdot \eta \leq 1, \quad (4.160)$$

as a function of η for fixed ξ is a continuous function on Ω. For integers $m \geq 1$, the derivative

$$(\partial_1 \cdot \ldots \cdot \partial_m)_\eta \; G(\partial_{0,\ldots,m}; \xi \cdot \eta) = G(\partial_0; \xi \cdot \eta), \quad -1 \leq \xi \cdot \eta < 1, \quad (4.161)$$

as a function of η possesses a logarithmic singularity in $\xi \in \Omega$. These properties are of basic interest in spherical spline settings corresponding to iterated Beltrami derivatives (see W. Freeden (1981a), W. Freeden et al. (1998)).

Observing the fact that the pth convolution of the Green function with respect to ∂_n coincides with the Green function with respect to ∂_n^p, i.e.,

$$G^{(p)}(\partial_n; \xi \cdot \eta) = G((\partial_n)^p; \xi \cdot \eta), \; p \geq 1, n \geq 0, \quad (4.162)$$

we obtain more generally

$$F(\xi) = \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\xi \cdot \eta) \, d\omega(\eta) + \int_\Omega G(\partial_n^p; \xi \cdot \eta)((\partial_n^p)_\eta F(\eta)) \, d\omega(\eta) \quad (4.163)$$

provided that F is of class $\mathrm{C}^{(2p)}(\Omega)$, where

$$G(\partial_n^p; \xi \cdot \eta) = \frac{1}{4\pi} \sum_{(\partial_n)^\wedge(k) \neq 0} \frac{2k+1}{((\partial_n)^\wedge(k))^p} P_k(\xi \cdot \eta). \quad (4.164)$$

Hence, we are able to compare a function $F \in \mathrm{C}^{(2p)}(\Omega)$ with the nth degree term of its orthogonal expansion in terms of spherical harmonics.

By use of the Green function $G(\partial_0 \partial_1; \cdot, \cdot)$ with respect to the operator $\partial_0 \partial_1$, we are able to generalize the second fundamental theorem. Observing the recursion property

$$(\partial_1)_\eta G(\partial_0 \partial_1; \xi \cdot \eta) = G(\partial_0; \xi \cdot \eta) - \frac{3}{4\pi (\partial_0)^\wedge(1)} P_1(\xi \cdot \eta) \quad (4.165)$$

we obtain

$$\begin{aligned} &\int_\Omega G(\partial_0; \xi \cdot \eta)\big((\partial_0)_\eta F(\eta)\big) \, d\omega(\eta) \\ &\quad = \int_\Omega (\partial_1)_\eta G(\partial_0 \partial_1; \xi \cdot \eta)\big((\partial_0)_\eta F(\eta)\big) \, d\omega(\eta) \\ &\qquad + \frac{3}{4\pi (\partial_0)^\wedge(1)} \int_\Omega P_1(\xi \cdot \eta)\big((\partial_0)_\eta F(\eta)\big) \, d\omega(\eta). \end{aligned} \quad (4.166)$$

Integration by parts, i.e., application of the Second Green Surface Theorem, yields for a function $F \in \mathrm{C}^{(4)}(\Omega)$

$$\begin{aligned} &\int_\Omega (\partial_1)_\eta G(\partial_0\partial_1; \xi\cdot\eta)\big((\partial_0)_\eta F(\eta)\big)\, d\omega(\eta) \\ &\quad = \int_\Omega G(\partial_0\partial_1; \xi\cdot\eta)\big((\partial_0\partial_1)_\eta F(\eta)\big)\, d\omega(\eta). \end{aligned} \tag{4.167}$$

In the same way, we get

$$\frac{3}{4\pi(\partial_0)^\wedge(1)} \int_\Omega P_1(\xi\cdot\eta)\left((\partial_0)_\eta F(\eta)\right)\, d\omega(\eta) = \frac{3}{4\pi}\int_\Omega P_1(\xi\cdot\eta)F(\eta)\, d\omega(\eta). \tag{4.168}$$

Therefore, we have

$$\begin{aligned} &\int_\Omega G(\partial_0; \xi\cdot\eta)\big((\partial_0)_\eta F(\eta)\big)\, d\omega(\eta) \\ &\quad = \int_\Omega G(\partial_0\partial_1; \xi\cdot\eta)\big((\partial_0\partial_1)_\eta F(\eta)\big)\, d\omega(\eta) + \frac{3}{4\pi}\int_\Omega F(\eta)P_1(\xi\cdot\eta)\, d\omega(\eta) \end{aligned} \tag{4.169}$$

provided that F is a four times continuously differentiable function on Ω. Thus, by combination of Theorem 4.15 and (4.169), we have, for all functions, $F \in \mathrm{C}^{(4)}(\Omega)$

$$\begin{aligned} F(\xi) &= \sum_{n=0}^{1} \frac{2n+1}{4\pi}\int_\Omega F(\eta)P_n(\xi\cdot\eta)\, d\omega(\eta) \\ &\quad + \int_\Omega G(\partial_0\partial_1; \xi\cdot\eta)\big((\partial_0\partial_1)_\eta F(\eta)\big)\, d\omega(\eta). \end{aligned} \tag{4.170}$$

More generally, by successive integration by parts, we obtain in connection with the definition of $G(\partial_{0,\ldots,m}; \cdot, \cdot)$ the following integral formulas.

Theorem 4.26. *Let m be a non-negative integer and ξ be a fixed point of the unit sphere Ω. Let F be a $(2m+2)$–times continuously differentiable function on Ω. Then*

$$\begin{aligned} F(\xi) &= \sum_{n=0}^{m} \frac{2n+1}{4\pi}\int_\Omega F(\eta)P_n(\xi\cdot\eta)\, d\omega(\eta) \\ &\quad + \int_\Omega G(\partial_{0,\ldots,m}; \xi\cdot\eta)\big((\partial_{0,\ldots,m})_\eta F(\eta)\big)\, d\omega(\eta). \end{aligned} \tag{4.171}$$

If F is $(2m+1)$–times continuously differentiable on Ω, then

$$\begin{aligned} F(\xi) &= \sum_{n=0}^{m} \frac{2n+1}{4\pi}\int_\Omega F(\eta)P_n(\xi\cdot\eta)\, d\omega(\eta) \\ &\quad - \int_\Omega \big(\nabla^*_\eta G(\partial_{0,\ldots,m}; \xi\cdot\eta)\big)\cdot\big(\nabla^*_\eta(\partial_{1,\ldots,m})_\eta F(\eta)\big)\, d\omega(\eta), \end{aligned} \tag{4.172}$$

where

$$\nabla^*_\eta G(\partial_{0,\ldots,m};\xi\cdot\eta) = \Big(\frac{1}{4\pi}\sum_{k=m+1}^{\infty}\frac{2k+1}{(\partial_{0,\ldots,m})^\wedge(k)}P'_k(\xi\cdot\eta)\Big)(\xi-(\xi\cdot\eta)\eta).$$

Analogously, if F is $(2m+1)$–times continuously differentiable on Ω, then

$$\begin{aligned} F(\xi) &= \sum_{n=0}^{m}\frac{2n+1}{4\pi}\int_\Omega F(\eta)P_n(\xi\cdot\eta)\,d\omega(\eta) \\ &\quad - \int_\Omega \mathrm{L}^*_\eta G(\partial_{0,\ldots,m};\xi\cdot\eta)\cdot \mathrm{L}^*_\eta(\partial_{1,\ldots,m})_\eta F(\eta)\,d\omega(\eta), \end{aligned}$$

where

$$\mathrm{L}^*_\eta G(\partial_{0,\ldots,m};\xi\cdot\eta) = \left(\frac{1}{4\pi}\sum_{k=m+1}^{\infty}\frac{2k+1}{(\partial_{0,\ldots,m})^\wedge(k)}P'_k(\xi\cdot\eta)\right)(\xi\wedge\eta).$$

Inserting the addition theorem of spherical harmonics, we find from (4.171) for all functions $F\in \mathrm{C}^{(2m+2)}(\Omega)$

$$F(\xi) = \sum_{n=0}^{m}\sum_{j=1}^{2n+1} Y_{n,j}(\xi)F^\wedge(n,j) \tag{4.173}$$

$$+\int_\Omega G(\partial_{0,\ldots,m};\xi\cdot\eta)\big((\partial_{0,\ldots,m})_\eta F(\eta)\big)\,d\omega(\eta). \tag{4.174}$$

This formula gives a comparison between the mth partial sum of the Fourier expansion of F into spherical harmonics and the functional value of F with explicit knowledge of the remainder term.

More general, by iterated application of the Second Green Surface Theorem, we obtain

$$\begin{aligned} F(\xi) &= \sum_{n=0}^{m}\sum_{j=1}^{2n+1} Y_{n,j}(\xi)F^\wedge(n,j) \qquad (4.175)\\ &\quad + \int_\Omega G(\partial_0^{p_0}\cdot\ldots\partial_m^{p_m};\xi\cdot\eta)((\partial_0^{p_0}\cdot\ldots\partial_m^{p_m})F(\eta))\,d\omega(\eta) \end{aligned}$$

provided that $p_l\geq 1,\ l=0,\ldots,m$, and F is $2(p_0+\ldots+p_m)$–times continuously differentiable on Ω.

The identity (4.171) can be written as follows:

$$\begin{aligned} &F(\xi) - \sum_{n=0}^{m}\sum_{j=1}^{2n+1} Y_{n,j}(\xi)F^\wedge(n,j) \\ &\quad = \int_\Omega (\partial_{0,\ldots,m})_\eta G(\partial^2_{0,\ldots,m};\xi\cdot\eta)\big((\partial_{0,\ldots,m})_\eta F(\eta)\big)\,d\omega(\eta), \qquad (4.176) \end{aligned}$$

where $\partial^2_{0,\ldots,m} = \partial_{0,\ldots,m}\partial_{0,\ldots,m}$ and $G(\partial^2_{0,\ldots,m};\cdot,\cdot)$ is given by the convolution integral (for a graphical impression, see Fig. 4.9)

$$\begin{aligned} G(\partial^2_{0,\ldots,m};\xi\cdot\eta) &= G^{(2)}(\partial_{0,\ldots,m};\xi\cdot\eta) \\ &= \int_\Omega G(\partial_{0,\ldots,m};\xi\cdot\zeta)G(\partial_{0,\ldots,m},\zeta\cdot\eta)\,d\omega(\zeta). \end{aligned} \tag{4.177}$$

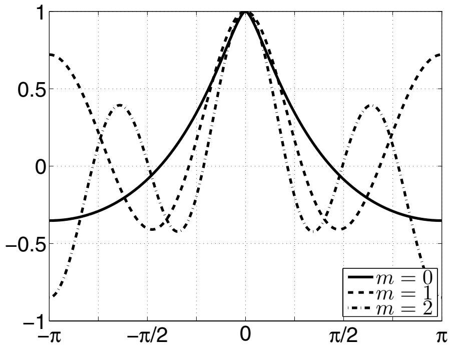


Fig. 4.9: Green's functions $\vartheta \mapsto G(\partial^2_{0,\ldots,m};\cos\vartheta)$ for $m = 0,\ 1,\ 2$ (normalized).

Clearly, $G(\partial^2_{0,\ldots,m};\cdot,\cdot)$ allows the bilinear expansion

$$G(\partial^2_{0,\ldots,m};\xi\cdot\eta) = \frac{1}{4\pi}\sum_{k=m+1}^{\infty}\frac{2k+1}{(\partial^2_{0,\ldots,m})^\wedge(k)}P_k(\xi\cdot\eta), \quad \xi,\eta\in\Omega, \tag{4.178}$$

with

$$(\partial^2_{0,\ldots,m})^\wedge(k) = ((\partial_{0,\ldots,m})^\wedge(k))^2. \tag{4.179}$$

From (4.176) we obtain, for example,

$$\begin{aligned} &\left\| F - \sum_{n=0}^{m}\sum_{j=1}^{2n+1}(F,Y_{n,j})_{\mathrm{L}^2(\Omega)}Y_{n,j}\right\|_{\mathrm{L}^2(\Omega)} \\ &\quad\leq \left(\frac{1}{4\pi}\sum_{k=m+1}^{\infty}\frac{2k+1}{(\partial^2_{0,\ldots,m})^\wedge(k)}\right)^{\frac{1}{2}} \|(\partial_{0,\ldots,m})F\|_{\mathrm{L}^2(\Omega)}. \end{aligned} \tag{4.180}$$

We omit $\mathrm{L}^p(\Omega)$-estimates for $p\neq 2$ and $\mathrm{C}(\Omega)$-estimates.

4.9 Differential Equations Involving Green's Function with Respect to Iterated Beltrami Operators $\partial_{0,\ldots,m} = \partial_0 \ldots \partial_m$

According to our nomenclature, $\mathrm{Harm}_{0,\ldots,m}$, $m \geq 0$, denotes the space of all spherical harmonics of degree $\leq m$ so that in the sense of the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$

$$\mathrm{Harm}_{0,\ldots,m} = \bigoplus_{l=0}^{m} \mathrm{Harm}_l. \tag{4.181}$$

Consequently, $\mathrm{Harm}_{0,\ldots,m}$ possesses the dimension

$$M = d(\mathrm{Harm}_{0,\ldots,m}) = \sum_{j=0}^{m} d(\mathrm{Harm}_j) = \sum_{j=0}^{m} (2j+1) = (m+1)^2. \tag{4.182}$$

The space $\mathrm{Harm}_{0,\ldots,m}$ equipped with the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Omega)}$ is an M-dimensional Hilbert space with the reproducing kernel $K_{\mathrm{Harm}_{0,\ldots,m}}(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}$ given by

$$\begin{aligned} K_{\mathrm{Harm}_{0,\ldots,m}}(\xi \cdot \eta) &= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) \\ &= \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta). \end{aligned} \tag{4.183}$$

Note that the recursion relation

$$(n+1)(P_{n+1}(t) - P_n(t)) - n(P_n(t) - P_{n-1}(t)) = (2n+1)(t-1)P_n(t) \tag{4.184}$$

implies

$$(\xi \cdot \eta - 1) K_{\mathrm{Harm}_{0,\ldots,m}}(\xi \cdot \eta) = \frac{m+1}{4\pi} \left(P_{m+1}(\xi \cdot \eta) - P_m(\xi \cdot \eta) \right) \tag{4.185}$$

for all $(\xi, \eta) \in \Omega \times \Omega$.

Let Y be an element of class $\mathrm{Harm}_{0,\ldots,m}$ of spherical harmonics of degree $\leq m$:

$$Y(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} Y^{\wedge}(n,j) Y_{n,j}(\eta), \quad \eta \in \Omega. \tag{4.186}$$

Then, observing the differential equation

$$(\partial_n)_\eta Y_{n,j}(\eta) = 0, \quad \eta \in \Omega, \tag{4.187}$$

for $n = 0, ..., m,\ j = 1, ..., 2n + 1$ we have

$$\begin{aligned}(\partial_{0,...,m})_\eta Y(\eta) &= (\partial_0 \cdot ... \cdot \partial_m)_\eta Y(\eta) \\ &= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} Y^\wedge(n,j)(\partial_0 \cdot ... \cdot \partial_m)_\eta Y_{n,j}(\eta) \\ &= 0 \qquad (4.188)\end{aligned}$$

for all $\eta \in \Omega$. On the other hand, we know from (4.174) that any solution $Y \in \mathrm{C}^{(2m+2)}(\Omega)$ of the homogeneous differential equation

$$(\partial_{0,...,m})_\eta Y(\eta) = 0, \quad \eta \in \Omega, \qquad (4.189)$$

is representable in the form

$$Y(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} Y^\wedge(n,j) Y_{n,j}(\eta), \quad \eta \in \Omega. \qquad (4.190)$$

But this means that $\mathrm{Harm}_{0,...,m}$ is the null space of the operator $\partial_{0,...,m}$.

For given $H \in \mathrm{C}(\Omega)$, the integral formula (Theorem 4.26) can be used to discuss the general differential equation

$$(\partial_{0,...,m})_\eta F(\eta) = H(\eta), \quad \eta \in \Omega. \qquad (4.191)$$

By virtue of the Green surface identity, we first see that

$$\int_\Omega \big((\partial_{0,..,m})_\eta F(\eta)\big) Y(\eta)\, d\omega(\eta) = \int_\Omega \big((\partial_{0,..,m})_\eta Y(\eta)\big) F(\eta)\, d\omega(\eta) = 0 \quad (4.192)$$

for all elements $Y \in \mathrm{Harm}_{0,...,m}$. From the considerations given above, it is clear that any function $Y \in \mathrm{Harm}_{0,...,m}$ can be added to F without changing the differential equation (4.191). However, if we require that F is orthogonal to the null space $\mathrm{Harm}_{0,...,m}$ of $\partial_{0,..,m}$, then the differential equation is uniquely solvable. This finally leads us to the following result.

Theorem 4.27. *Let H be a continuous function on Ω orthogonal to* $\mathrm{Harm}_{0,...,m}$, *i.e.,*

$$\int_\Omega H(\eta) Y_{n,j}(\eta)\, d\omega = 0 \qquad (4.193)$$

for $n = 0, ..., m,\ j = 1, ..., 2n + 1$. Then the function F given by

$$F(\xi) = \int_\Omega G(\partial_{0,..,m}; \xi \cdot \eta) H(\eta)\, d\omega(\eta),\ \xi \in \Omega, \qquad (4.194)$$

represents the only $(2m + 2)$*–times continuously differentiable solution of the differential equation* (4.191) *on Ω which is orthogonal to* $\mathrm{Harm}_{0,...,m}$.

This result turns out to be useful for the decomposition of spherical vector and tensor field into normal and tangential components.

Of course, spline (integration) methods can be introduced in the same way for the operators $\partial_{0,\ldots,m}$ as instead of the Beltrami operator (see W. Freeden (1981c)).

4.10 Bibliographical Notes

The Green functions with respect to iterated Beltrami operators have been investigated in detail by W. Freeden (1978); W. Freeden (1979a); W. Freeden (1980b), G.E. Backus et al. (1996) for the three-dimensional case, and R. Reuter (1982) in the multi-dimensional case. The integral formulas are due to W. Freeden (1978); W. Freeden (1979a); W. Freeden (1980b, 1981a). Space regularizations of Green functions and their use in (multiscale approximation of) geodetic problems can be found in a note due to W. Freeden, M. Schreiner (2006), T. Fehlinger et al. (2007a), T. Fehlinger et al. (2007b), and W. Freeden, K. Wolf (2008). The explicit representation stated in Lemma 4.25 was presented by W. Freeden, M. Gutting (2008). Related formulas in Euclidean spaces $\mathbb{R}^n$ are multidimensional Euler summation and cubature formulas (cf. W. Freeden, J. Fleck (1987) and the references therein). Green functions with respect to iterated Beltrami operators are essential tools in the Helmholtz decomposition theorems for spherical vector and tensor fields (see W. Freeden et al. (1998) and the considerations given in Chapters 5 and 6).

5 Vector Spherical Harmonics

Various applications imply different formulations of the definition of vector spherical harmonics, putting the accent on different issues. What is our understanding in this context? One important aspect is the easy transition from scalar spherical harmonics to the vectorial ones. A simple approach is to formulate the vectorial problem in terms of cartesian components. However, we already know that this procedure leads back to anisotropic scalar component equations, so that the physical relevance is difficult to realize, the mathematical formulation is lengthy, and the numerical modeling often becomes too complicated. In a large number of physically motivated applications, it turns out that a separation into normal and tangential vector fields is of advantage where the underlying differential equations are separable in spherical coordinates. No doubt, the definition of vector spherical harmonics should take into account this separation. Another important feature is that all geosciences are becoming increasingly interested in a unifying concept to handle consistently scalar spherical functions together with spherical vector and tensor fields. The goal is to combine different types of data derived from various sources such as terrestrial, airborne, and satellite observations.

In fact, most of the aforementioned geophysically motivated aspects are guaranteed adequately within a vectorial framework, transforming scalar functions into vector fields by use of certain operators $o^{(i)}, i = 1, 2, 3$. The operator $o^{(1)}$ separates the normal part of a vector field from the tangential part; $o^{(2)}$ defines a (tangential) surface gradient field that is curl-free, while $o^{(3)}$ yields a (tangential) surface curl gradient field that is divergence-free. In doing so, we are led to definitions that are independent of any particular choice of spherical harmonics and do not relate to any particular choice of a spherical coordinate system. Moreover, the rotational symmetry, i.e., the isotropy can be reflected in suitable (vectorial) manner.

The layout of this chapter on vector spherical harmonics is as follows: Section 5.1 is concerned with the separation of vector fields into normal and tangential parts. In Section 5.2, we introduce the vector spherical harmonics based on the properties of the operators $o^{(i)}$, $i = 1, 2, 3$. Section 5.3 is dedicated to the Helmholtz decomposition formula for spherical vector fields by use of the Green function with respect to the Beltrami operator. Section 5.4 shows the closure and completeness of vector spherical harmonics intrinsically on the sphere based on Bernstein summability. The interrelations between vector spherical harmonics and homogeneous harmonic vector polynomials are investigated in more detail in Section 5.5. It follows (in Section 5.6) the exact computation of orthogonal systems of vector spherical harmonics. Section 5.7 deals with the orthogonal invariance. Section 5.8 shows us that the vector spherical harmonics can be regarded as eigenfunctions of a vectorial analogue of the Beltrami operator. Section 5.9 presents the formulation of the addition theorem in terms of vector spherical harmonics thereby introducing appropriate counterparts of the Legendre polynomial. In Section 5.10, we prove vectorial versions of the Funk–Hecke formula. Vectorial counterparts of the Legendre polynomial are introduced in Section 5.11. Degree and order variances are discussed in Section 5.12. After a deeper insight into counterparts of Legendre polynomials within the vectorial context and the degree and order variances, we consider (in Section 5.13) an alternative system of vector spherical harmonics directly related to homogeneous harmonic vector polynomials, and present another system in Section 5.14. Finally, we summarize the methods for expanding vector fields using different convolution processes in Section 5.15.

5.1 Normal and Tangential Fields

In order to separate continuous vector fields into their tangential and normal parts, we introduce the projection operators p_{nor} and p_{tan} by

$$p_{\mathrm{nor}} f(\xi) = f_{\mathrm{nor}}(\xi) = (f(\xi)\cdot\xi)\,\xi, \quad \xi\in\Omega, f\in \mathrm{c}(\Omega), \tag{5.1}$$

$$p_{\mathrm{tan}} f(\xi) = f_{\mathrm{tan}}(\xi) = f(\xi) - p_{\mathrm{nor}} f(\xi), \quad \xi\in\Omega, f\in \mathrm{c}(\Omega). \tag{5.2}$$

It is easy to see that, for all $\xi\in\Omega$,

$$p_{\mathrm{tan}}(\xi\wedge f(\xi)) = \xi\wedge p_{\mathrm{tan}} f(\xi), \tag{5.3}$$

$$p_{\mathrm{tan}} f(\xi) = -\xi\wedge(\xi\wedge f(\xi)) = -\xi\wedge(\xi\wedge p_{\mathrm{tan}} f(\xi)). \tag{5.4}$$

Obviously, for $\xi\in\Omega$ and $f, g\in \mathrm{c}(\Omega)$,

$$f(\xi)\cdot g(\xi) = p_{\mathrm{nor}} f(\xi)\cdot p_{\mathrm{nor}} g(\xi) + p_{\mathrm{tan}} f(\xi)\cdot p_{\mathrm{tan}} g(\xi). \tag{5.5}$$

Furthermore, from Lemma 2.6, we know that $p_{\mathrm{tan}} f(\xi) = 0$ if and only if $f(\xi)\cdot\tau_\xi = 0$ for every unit vector τ_ξ that is perpendicular to ξ (note that $f(\xi)\cdot\tau_\xi = (p_{\mathrm{nor}} f(\xi) + p_{\mathrm{tan}} f(\xi))\cdot\tau_\xi$).

We let

$$\begin{aligned} \mathrm{c}_{\mathrm{nor}}(\Omega) &= \{f \in \mathrm{c}(\Omega) |\ f = p_{\mathrm{nor}} f\}, && (5.6)\\ \mathrm{c}_{\mathrm{tan}}(\Omega) &= \{f \in \mathrm{c}(\Omega) |\ f = p_{\mathrm{tan}} f\}. && (5.7)\end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{l}^2_{\mathrm{nor}}(\Omega) &= \overline{\mathrm{c}_{\mathrm{nor}}(\Omega)}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}}, && (5.8)\\ \mathrm{l}^2_{\mathrm{tan}}(\Omega) &= \overline{\mathrm{c}_{\mathrm{tan}}(\Omega)}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}}, && (5.9)\end{aligned}$$

We say $f \in \mathrm{l}^2(\Omega)$ is *normal* if $f = p_{\mathrm{nor}} f$ and *tangential* if $f = p_{\mathrm{tan}} f$, respectively. Clearly, we have the orthogonal decomposition

$$\mathrm{l}^2(\Omega) = \mathrm{l}^2_{\mathrm{nor}}(\Omega) \oplus \mathrm{l}^2_{\mathrm{tan}}(\Omega). \tag{5.10}$$

The spaces $\mathrm{c}^{(p)}_{\mathrm{nor}}(\Omega)$ and $\mathrm{c}^{(p)}_{\mathrm{tan}}(\Omega), 0 \leq p \leq \infty$, are defined in the same fashion.

The projection of the *identity tensor*

$$\mathbf{i} = \sum_{i=1}^{3} \varepsilon^i \otimes \varepsilon^i \tag{5.11}$$

onto the tangential components at a point $\xi \in \Omega$ defines the *surface identity tensor field* $\mathbf{i}_{\mathrm{tan}}$ given by

$$\mathbf{i}_{\mathrm{tan}}(\xi) = \mathbf{i} - \xi \otimes \xi, \quad \xi \in \Omega. \tag{5.12}$$

Moreover, we define the *surface rotation (tensor) field* $\mathbf{j}_{\mathrm{tan}}$ by

$$\mathbf{j}_{\mathrm{tan}}(\xi) = \xi \wedge \mathbf{i} = \sum_{i=1}^{3} (\xi \wedge \varepsilon^i) \otimes \varepsilon^i, \qquad \xi \in \Omega. \tag{5.13}$$

Obviously,

$$\begin{aligned} \mathbf{i}_{\mathrm{tan}}(\xi)\ \xi &= 0, \quad \mathbf{j}_{\mathrm{tan}}(\xi)\ \xi = 0, && (5.14)\\ \mathbf{i}_{\mathrm{tan}}(\xi)\ u_\xi &= u_\xi, \quad \mathbf{j}_{\mathrm{tan}}(\xi)\ u_\xi = \xi \wedge u_\xi, && (5.15)\end{aligned}$$

if $u_\xi \in \mathbb{R}^3,\ u_\xi \cdot \xi = 0$.

5.2 Definition of Vector Spherical Harmonics

The abbreviation

$$0_i = \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3 \end{cases} \tag{5.16}$$

will simplify our following considerations:

Assume that F is of class $\mathrm{C}^{(0_i)}(\Omega)$, $i = 1,2,3$. We define operators $o^{(i)}$: $\mathrm{C}^{(0_i)}(\Omega) \to \mathrm{c}(\Omega)$, respectively, as follows:

$$o_\xi^{(1)} F(\xi) = \xi F(\xi), \quad \xi \in \Omega, \tag{5.17}$$

$$o_\xi^{(2)} F(\xi) = \nabla_\xi^* F(\xi), \quad \xi \in \Omega, \tag{5.18}$$

$$o_\xi^{(3)} F(\xi) = \mathrm{L}_\xi^* F(\xi), \quad \xi \in \Omega. \tag{5.19}$$

It is clear that $o^{(1)}F$ is a normal vector field, whereas $o^{(2)}F$ and $o^{(3)}F$ are tangential. Moreover, it is not difficult to prove the following results:

$$o_\xi^{(1)} F(\xi) \cdot o_\xi^{(2)} F(\xi) = 0, \quad F \in \mathrm{C}^{(1)}(\Omega), \tag{5.20}$$

$$o_\xi^{(1)} F(\xi) \cdot o_\xi^{(3)} F(\xi) = 0, \quad F \in \mathrm{C}^{(1)}(\Omega), \tag{5.21}$$

$$o_\xi^{(2)} F(\xi) \cdot o_\xi^{(3)} F(\xi) = 0, \quad F \in \mathrm{C}^{(1)}(\Omega). \tag{5.22}$$

Green's integral formulas, i.e., partial integration on the sphere, help us to introduce the operators $O^{(i)}$ which are adjoint to $o^{(i)}$. In more detail, for $f \in \mathrm{c}^{(0_i)}(\Omega)$ and $G \in \mathrm{C}^{(0_i)}(\Omega)$, we have

$$(o^{(i)}G, f)_{\mathrm{l}^2(\Omega)} = (G, O^{(i)} f)_{\mathrm{L}^2(\Omega)}, \tag{5.23}$$

$i = 1,2,3$. Explicitly written out, this means that

$$\int_\Omega o^{(i)} G(\xi) \cdot f(\xi) \, d\omega(\xi) = \int_\Omega G(\xi) O^{(i)} f(\xi) \, d\omega(\xi). \tag{5.24}$$

We easily find

$$O_\xi^{(1)} f(\xi) = \xi \cdot p_{\mathrm{nor}} f(\xi), \quad \xi \in \Omega, \tag{5.25}$$

$$O_\xi^{(2)} f(\xi) = -\nabla_\xi^* \cdot p_{\mathrm{tan}} f(\xi), \quad \xi \in \Omega, \tag{5.26}$$

$$O_\xi^{(3)} f(\xi) = -\mathrm{L}_\xi^* \cdot p_{\mathrm{tan}} f(\xi), \quad \xi \in \Omega, \tag{5.27}$$

provided that f is of class $\mathrm{c}^{(0_i)}(\Omega)$, $i \in \{1,2,3\}$.

It can be readily shown that the $O^{(i)}$–operators satisfy the following relations.

Lemma 5.1. *Suppose that F is of class $\mathrm{C}^{(2)}(\Omega)$. Then the following statements hold true:*

(i) *If $i \neq j$, $i, j \in \{1,2,3\}$, then $O_\xi^{(i)} o_\xi^{(j)} F(\xi) = 0, \quad \xi \in \Omega.$*

(ii)

$$O_\xi^{(i)} o_\xi^{(i)} F(\xi) = \begin{cases} F(\xi), & i = 1, \\ -\Delta_\xi^* F(\xi), & i = 2, 3. \end{cases}$$

The definition of vector spherical harmonics can be given without using any local coordinate system on the sphere only by aid of the operators $o^{(i)}, O^{(i)}, i \in \{1, 2, 3\}$.

Definition 5.2. Any vector field of the form

$$y_n^{(i)} = o^{(i)} Y_n, \ n \geq 0_i, \ Y_n \in \mathrm{Harm}_n, \quad i = 1, 2, 3,$$

is called a vector spherical harmonic of degree n and type i (with respect to the dual system $o^{(i)}, O^{(i)}$).

$o^{(1)} Y_n$ represents a normal field, while $o^{(2)} Y_n$, $o^{(3)} Y_n$ are tangential fields of degree n.

Obviously, according to our construction, we have (see Fig. 5.1)

$$\xi \wedge (o^{(1)} Y_n)(\xi) = 0, \quad \xi \cdot (o^{(2)} Y_n)(\xi) = 0, \quad \xi \cdot (o^{(3)} Y_n)(\xi) = 0, \tag{5.28}$$

$$\mathrm{L}_\xi^* \cdot (o^{(2)} Y_n)(\xi) = 0, \quad \nabla_\xi^* \cdot (o^{(3)} Y_n)(\xi) = 0. \tag{5.29}$$

The orthogonality of scalar spherical harmonics helps us to show the orthogonality of vector spherical harmonics : First, for degrees n, m with $n \neq m$, it follows that

$$\begin{aligned} \int_\Omega o_\xi^{(1)} Y_n(\xi) \cdot o_\xi^{(1)} Y_m(\xi) \, d\omega(\xi) &= \int_\Omega Y_n(\xi) Y_m(\xi) (\xi \cdot \xi) \, d\omega(\xi) \quad (5.30) \\ &= \int_\Omega Y_n(\xi) Y_m(\xi) \, d\omega(\xi) = 0. \end{aligned}$$

In connection with (2.161), we obtain for $n \neq m$, $(n, m \geq 1)$

$$\begin{aligned} &\int_\Omega o_\xi^{(2)} Y_n(\xi) \cdot o_\xi^{(2)} Y_m(\xi) \, d\omega(x) \quad (5.31) \\ &= \int_\Omega \nabla_\xi^* Y_n(\xi) \cdot \nabla_\xi^* Y_m(\xi) \, d\omega(\xi) \\ &= -\int_\Omega Y_n(\xi) \Delta_\xi^* Y_m(\xi) \, d\omega(\xi) \\ &= m(m+1) \int_\Omega Y_n(\xi) Y_m(\xi) \, d\omega(\xi) \\ &= 0. \end{aligned}$$

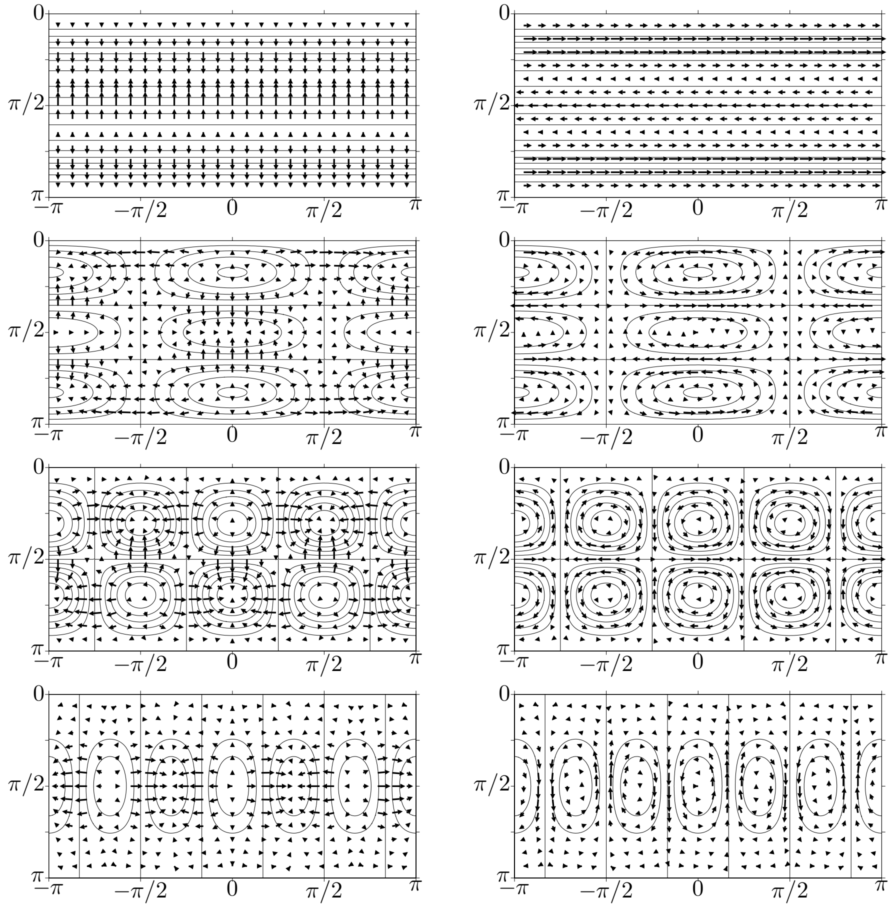


Fig. 5.1: Vector spherical harmonics $y_{n,j}^{(i)}$ of type $i = 2$ (*left*) and $i = 3$ (*right*) for the degree 3 and orders $0, 1, 2, 3$ (from *top* to *bottom*). The contour lines represent the scalar spherical harmonic $Y_{n,j}$ from which the vector spherical harmonics are generated.

Finally, for $n \neq m$ $(n, m \geq 1)$, we find with (2.163)

$$\begin{aligned}\int_\Omega o_\xi^{(3)} Y_n(\xi) o_\xi^{(3)} Y_m(\xi)\ d\omega(\xi) &= \int_\Omega \mathrm{L}_\xi^* Y_n(\xi) \cdot \mathrm{L}_\xi^* Y_m(\xi)\ d\omega(\xi) \quad (5.32)\\ &= -\int_\Omega Y_n(\xi) \Delta_\xi^* Y_m(\xi)\ d\omega(\xi)\\ &= 0.\end{aligned}$$

In other words, for $i \in \{1,2,3\}$ and $n \neq m$, with $n, m \geq 0_i$ we have

$$\begin{aligned} \int_\Omega o_\xi^{(i)} Y_n(\xi) \cdot o_\xi^{(i)} Y_m(\xi) \, d\omega(\xi) &= \int_\Omega Y_n(\xi) O_\xi^{(i)} o_\xi^{(i)} Y_m(\xi) \, d\omega(\xi) \\ &= 0. \end{aligned} \tag{5.33}$$

By $\mathrm{harm}_n^{(i)}$, we denote the set of all vector spherical harmonics of degree n and type i. Furthermore, we let

$$\mathrm{harm}_0 = \mathrm{harm}_0^{(1)}, \tag{5.34}$$

$$\mathrm{harm}_n = \bigoplus_{i=1}^{3} \mathrm{harm}_n^{(i)}, \quad n \geq 1. \tag{5.35}$$

We know from the orthogonality of the vector spherical harmonics of different degrees that harm_n is orthogonal to harm_m whenever $n \neq m$.

If $\{Y_{n,j}\}_{n=0,1,\ldots,j=1,\ldots 2n+1}$ forms an $\mathrm{L}^2(\Omega)$-orthonormal set of (scalar) spherical harmonics, then

$$y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} \, o^{(i)} Y_{n,j}, \tag{5.36}$$

$i \in \{1,2,3\}, n \geq 0_i, j = 1, \ldots, 2n+1$, forms an $\mathrm{l}^2(\Omega)$-orthonormal system of vector spherical harmonics (with respect to the dual system $o^{(i)}, O^{(i)}$), provided that the values $\mu_n^{(i)}$ are chosen in such a way that

$$\mu_n^{(i)} = \| O^{(i)} o^{(i)} Y_{n,j} \|_{\mathrm{L}^2(\Omega)}, \tag{5.37}$$

i.e.,

$$\mu_n^{(i)} = \begin{cases} 1, & i = 1 \\ -(\Delta^*)^\wedge(n) = n(n+1), & i = 2,3 \,. \end{cases}$$

Altogether we find

$$\int_\Omega y_{n,j}^{(i)}(\xi) \cdot y_{m,l}^{(k)}(\xi) \, d\omega(\xi) = \delta_{ik} \delta_{nm} \delta_{jl}. \tag{5.38}$$

Obviously, vector spherical harmonics can be calculated from the representations of scalar spherical harmonics. Illustrations of the tangential vector spherical harmonics are given in Fig. 5.1.

Example 5.3. Observing the well-known representations

$$Y_{0,1}(\xi) = \frac{1}{\sqrt{4\pi}}, \quad \xi \in \Omega, \tag{5.39}$$

$$Y_{1,j}(\xi) = \sqrt{\frac{3}{4\pi}} (\xi \cdot \varepsilon^j), \quad \xi \in \Omega, \; j = 1, \ldots, 3, \tag{5.40}$$

it is not difficult to see, that the vector fields

$$y^{(1)}_{0,1}(\xi) = \frac{1}{\sqrt{4\pi}}\xi, \quad \xi \in \Omega, \tag{5.41}$$

$$y^{(1)}_{1,j}(\xi) = \sqrt{\frac{3}{4\pi}}(\xi \cdot \varepsilon^j)\xi, \quad \xi \in \Omega,\ j = 1, \ldots, 3, \tag{5.42}$$

$$y^{(2)}_{1,j}(\xi) = \sqrt{\frac{3}{8\pi}}(\varepsilon^j - (\xi \cdot \varepsilon^j)\xi), \quad \xi \in \Omega, j = 1, \ldots, 3, \tag{5.43}$$

$$y^{(3)}_{1,j}(\xi) = \sqrt{\frac{3}{8\pi}}(\xi \wedge \varepsilon^j), \quad \xi \in \Omega,\ j = 1, \ldots, 3 \tag{5.44}$$

form an $\mathrm{l}^2(\Omega)$-orthonormal system of vector spherical harmonic of degree $0, 1$.

5.3 Helmholtz Decomposition Theorem for Spherical Vector Fields

The motivation for the $o^{(i)}$-operators is based on the fact that any vector field $f \in \mathrm{c}^{(1)}(\Omega)$ can be explicitly written out as

$$f(\xi) = \sum_{i=1}^{3} o^{(i)} F_i(\xi), \quad \xi \in \Omega, \tag{5.45}$$

with uniquely determined (scalar) functions $F_i : \Omega \to \mathbb{R}$.

In what follows, we formulate the decomposition theorem in a rigorous sense. Our particular purpose is to show how the scalar functions F_i can be determined in an explicit way by use of the concept of the Green function with respect to the Beltrami operator. An example is shown in Figs. 5.2 and 5.3.

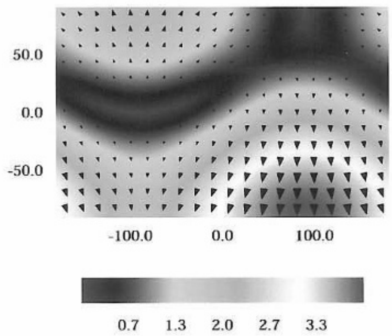


Fig. 5.2: Tangential vector field.

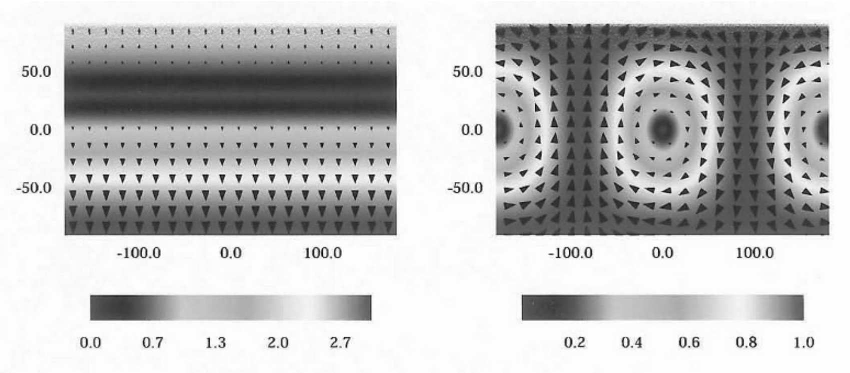


Fig. 5.3: Curl-free part (left picture) and divergence-free part (right picture) of the tangential vector field illustrated in Fig. 5.2.

Theorem 5.4. *(Helmholtz Decomposition Theorem) Let $f : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Then there exist uniquely determined functions $F_1 \in \mathrm{C}^{(1)}(\Omega)$ and $F_2, F_3 \in \mathrm{C}^{(2)}(\Omega)$ satisfying*

$$\int_\Omega F_i(\xi)\, d\omega(\xi) = 0, \quad i = 2, 3, \tag{5.46}$$

such that

$$f(\xi) = \sum_{i=1}^{3} o^{(i)} F_i(\xi) = F_1(\xi)\xi + \nabla^*_\xi F_2(\xi) + \mathrm{L}^*_\xi F_3(\xi), \quad \xi \in \Omega.$$

The functions F_i are given by

$$\begin{aligned} F_1(\xi) &= O^{(1)}_\xi f(\xi), \quad \xi \in \Omega, \\ F_2(\xi) &= -\int_\Omega G(\Delta^*; \xi \cdot \eta) O^{(2)}_\eta f(\eta)\, d\omega(\eta), \quad \xi \in \Omega, \\ F_3(\xi) &= -\int_\Omega G(\Delta^*; \xi \cdot \eta) O^{(3)}_\eta f(\eta)\, d\omega(\eta), \quad \xi \in \Omega. \end{aligned}$$

Proof. Any vector field f can be written as

$$f = f_{\mathrm{nor}} + f_{\mathrm{tan}}, \tag{5.47}$$

where

$$f_{\mathrm{nor}} = p_{\mathrm{nor}} f \in \mathrm{c}^{(1)}_{\mathrm{nor}}(\Omega), \tag{5.48}$$
$$f_{\mathrm{tan}} = p_{\mathrm{tan}} f \in \mathrm{c}^{(1)}_{\mathrm{tan}}(\Omega). \tag{5.49}$$

Clearly, we have $f_{\mathrm{nor}}(\xi) = o^{(1)} F_1(\xi)$ with $F_1(\xi) = O^{(1)} f(\xi), \xi \in \Omega$.

The tangential vector field f_{tan} can be represented in the form

$$f_{\mathrm{tan}} = \nabla^* F_2 + \mathrm{L}^* F_3. \tag{5.50}$$

Applying the operators $\nabla^* \cdot$ and $\mathrm{L}^* \cdot$, respectively, to f_{tan} we obtain the Beltrami differential equations

$$\Delta^* F_2 = \nabla^* \cdot f_{\mathrm{tan}}, \tag{5.51}$$
$$\Delta^* F_3 = \mathrm{L}^* \cdot f_{\mathrm{tan}}. \tag{5.52}$$

Since

$$\int_\Omega \nabla^*_\xi \cdot f_{\mathrm{tan}}(\xi)\, d\omega(\xi) = \int_\Omega \mathrm{L}^*_\xi \cdot f_{\mathrm{tan}}(\xi)\, d\omega(\xi) = 0, \tag{5.53}$$

we get the required decomposition from the fundamental theorem for the Beltrami operator and the definition of the adjoint operators $O^{(i)}$ to $o^{(i)}$. Furthermore, (5.46) is valid.

In order to prove the uniqueness of F_i, $i = 1, 2, 3$, assume that there exists another triple of functions G_i, $i = 1, 2, 3$, such that

$$f = o^{(1)} F_1 + o^{(2)} F_2 + o^{(3)} F_3, \tag{5.54}$$
$$f = o^{(1)} G_1 + o^{(2)} G_2 + o^{(3)} G_3. \tag{5.55}$$

Then, it follows that $F_1 = O^{(1)} f = G_1$, thus, F_1 is uniquely defined. Applications of $O^{(2)}$ and $O^{(3)}$ to (5.54) and (5.55) yields

$$\Delta^* F_2 = \Delta^* G_2, \tag{5.56}$$
$$\Delta^* F_3 = \Delta^* G_3. \tag{5.57}$$

Hence, the normalization conditions (5.46) for F_i and G_i, $i = 2, 3$, imply $F_2 = G_2$, $F_3 = G_3$, as required (see Theorem 4.21). □

According to the Helmholtz decomposition theorem, an arbitrary vector field $f \in \mathrm{c}^{(1)}(\Omega)$ can be written uniquely in the form

$$f = F_1(\xi)\xi + \nabla^*_\xi F_2(\xi) + \mathrm{L}^*_\xi F(\eta), \quad \xi \in \Omega \tag{5.58}$$

where $F_1 \in \mathrm{C}^{(1)}(\Omega)$ and $F_2, F_2 \in \mathrm{C}^{(2)}(\Omega)$ satisfy the identities

$$F_1(\xi) = f(\xi) \cdot \xi \tag{5.59}$$

and

$$\int_\Omega F_2(\xi)\ d\omega(\xi) = \int_\Omega F_3(\xi)\ d\omega(\xi) = 0. \tag{5.60}$$

Usually, a vector field of the special form

$$\xi \mapsto F_1(\xi)\xi + \nabla^*_\xi F_2(\xi), \quad \xi \in \Omega, \tag{5.61}$$

is said to be *spheroidal*, whereas one of the form

$$\xi \mapsto \mathrm{L}^*_\xi F_3(\xi), \quad \xi \in \Omega, \tag{5.62}$$

is said to be *toroidal*. Thus, the Helmholtz theorem represents the decomposition of an arbitrary vector field of class $\mathrm{c}^{(1)}(\Omega)$ into its spheroidal and toroidal parts. Note that a spheroidal field has both radial and tangential components, whereas a toroidal field is purely tangential. To our knowledge, there is no commonly accepted name for the pureley tangential surface gradient field $\nabla^* F_2$ of a spheroidal field; G.E. Backus (1966); G.E. Backus (1986) suggests calling it *consoidal*.

The space harm_n of vector spherical harmonics of degree n, therefore, lead naturally to radial, consoidal, and toroidal subspaces $\mathrm{harm}_n^{(1)}$, $\mathrm{harm}_n^{(2)}$ and $\mathrm{harm}_n^{(3)}$. The spaces are mutually orthogonal. In addition, they are orthogonally invariant and irreducible.

A useful consequence of the Helmholtz representation is that the surface divergence of a toroidal field is zero, and that a spheroidal field has a toroidal surface curl, and vice versa. The problem of getting back a consoidal and toroidal field from its generator (in terms of momentum (frequency) and/or space regularizations) leads back to Chapter 4.

The Helmholtz decomposition theorem (Theorem 5.4) also implies an orthogonal decomposition of the space $\mathrm{c}^{(\infty)}(\Omega)$. In fact,

$$\mathrm{c}^{(\infty)}(\Omega) = \mathrm{c}^{(\infty)}_{(1)}(\Omega) \oplus \mathrm{c}^{(\infty)}_{(2)}(\Omega) \oplus \mathrm{c}^{(\infty)}_{(3)}(\Omega), \tag{5.63}$$

where

$$\mathrm{c}^{(\infty)}_{(1)}(\Omega) = \mathrm{c}^{(\infty)}_{\mathrm{nor}}(\Omega), \tag{5.64}$$

$$\mathrm{c}^{(\infty)}_{(2)}(\Omega) = \{f \in \mathrm{c}^{(\infty)}(\Omega) \mid O^{(1)} f = O^{(3)} f = 0\}, \tag{5.65}$$

$$\mathrm{c}^{(\infty)}_{(3)}(\Omega) = \{f \in \mathrm{c}^{(\infty)}(\Omega) \mid O^{(1)} f = O^{(2)} f = 0\}. \tag{5.66}$$

Of course, these definitions can be extended as well to the spaces $c^{(k)}(\Omega)$, $0 \leq k \leq \infty$, or $l^2(\Omega)$. In the case of $l^2(\Omega)$, we are able to write

$$l^2_{(i)}(\Omega) = \overline{\{o^{(i)}F \mid F \in C^{(\infty)}(\Omega)\}}^{\|\cdot\|_{l^2(\Omega)}}. \tag{5.67}$$

Thus, we end up with the orthogonal decompositions

$$l^2(\Omega) = l^2_{\mathrm{nor}}(\Omega) \oplus l^2_{\mathrm{tan}}(\Omega), \tag{5.68}$$

$$l^2_{\mathrm{tan}}(\Omega) = l^2_{(2)}(\Omega) \oplus l^2_{(3)}(\Omega). \tag{5.69}$$

5.4 Orthogonal (Fourier) Expansions

Next, we prove the closure and completeness of vector spherical harmonics intrinsically on the sphere (note that a non-intrinsic proof follows from the arguments of Section 5.5). For our purpose here, we use vectorial variants of the scalar zonal Bernstein kernels. Although the approximation of functions by using Bernstein polynomials is one of the classical research topics and their theory is a rich one, their application within the vector theory of spherical harmonics seems to go back to W. Freeden, M. Gutting (2008). Indeed, the vector zonal Bernstein kernel approximations can be shown to guarantee the closure property in the space of continuous spherical normal as well as tangential vector fields, respectively. In consequence, they also assure closure and completeness in the Hilbert space of (Lebesgue-)square-integrable vector fields. Essential tools are the theory of the Green function with respect to the (iterated) Beltrami operator and the Helmholtz decomposition theorem.

We begin our considerations by convolving the Green function with respect to the Beltrami operator against the Bernstein kernel

$$BG_n(\xi \cdot \eta) = \int_\Omega G(\Delta^*; \xi \cdot \alpha) B_n(\alpha \cdot \eta) d\omega(\alpha).$$

Written in terms of a Legendre series, we find the finite sum

$$BG_n(\xi \cdot \eta) = \sum_{k=1}^{n} \frac{2k+1}{4\pi} \frac{B_n^\wedge(k)}{-k(k+1)} P_k(\xi \cdot \eta).$$

Note that the Bernstein kernel is a polynomial and the Green function is of class $L^1[-1,1]$, hence, the existence of the convolution integral as bandlimited Legendre expansion is obvious.

Next, we are interested in the Bernstein summability of Fourier expansions in terms of vector spherical harmonics. To this end, we need some preparatory material (more precisely, Lemma 5.5 and Lemma 5.6). Essential tool of our considerations is the Green function with respect to the Beltrami operator (cf. W. Freeden, M. Gutting (2008)).

Lemma 5.5. *For* $i \in \{1,2,3\}$

$$\lim_{n\to\infty} \|F_i - F_i^{(n)}\|_{C(\Omega)} = 0.$$

where F_i, $i = 1,2,3$, *are the functions occuring in the Helmholtz decomposition theorem*

$$\begin{aligned} F_1(\xi) &= O_\xi^{(1)} f(\xi), \\ F_i(\xi) &= -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(i)} f(\eta)\ d\omega(\eta), \quad i = 2,3 \end{aligned}$$

and $F_i^{(n)}$, $i = 1,2,3$, *are given by*

$$\begin{aligned} F_1^{(n)}(\xi) &= \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(1)} f(\eta)\ d\omega(\eta), \\ F_i^{(n)}(\xi) &= -\int_\Omega BG_n(\xi\cdot\eta) O_\eta^{(i)} f(\eta)\ d\omega(\eta), \quad i = 2,3. \end{aligned}$$

Proof. Clearly, the case $i = 1$ of Lemma 5.5 is easy to handle. It follows immediately from the scalar theory. Thus, it remains to study the cases $i = 2,3$. We start from the convolution integrals

$$F_i^{(n)}(\xi) = -\left(BG_n * O^{(i)} f\right)(\xi) = -\int_\Omega BG_n(\xi\cdot\eta) O_\eta^{(i)} f(\eta) d\omega(\eta), \quad (5.70)$$

$i = 2,3$. It is not difficult to see that

$$\begin{aligned} \|F_i - F_i^{(n)}\|_{C^{(0)}(\Omega)} &= \|G(\Delta^*;\cdot) * O^{(i)} f - BG_n * O^{(i)} f\|_{C(\Omega)} \\ &\le \|O^{(i)} f\|_{C^{(0)}(\Omega)} \|G(\Delta^*;\cdot) - BG_n\|_{L^1[-1,1]}. \end{aligned}$$

Since both kernels $G(\Delta^*;\cdot)$ and BG_n are of class $L^2[-1,1]$ and, for all $k \in \mathbb{N}_0$, the Legendre coefficients of the Bernstein kernel $B_n^\wedge(k)$ converge to 1 for n tending to infinity, we are able to deduce that

$$\lim_{n\to\infty} \|G(\Delta^*;\cdot) - BG_n\|_{L^2[-1,1]} = 0.$$

Obviously, this implies L^1-convergence as well as $\|F_i - F_i^{(n)}\|_{C(\Omega)} \to 0$ for $i = 1,2,3$ and $n \to \infty$, as required. □

Considering the $o^{(i)}$-derivatives, we have to verify the following lemma.

Lemma 5.6. *For* $i \in \{1,2,3\}$

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| o^{(i)}_\xi F_i(\xi) - o^{(i)}_\xi F_i^{(n)}(\xi) \right| = 0.$$

Proof. It is not hard to see that for $i \in \{2,3\}$

$$\begin{aligned}
&\| o^{(i)}_\xi F_i(\xi) - o^{(i)}_\xi F_i^{(n)}(\xi) \|_{c(\Omega)} \qquad (5.71)\\
&= \sup_{\xi\in\Omega} \left| o^{(i)}_\xi \int_\Omega G(\Delta^*;\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) - o^{(i)}_\xi \int_\Omega BG_n(\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) \right| \\
&= \sup_{\xi\in\Omega} \left| \int_\Omega o^{(i)}_\xi G(\Delta^*;\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) - \int_\Omega o^{(i)}_\xi BG_n(\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) \right|,
\end{aligned}$$

where it is clear that the operator $o^{(i)}$ can be drawn inside the two integrals. This leads us to following estimate:

$$\begin{aligned}
&\sup_{\xi\in\Omega} \left| \int_\Omega o^{(i)}_\xi G(\Delta^*;\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) - \int_\Omega o^{(i)}_\xi BG_n(\xi\cdot\eta) O^{(i)}_\eta f(\eta) d\omega(\eta) \right| \\
&\quad \le \sup_{\xi\in\Omega} \int_\Omega \left| o^{(i)}_\xi G(\Delta^*;\xi\cdot\eta) - o^{(i)}_\xi BG_n(\xi\cdot\eta) \right| \left| O^{(i)}_\eta f(\eta) \right| d\omega(\eta) \\
&\quad \le \| O^{(i)} f \|_{C(\Omega)} \int_\Omega \left| o^{(i)}_\xi G(\Delta^*;\xi\cdot\eta) - o^{(i)}_\xi BG_n(\xi\cdot\eta) \right| d\omega(\eta). \qquad (5.72)
\end{aligned}$$

We have to study the convergence of the last integral. In more detail, we are interested in proving that

$$\lim_{n\to\infty} \int_\Omega \left| o^{(i)}_\xi G(\Delta^*;\xi\cdot\eta) - o^{(i)}_\xi BG_n(\xi\cdot\eta) \right| d\omega(\eta) = 0. \qquad (5.73)$$

For that purpose, we notice that the Bernstein kernels $o^i_\xi BG_n(\xi\cdot\eta),\ i = 2,3$, admit the following (Legendre) series expansions

$$o^{(2)}_\xi BG_n(\xi\cdot\eta) = \sum_{k=1}^{n} \frac{2k+1}{4\pi} \frac{B_n^\wedge(k)}{-k(k+1)} P_k'(\xi\cdot\eta)\,(\eta - (\xi\cdot\eta)\xi)\,, \qquad (5.74)$$

$$o^{(3)}_\xi BG_n(\xi\cdot\eta) = \sum_{k=1}^{n} \frac{2k+1}{4\pi} \frac{B_n^\wedge(k)}{-k(k+1)} P_k'(\xi\cdot\eta)\,(\xi\wedge\eta)\,. \qquad (5.75)$$

Moreover, an easy calculation shows that the application of the $o^{(i)}$-operators, $i = 2,3$, to the Green function with respect to the Beltrami operator leads us to the identities

$$o^{(2)}_\xi G(\Delta^*;\xi\cdot\eta) = -\frac{1}{4\pi}\frac{\eta-(\eta\cdot\xi)\xi}{1-\eta\cdot\xi}, \qquad o^{(3)}_\xi G(\Delta^*;\xi\cdot\eta) = -\frac{1}{4\pi}\frac{\xi\wedge\eta}{1-\eta\cdot\xi}.$$

Consequently, for $i = 2, 3$, our integral can be expressed in the form

$$\begin{aligned}
&\int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi\cdot\eta) - o_\xi^{(i)} BG_n(\xi\cdot\eta)\right| d\omega(\eta) \qquad (5.76)\\
&= \int_\Omega \left| \frac{-1}{4\pi}\frac{o_\xi^{(i)}(\xi\cdot\eta)}{1-\xi\cdot\eta} - \frac{-1}{4\pi}\sum_{k=1}^{n}\frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(\xi\cdot\eta) o_\xi^{(i)}(\xi\cdot\eta)\right| d\omega(\eta)\\
&= \frac{1}{4\pi}\int_\Omega \left| o_\xi^{(i)}(\xi\cdot\eta)\right| \left| \frac{1}{1-\xi\cdot\eta} - \sum_{k=1}^{n}\frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(\xi\cdot\eta)\right| d\omega(\eta)\\
&= \frac{1}{2}\int_{-1}^{1}\sqrt{1-t^2}\left|\frac{1}{1-t} - \sum_{k=1}^{n}\frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(t)\right| dt.
\end{aligned}$$

At this point, we use the recurrence relation (Lemma 3.192) for the Legendre polynomials. This gives us the identity

$$\begin{aligned}
&\int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi\cdot\eta) - o_\xi^{(i)} BG_n(\xi\cdot\eta)\right| d\omega(\eta) \qquad (5.77)\\
&= \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left| 1-(1-t)\sum_{k=1}^{n} B_n^\wedge(k)\frac{P_{k+1}(t)-P_{k-1}(t)}{t^2-1}\right| dt\\
&= \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left| 1+\frac{1}{1+t}\sum_{k=1}^{n} B_n^\wedge(k)\left(P_{k+1}(t)-P_{k-1}(t)\right)\right| dt\,.
\end{aligned}$$

For the occurring sum, it follows that

$$\begin{aligned}
&\sum_{k=1}^{n} B_n^\wedge(k)\left(P_{k+1}(t)-P_{k-1}(t)\right) = \\
&\qquad B_n^\wedge(n)P_{n+1}(t) + B_n^\wedge(n-1)P_n(t) - B_n^\wedge(2)P_1(t) - B_n^\wedge(1)P_0(t)\\
&\qquad + \sum_{k=2}^{n-1}\left(B_n^\wedge(k-1) - B_n^\wedge(k+1)\right) P_k(t)\,, \qquad (5.78)
\end{aligned}$$

where a simple calculation shows that

$$B_n^\wedge(k-1) - B_n^\wedge(k+1) = B_{n+1}^\wedge(k)(2k+1)\frac{2}{(n+2)}. \qquad (5.79)$$

We plug (5.79) into (5.78) getting the following result

$$\begin{aligned}\sum_{k=1}^{n} & B_n^{\wedge}(k)\,(P_{k+1}(t) - P_{k-1}(t)) \qquad (5.80)\\ &= B_n^{\wedge}(n)P_{n+1}(t) + B_n^{\wedge}(n-1)P_n(t) - B_n^{\wedge}(2)\,P_1(t) - B_n^{\wedge}(1)P_0(t)\\ &\quad + \frac{2}{n+2}\sum_{k=2}^{n-1} B_{n+1}^{\wedge}(k)(2k+1)P_k(t)\\ &= \frac{2}{n+2}\sum_{k=0}^{n+1} B_{n+1}^{\wedge}(k)(2k+1)P_k(t) - (1+t)\\ &= \frac{2}{n+2}(n+1)\left(\frac{1+t}{2}\right)^{n+1} - (1+t)\,.\end{aligned}$$

Keeping this result in mind, we return to the integral (5.77). As a matter of fact, the identity (5.77) can be rewritten in the form

$$\begin{aligned}&\frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left|1 + \frac{1}{1+t}\sum_{k=1}^{n} B_n^{\wedge}(k)\,(P_{k+1}(t) - P_{k-1}(t))\right| dt \qquad (5.81)\\ &= \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left|\frac{n+1}{n+2}\left(\frac{1+t}{2}\right)^{n}\right| dt.\end{aligned}$$

Clearly, as the Bernstein kernel is non-negative, we are left with the integral expression

$$\begin{aligned}\int_{\Omega}\left|o_{\xi}^{(i)}G(\Delta^*;\xi\cdot\eta) - o_{\xi}^{(i)}BG_n(\xi\cdot\eta)\right| d\omega(\eta) &= \frac{1}{2}\frac{n+1}{n+2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{1+t}{2}\right)^{n} dt\\ &= \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+2)}, \qquad (5.82)\end{aligned}$$

(which follows by induction). It is well-known that the value of our integral can be estimated as follows:

$$\frac{1}{\sqrt{2n+2}} < \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+2)} < \frac{2}{\sqrt{2n+2}}\,. \qquad (5.83)$$

Therefore, we immediately obtain the convergence of our integral for $n \to \infty$. In addition, we get information about the speed of the convergence, i.e.,

$$\int_{\Omega}\left|o_{\xi}^{(i)}G(\Delta^*;\xi\cdot\eta) - o_{\xi}^{(i)}BG_n(\xi\cdot\eta)\right| d\omega(\eta) = O(n^{-1/2}).$$

□

After these preparations, we are now in a position to establish the *'Bernstein summability' of the Fourier series in terms of vector spherical harmonics.*

Theorem 5.7. *For any vector field f of class* $\mathrm{c}^{(1)}(\Omega)$,

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left| f(\xi) - \sum_{i=1}^{3}\sum_{k=0_i}^{n}\sum_{j=1}^{2k+1} B_n^{\wedge}(k)\left(f^{(i)}\right)^{\wedge}(k,j) y_{k,j}^{(i)}(\xi)\right| = 0,$$

where, as usual, $0_1 = 0$ *and* $0_i = 1,\ i = 2,3.$

Proof. From Lemma 5.6, we know that for $f \in \mathrm{c}^{(1)}(\Omega)$

$$\begin{aligned}\lim_{n\to\infty}\sup_{\xi\in\Omega}\left| f(\xi) - \sum_{i=1}^{3} o_\xi^{(i)} F_i^{(n)}(\xi)\right| &= \lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\sum_{i=1}^{3} o_\xi^{(i)} F_i(\xi) - \sum_{i=1}^{3} o_\xi^{(i)} F_i^{(n)}(\xi)\right| \\ &\leq \sum_{i=1}^{3}\lim_{n\to\infty}\sup_{\xi\in\Omega}\left| o_\xi^{(i)} F_i(\xi) - o_\xi^{(i)} F_i^{(n)}(\xi)\right| = 0.\end{aligned} \tag{5.84}$$

The expression $o_\xi^{(1)} F_1^{(n)}(\xi)$ can be expressed in the form

$$\begin{aligned} o_\xi^{(1)} F_1^{(n)}(\xi) &= o_\xi^{(1)} \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(1)} f(\eta)\, d\omega(\eta) \\ &= \sum_{k=0}^{n} B_n^{\wedge}(k) o_\xi^{(1)} \sum_{j=1}^{2k+1}\left(O^{(1)} f\right)^{\wedge}(k,j) Y_{k,j}(\xi) \\ &= \sum_{k=0}^{n}\sum_{j=1}^{2k+1} B_n^{\wedge}(k)\left(O^{(1)} f\right)^{\wedge}(k,j) y_{k,j}^{(1)}(\xi), \end{aligned} \tag{5.85}$$

where we have

$$\begin{aligned}\left(O^{(1)} f\right)^{\wedge}(k,j) &= \int_\Omega O_\eta^{(1)} f(\eta) Y_{k,j}(\eta)\, d\omega(\eta) \\ &= \int_\Omega f(\eta)\cdot \underbrace{o_\eta^{(1)} Y_{k,j}(\eta)}_{=y_{k,j}^{(1)}(\eta)}\, d\omega(\eta) = \left(f^{(1)}\right)^{\wedge}(k,j).\end{aligned} \tag{5.86}$$

Furthermore, for $i = 2, 3$, it is not difficult to see that

$$\begin{aligned} o_\xi^{(1)} F_i^{(n)}(\xi) &= -o_\xi^{(i)} \int_\Omega BG_n(\xi \cdot \eta) O_\eta^{(1)} f(\eta) d\omega(\eta) \\ &= \sum_{k=1}^{n} \frac{B_n^\wedge(k)}{k(k+1)} o_\xi^{(i)} \sum_{j=1}^{2k+1} \left(O^{(i)} f\right)^\wedge (k,j) Y_{k,j}(\xi) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{2k+1} \frac{B_n^\wedge(k)}{\sqrt{k(k+1)}} \left(O^{(i)} f\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi). \end{aligned} \tag{5.87}$$

Taking a look at the coefficients $\left(O^{(i)} f\right)^\wedge (k,j)$, we find

$$\begin{aligned} \left(O^{(i)} f\right)^\wedge (k,j) &= \int_\Omega O_\eta^{(i)} f(\eta) Y_{k,j}(\eta) d\omega(\eta) = \int_\Omega f(\eta) \cdot o_\eta^{(i)} Y_{k,j}(\eta) d\omega(\eta) \\ &= \sqrt{k(k+1)} \int_\Omega f(\eta) \cdot y_{k,j}^{(i)}(\eta) d\omega(\eta) = \sqrt{k(k+1)} \left(f^{(i)}\right)^\wedge (k,j). \end{aligned} \tag{5.88}$$

The identities (5.85) and (5.86) as well as (5.87) and (5.88) allow us to conclude

$$o_\xi^{(i)} F_i^{(n)}(\xi) = \sum_{k=1}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(f^{(i)}\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi), \qquad i = 1, 2, 3. \tag{5.89}$$

In connection with (5.84), we therefore obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} o_\xi^{(i)} F_i^{(n)}(\xi) \right| \\ &= \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(f^{(i)}\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi) \right| = 0, \end{aligned} \tag{5.90}$$

provided that $f \in \mathrm{c}^{(1)}(\Omega)$. This is the desired result. □

Next, a well-known density argument enables us to verify the *closure of the vector spherical harmonics* $\left\{y_{k,j}^{(i)}\right\}_{i,k,j}$ *in the space* $\mathrm{c}(\Omega)$.

Theorem 5.8. *For any given* $\varepsilon > 0$ *and each* $f \in \mathrm{c}(\Omega)$*, there exists a linear combination* $\sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} d_{k,j}^{(i)} y_{k,j}^{(i)}$*, such that*

$$\left\| f - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} d_{k,j}^{(i)} y_{k,j}^{(i)} \right\|_{\mathrm{c}(\Omega)} \leq \varepsilon.$$

Indeed, if we take any $g \in c^{(0)}(\Omega)$ and any $\varepsilon > 0$, we find a field $f \in c^{(1)}(\Omega)$ such that $\sup_{\xi\in\Omega} |g(\xi) - f(\xi)| < \frac{\varepsilon}{2}$. Due to Theorem 5.7, there also exists an integer N with

$$\sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(f^{(i)}\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi) \right| < \frac{\varepsilon}{2}.$$

Combining both inequalities, we therefore obtain

$$\sup_{\xi\in\Omega} \left| g(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} \underbrace{B_n^\wedge(k) \left(f^{(i)}\right)^\wedge (k,j)}_{d_{k,j}^{(i)}} y_{k,j}^{(i)}(\xi) \right| < \varepsilon.$$

By standard arguments, this immediately gives us the closure in $c(\Omega)$ with respect to $\|\cdot\|_{l^2(\Omega)}$ as well as in $l^2(\Omega)$ which in turn leads to completeness of the system $\left\{y_{k,j}^{(i)}\right\}_{i,k,j}$ in $l^2(\Omega)$.

Summarizing our results, we therefore obtain the following theorem.

Theorem 5.9. *Let* $\{y_{n,j}^{(i)}\}_{\substack{i=1,2,3 \\ n=0_i,\ldots,\ j=1,\ldots,2n+1}}$ *be defined as in (5.36). Then, the following statements are valid:*

(i) *The system of vector spherical harmonics is closed in* $c(\Omega)$ *with respect to* $\|\cdot\|_{c(\Omega)}$.

(ii) *The system is complete in* $l^2(\Omega)$ *with respect to* $(\cdot,\cdot)_{l^2(\Omega)}$.

Once more, part (i) of this theorem says that any continuous vector field on Ω can be approximated arbitrarily close by finite linear combinations of vector spherical harmonics, while part (ii) is equivalent (cf. Theorem 3.54) to the fact that every vector field in $l^2(\Omega)$ can be represented by its Fourier (orthogonal) series in terms of the $l^2(\Omega)$-orthogonal system $\{y_{n,j}^{(i)}\}$:

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) y_{n,j}^{(i)}. \tag{5.91}$$

To be more specific, for $f \in l^2(\Omega)$, we have

$$\lim_{N\to\infty} \left\| f - \sum_{i=1}^{3} \sum_{n=0_i}^{N} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) y_{n,j}^{(i)} \right\|_{l^2(\Omega)} = 0, \tag{5.92}$$

where the *'Fourier coefficients'* are given by

$$(f^{(i)})^\wedge(n,j) = (f, y^{(i)}_{n,j})_{\mathrm{l}^2(\Omega)} = \int_\Omega f(\xi) \cdot y^{(i)}_{n,j}(\xi)\, d\omega(\xi). \tag{5.93}$$

5.5 Homogeneous Harmonic Vector Polynomials

The property of scalar spherical harmonics of being restrictions of homogeneous harmonic polynomials to the unit sphere Ω has been of tremendous importance for many results in scalar theory. In what follows, similar relations between vector spherical harmonics and homogeneous harmonic vector polynomials are developed. Unfortunately, the different nature of the separation into normal/tangential components on the one hand, and into cartesian components, on the other hand, does not provide us with relations of comparable simplicity. Nevertheless, these interdependencies help us to recognize significant results on the role of vector spherical harmonics as trial functions in constructive approximation, viz. the closure and completeness of vector spherical harmonics.

Definition 5.10. A vector field $h_n : \mathbb{R}^3 \to \mathbb{R}^3, n \in \mathbb{N}_0$, is called a *homogeneous harmonic vector polynomial of degree* n if $h_n \cdot \varepsilon^i$ is a scalar homogeneous harmonic polynomial of degree n for every index $i \in \{1,2,3\}$.

Using the abbreviation,

$$\mathrm{Harm}_n(\mathbb{R}^3)\varepsilon^i = \{H_n\varepsilon^i \mid H_n \in \mathrm{Harm}_n(\mathbb{R}^3)\}, \tag{5.94}$$

the space of all homogeneous harmonic vector polynomials of degree n is characterized by

$$\bigoplus_{i=1}^{3} \mathrm{Harm}_n(\mathbb{R}^3)\varepsilon^i. \tag{5.95}$$

The restriction of a homogeneous harmonic vector polynomial of degree n to the unit sphere Ω does – *in contrast to the scalar case* – in general, not yield a spherical harmonic of degree n. But we shall see later on that each member of

$$\bigoplus_{i=1}^{3} \mathrm{Harm}_n(\Omega)\varepsilon^i \tag{5.96}$$

is expressible as a linear combination of vector spherical harmonics of different orders.

Suppose that H_n is of class $\mathrm{Harm}_n(\mathbb{R}^3)$. Then, it is immediately clear that the field ∇H_n is a homogeneous harmonic vector polynomial of degree $n-1$. A simple calculation shows that $x \mapsto x \wedge \nabla_x H_n(x)$, $x \in \mathbb{R}^3$, yields a homogeneous harmonic vector polynomial of degree n. The field $x \mapsto xH_n(x), x \in \mathbb{R}^3$, is, in general, not harmonic, but it follows easily that $x \mapsto ((2n+1)x - |x|^2\nabla_x)H_n(x)$ is a homogeneous harmonic vector polynomial of degree $n+1$.

These preparations motivate the following definition:

Definition 5.11. For $n \in \mathbb{N}_0, H_n \in \mathrm{Hom}_n(\mathbb{R}^3)$, and $i \in \{1,2,3\}$ the operators $\tilde{o}_n^{(i)}, i = 1,2,3$, are defined by

$$\begin{aligned}
\tilde{o}_n^{(1)} H_n(x) &= ((2n+1)x - |x|^2\nabla_x)H_n(x), \quad x \in \mathbb{R}^3, \\
\tilde{o}_n^{(2)} H_n(x) &= \nabla_x H_n(x), \quad x \in \mathbb{R}^3, \\
\tilde{o}_n^{(3)} H_n(x) &= x \wedge \nabla_x H_n(x), \quad x \in \mathbb{R}^3.
\end{aligned}$$

The aforementioned properties of the operators $\tilde{o}_n^{(i)}$, $i \in \{1,2,3\}$, introduced in Definition 5.10 are summarized in the following result.

Lemma 5.12. *Let $H_n \in \mathrm{Harm}_n(\mathbb{R}^3), n \in \mathbb{N}_0$. Then $\tilde{o}_n^{(i)} H_n$ is a homogeneous harmonic vector polynomial of degree $\deg^{(i)}(n)$, where*

$$\deg^{(i)}(n) = \begin{cases} n+1, & i=1 \\ n-1, & i=2 \\ n, & i=3. \end{cases} \tag{5.97}$$

(If $\deg^{(i)}(n) < 0$, then, by definition, $\tilde{o}_n^{(i)} H_n = 0$).

Proof. Since the other statements are straightforward, we only prove that the cartesian components of $\tilde{o}_n^{(1)} H_n(x) = ((2n+1)x - |x|^2\nabla_x)H_n(x)$ constitute homogeneous harmonic polynomials of degree $n+1$. To be more specific, observe that the components of $\tilde{o}_n^{(1)} H_n$ are homogeneous of degree $n+1$. Since, for $j \in \{1,2,3\}$, the function $\varepsilon^j \cdot \nabla H_n$ is a homogeneous harmonic polynomial of degree $n-1$, we obtain by elementary calculations

$$\begin{aligned}
&\Delta_x(\tilde{o}_n^{(1)} H_n(x)) \cdot \varepsilon^j \qquad (5.98)\\
&= \Delta_x \left((2n+1)x_j H_n(x) - |x|^2 \frac{\partial}{\partial x^j} H_n(x) \right) \\
&= 2(2n+1)\varepsilon^j \cdot \nabla H_n(x) \\
&\quad - \left(\Delta_x |x|^2\right) \frac{\partial}{\partial x^j} H_n(x) - \left(2\nabla_x |x|^2\right) \cdot \left(\nabla_x \frac{\partial}{\partial x^j} H_n(x) \right) \\
&= 2(2n+1)\frac{\partial}{\partial x^j} H_n(x) - 6\frac{\partial}{\partial x^j} H_n(x) - 4x \cdot \nabla_x \frac{\partial}{\partial x^j} H_n(x) \\
&= 4(n-1)\frac{\partial}{\partial x^j} H_n(x) - 4(n-1)\frac{\partial}{\partial x^j} H_n(x) \\
&= 0.
\end{aligned}$$

This yields the desired result. □

With the notation $x = r\xi, r = |x|$, and the known representation of the gradient ∇ in polar coordinates, it follows that

$$((2n+1)x - |x|^2\nabla_x) = \xi\left((2n+1)r - r^2\frac{\partial}{\partial r}\right) - r\nabla^*_\xi. \tag{5.99}$$

Thus, for $Y_n \in \mathrm{Harm}_n$ and $x = r\xi, r = |x| > 0$, we see that

$$\begin{aligned}
\tilde{o}_n^{(1)} r^n Y_n(\xi) &= (n+1)r^{n+1}Y_n(\xi)\xi - r^{n+1}\nabla^*_\xi Y_n(\xi), && (5.100)\\
\tilde{o}_n^{(2)} r^n Y_n(\xi) &= nr^{n-1}Y_n(\xi)\xi + r^{n-1}\nabla^*_\xi Y_n(\xi), && (5.101)\\
\tilde{o}_n^{(3)} r^n Y_n(\xi) &= r^n \mathrm{L}^*_\xi Y_n(\xi). && (5.102)
\end{aligned}$$

But this shows us that the restrictions of $r\xi \mapsto \tilde{o}_n^{(i)} r^n Y_n(\xi)$ to the unit sphere Ω can be written as linear combinations of vector spherical harmonics $o^{(i)}Y_n$. More explicitly,

$$\begin{aligned}
\tilde{o}_n^{(1)} H_n(x)|_{r=1} &= (n+1)o_\xi^{(1)}Y_n(\xi) - o_\xi^{(2)}Y_n(\xi), && (5.103)\\
\tilde{o}_n^{(2)} H_n(x)|_{r=1} &= no_\xi^{(1)}Y_n(\xi) + o_\xi^{(2)}Y_n(\xi), && (5.104)\\
\tilde{o}_n^{(3)} H_n(x)|_{r=1} &= o_\xi^{(3)}Y_n(\xi), && (5.105)
\end{aligned}$$

where $H_n(x) = r^n Y_n(\xi), x = r\xi$.

By inverting the equations (5.103) –(5.105), we find

$$\begin{aligned}
o_\xi^{(1)}Y_n(\xi) &= \frac{1}{2n+1}\left(\tilde{o}_n^{(1)}H_n(x)|_{r=1}\right) + \frac{1}{2n+1}\left(\tilde{o}_n^{(2)}H_n(x)|_{r=1}\right), && (5.106)\\
o_\xi^{(2)}Y_n(\xi) &= \frac{-n}{2n+1}\left(\tilde{o}_n^{(1)}H_n(x)|_{r=1}\right) + \frac{n+1}{2n+1}\left(\tilde{o}_n^{(2)}H_n(x)|_{r=1}\right), && (5.107)\\
o_\xi^{(3)}Y_n(\xi) &= \left(\tilde{o}_n^{(3)}H_n(x)|_{r=1}\right). && (5.108)
\end{aligned}$$

By virtue of Lemma 5.12, it follows that the cartesian components of any vector spherical harmonic of degree n and type 1 and 2 can be expressed as linear combinations involving scalar spherical harmonics of degrees $n-1$ and $n+1$, whereas the cartesian components of a vector spherical harmonic of degree n and type 3 are a linear combination in terms of scalar spherical harmonics of degree n.

In more detail,

$$\begin{aligned}
\mathrm{harm}_n^{(i)} &\subset \bigoplus_{j=1}^{3}\mathrm{Harm}_{n-1}\varepsilon^j \oplus \bigoplus_{j=1}^{3}\mathrm{Harm}_{n+1}\varepsilon^j, \quad i = 1,2, && (5.109)\\
\mathrm{harm}_n^{(3)} &\subset \bigoplus_{j=1}^{3}\mathrm{Harm}_n\varepsilon^j. && (5.110)
\end{aligned}$$

An immediate consequence is the fact that

$$y_n^{(i)}(-\xi) = (-1)^{n+1} y_n^{(i)}(\xi), \quad \xi \in \Omega,\ y_n^{(i)} \in \mathrm{harm}_n^{(i)},\ i = 1, 2, \tag{5.111}$$

and

$$y_n^{(3)}(-\xi) = (-1)^n y_n^{(3)}(\xi), \quad \xi \in \Omega,\ y_n^{(3)} \in \mathrm{harm}_n^{(3)}. \tag{5.112}$$

Furthermore, the following orthogonality relations are readily obtainable from (5.109) and (5.110).

Lemma 5.13. *Let* $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ *and* $Y_m \in \mathrm{Harm}_m$. *Then*

$$\int_\Omega y_n^{(i)}(\xi) Y_m(\xi)\, d\omega(\xi) = 0,$$

whenever $i \in \{1, 2\}$ *and* $m \notin \{n-1, n+1\}$ *or* $i = 3$ *and* $m \neq n$.

Next, we are interested in closure and completeness properties of vector spherical harmonics. It is obvious from the corresponding results of (scalar) spherical harmonics that $\bigoplus_{m=0}^{\infty} \bigoplus_{i=1}^{3} \mathrm{Harm}_m \varepsilon^i$ is dense in $\mathrm{c}(\Omega)$ with respect to $\|\cdot\|_{\mathrm{c}(\Omega)}$ and in $\mathrm{l}^2(\Omega)$ with respect to $\|\cdot\|_{\mathrm{l}^2(\Omega)}$. On the other hand, one can readily show that

$$\mathrm{harm}_n \subset \bigoplus_{m=n-1}^{n+1} \bigoplus_{i=1}^{3} \mathrm{Harm}_m \varepsilon^i. \tag{5.113}$$

It is obvious that similar properties of the above sets $\bigoplus_{m=0}^{\infty} \mathrm{harm}_m^{(1)}$ and $\bigoplus_{m=1}^{\infty} (\mathrm{harm}_m^{(2)} \oplus \mathrm{harm}_m^{(3)})$ may be verified within the spaces $\mathrm{c_{nor}}(\Omega)$ and $\mathrm{l}^2_{\mathrm{nor}}(\Omega)$, respectively, $\mathrm{c_{tan}}(\Omega)$ and $\mathrm{l}^2_{\mathrm{tan}}(\Omega)$.

5.6 Exact Computation of Orthonormal Systems

In Chapter 3, an algorithm for generating $\mathrm{L}^2(\Omega)$-orthonormal systems of scalar spherical harmonics was indicated. In what follows, we are interested in a viable way for determining $\mathrm{l}^2(\Omega)$-orthonormal systems of vector spherical harmonics $\{y_{n,j}^{(i)}\}$. It should be noted that, we avoid problems arising from the singularities of a spherical coordinate system when using cartesian coordinate representations.

Let $H_{n,j}, j = 1, \ldots, 2n+1$, be an $(\cdot,\cdot)_{\mathrm{Hom}_n}$-orthonormal system of homogeneous harmonic polynomials of degree n, of the form

$$H_{n,j}(x) = \sum_{[\alpha]=n} B_\alpha^j x^\alpha \tag{5.114}$$

with known real numbers B^j_α (as described in Chapter 3). Then we know that $Y_{n,j}(\xi) = \sqrt{\mu_n} H_{n,j}(\xi)$, $\xi \in \Omega$, constitutes an $\mathrm{L}^2(\Omega)$-orthogonal system of spherical harmonics. Therefore, via the well-known procedure, by letting

$$y^{(i)}_{n,j} = (\mu^{(i)}_n)^{1/2} o^{(i)} Y_{n,j}, \quad j = 1, \ldots, 2n+1, \ i \in 1, 2, 3, \tag{5.115}$$

an $\mathrm{l}^2(\Omega)$-orthonormal system of vector spherical harmonics of kind i is found. More explicitly,

$$y^{(1)}_{n,j}(\xi) = \frac{1}{\sqrt{\mu_n}} h^{(1)}_{n,j}(x)|_{|x|=1} = \frac{1}{\sqrt{\mu_n}} h^{(1)}_{n,j}(\xi), \tag{5.116}$$

$$y^{(2)}_{n,j}(\xi) = \frac{1}{\sqrt{\mu_n}} \frac{1}{\sqrt{n(n+1)}} h^{(2)}_{n,j}(x)|_{|x|=1} = \frac{1}{\sqrt{\mu_n}} \frac{1}{\sqrt{n(n+1)}} h^{(2)}_{n,j}(\xi), \tag{5.117}$$

$$y^{(3)}_{n,j}(\xi) = \frac{1}{\sqrt{\mu_n}} \frac{1}{\sqrt{n(n+1)}} h^{(3)}_{n,j}(x)|_{|x|=1} = \frac{1}{\sqrt{\mu_n}} \frac{1}{\sqrt{n(n+1)}} h^{(3)}_{n,j}(\xi), \tag{5.118}$$

where

$$h^{(1)}_{n,j}(x) = H_{n,j}(x)x, \tag{5.119}$$

$$h^{(2)}_{n,j}(x) = x^2 \nabla_x H_{n,j}(x) - n H_{n,j}(x) x, \tag{5.120}$$

$$h^{(3)}_{n,j}(x) = x \wedge \nabla_x H_{n,j}(x) \tag{5.121}$$

$(x = r\xi, r = |x|, \xi \in \Omega)$.

Our purpose is to determine the vector spherical harmonics using exclusively exact integer arithmetic. We base our considerations on the representation

$$h^{(i)}_{n,j}(x) = \sum_{k=1}^{3} \varepsilon^k \left(\sum_{[\alpha]=m_i} D^{i,k}_{\alpha;j} x^\alpha \right), \tag{5.122}$$

where $m_1 = m_2 = n+1, m_3 = n$. Observing, the already known identities

$$y^{(1)}_{n,j}(\xi) = \sqrt{\frac{\mu_n}{\mu^{(1)}_n}} \frac{1}{2n+1} (\tilde{o}^{(1)}_n + \tilde{o}^{(2)}_n) H_{n,j}(r\xi)|_{r=1}, \tag{5.123}$$

$$y^{(2)}_{n,j}(\xi) = \sqrt{\frac{\mu_n}{\mu^{(2)}_n}} \frac{1}{2n+1} (-n\tilde{o}^{(1)}_n + (n+1)\tilde{o}^{(2)}_n) H_{n,j}(r\xi)|_{r=1}, \tag{5.124}$$

$$y^{(3)}_{n,j}(\xi) = \sqrt{\frac{\mu_n}{\mu^{(3)}_n}} \tilde{o}^{(3)}_n H_{n,j}(r\xi)|_{r=1}, \tag{5.125}$$

we find by observing the definition of the $k_n^{(i)}$-operators (see Definition 5.11)

$$y_{n,j}^{(1)}(\xi) = \sqrt{\frac{\mu_n}{\mu_n^{(1)}}} \sum_{[\alpha]=n} B_\alpha^j \begin{pmatrix} \xi_1^{\alpha_1+1}\xi_2^{\alpha_2}\xi_3^{\alpha_3} \\ \xi_1^{\alpha_1}\xi_2^{\alpha_2+1}\xi_3^{\alpha_3} \\ \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3+1} \end{pmatrix},$$

$$y_{n,j}^{(2)}(\xi) = \sqrt{\frac{\mu_n}{\mu_n^{(2)}}} \sum_{[\alpha]=n} B_\alpha^j \begin{pmatrix} \alpha_1\xi_1^{\alpha_1-1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}(\xi_1^2+\xi_2^2+\xi_3^2) - n\xi_1^{\alpha_1+1}\xi_2^{\alpha_2}\xi_3^{\alpha_3} \\ \alpha_2\xi_1^{\alpha_1}\xi_2^{\alpha_2-1}\xi_3^{\alpha_3}(\xi_1^2+\xi_2^2+\xi_3^2) - n\xi_1^{\alpha_1}\xi_2^{\alpha_2+2}\xi_3^{\alpha_3} \\ \alpha_3\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3-1}(\xi_1^2+\xi_2^2+\xi_3^2) - n\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3+1} \end{pmatrix},$$

$$y_{n,j}^{(3)}(\xi) = \sqrt{\frac{\mu_n}{\mu_n^{(3)}}} \sum_{[\alpha]=n} B_\alpha^j \begin{pmatrix} \alpha_3\xi_1^{\alpha_1}x_2^{\alpha_2+1}\xi_3^{\alpha_3-1} - \alpha_2\xi_1^{\alpha_1}\xi_2^{\alpha_2-1}\xi_3^{\alpha_3+1} \\ \alpha_1\xi_1^{\alpha_1-1}\xi_2^{\alpha_2}\xi_3^{\alpha_3+1} - \alpha_3\xi_1^{\alpha_1+1}\xi_2^{\alpha_2}\xi_3^{\alpha_3-1} \\ \alpha_2\xi_1^{\alpha_1+1}\xi_2^{\alpha_2-1}\xi_3^{\alpha_3} - \alpha_1\xi_1^{\alpha_1-1}\xi_2^{\alpha_1+1}\xi_3^{\alpha_3} \end{pmatrix}.$$

Hence, the coefficients $D_{\beta,j}^{i,k}$ occuring in (5.122) are found. We organize their computation by a matrix-matrix multiplication in the form

$$D_{\beta;j}^{i,k} = \sqrt{\frac{\mu_n}{\mu_n^{(i)}}} \sum_{[\alpha]=n} B_\alpha^j M_{\beta\alpha;j}^{i,k} \tag{5.126}$$

for $i,k = 1,2,3, j = 1,\ldots,2n+1, [\beta] = n+1$ in the cases $i = 1,2$ and $[\beta] = n$ if $i = 3$, respectively. The matrices $\mathbf{m}^{i,k} = (M_{\beta,\alpha}^{i,k})$ have $\binom{n+2}{2}$ rows and $\binom{n}{2}$ columns for $i = 1,2$ and they have $\binom{n}{2}$ rows and $\binom{n}{2}$ columns if $i = 3$. It is easy to see that

$$M_{\beta\alpha;j}^{1,1} = \begin{cases} 1 & \text{if } \beta-\alpha = (1,0,0)^T \\ 0 & \text{otherwise} \end{cases} \tag{5.127}$$

$$M_{\beta\alpha;j}^{1,2} = \begin{cases} 1 & \text{if } \beta-\alpha = (0,1,0)^T \\ 0 & \text{otherwise} \end{cases} \tag{5.128}$$

$$M_{\beta\alpha;j}^{1,3} = \begin{cases} 1 & \text{if } \beta-\alpha = (0,0,1)^T \\ 0 & \text{otherwise} \end{cases} \tag{5.129}$$

$$M_{\beta\alpha;j}^{2,1} = \begin{cases} \alpha_1 - n & \text{if } \beta-\alpha = (1,0,0)^T \\ \alpha_1 & \text{if } \beta-\alpha \in \{(-1,2,0)^T, (-1,0,2)^T\} \\ 0 & \text{otherwise} \end{cases} \tag{5.130}$$

$$M_{\beta\alpha;j}^{2,2} = \begin{cases} \alpha_2 - n & \text{if } \beta-\alpha = (0,1,0)^T \\ \alpha_2 & \text{if } \beta-\alpha \in \{(2,-1,0)^T, (0,-1,2)^T\} \\ 0 & \text{otherwise} \end{cases} \tag{5.131}$$

$$M_{\beta\alpha;j}^{2,3} = \begin{cases} \alpha_3 - n & \text{if } \beta-\alpha = (0,0,1)^T \\ \alpha_3 & \text{if } \beta-\alpha \in \{(2,0,-1)^T, (0,2,-1)^T\} \\ 0 & \text{otherwise} \end{cases} \tag{5.132}$$

$$M_{\beta\alpha;j}^{3,1} = \begin{cases} \alpha_3 - n & \text{if } \beta-\alpha = (0,1,-1)^T \\ -\alpha_2 & \text{if } \beta-\alpha \in (0,-1,1)^T \\ 0 & \text{otherwise} \end{cases} \tag{5.133}$$

$$M^{3,2}_{\beta\alpha;j} = \begin{cases} \alpha_1 & \text{if } \beta-\alpha = (-1,0,1)^T \\ -\alpha_3 & \text{if } \beta-\alpha \in (1,0,-1)^T \\ 0 & \text{otherwise} \end{cases} \tag{5.134}$$

$$M^{3,3}_{\beta\alpha;j} = \begin{cases} \alpha_2 & \text{if } \beta-\alpha = (1,-1,0)^T \\ -\alpha_1 & \text{if } \beta-\alpha = (-1,1,0)^T \\ 0 & \text{otherwise.} \end{cases} \tag{5.135}$$

Example 5.14. As example, we consider, for $n = 3$, the $(\cdot,\cdot)_{\mathrm{Hom}_3}$-orthonormal system

$$H_{3,1}(x) = \frac{1}{\sqrt{24}}({x_1}^3 - 3x_1{x_2}^2), \tag{5.136}$$

$$H_{3,2}(x) = x_1x_2x_3, \tag{5.137}$$

$$H_{3,3}(x) = \frac{1}{\sqrt{40}}({x_1}^3 + x_1{x_2}^2 - 4x_1{x_3}^2), \tag{5.138}$$

$$H_{3,4}(x) = \frac{1}{\sqrt{24}}(3{x_1}^2x_2 - {x_2}^3), \tag{5.139}$$

$$H_{3,5}(x) = \frac{1}{\sqrt{4}}({x_1}^2x_3 - {x_2}^2x_3), \tag{5.140}$$

$$H_{3,6}(x) = \frac{1}{\sqrt{40}}({x_1}^2x_2 + {x_2}^3 - 4x_2x_3), \tag{5.141}$$

$$H_{3,7}(x) = \frac{1}{\sqrt{60}}(3{x_1}^2x_3 + 3{x_2}^2x_3 - 2{x_3}^3) \tag{5.142}$$

of homogeneous harmonic polynomials of degree 3. Then we obtain

$$y^{(1)}_{3,1}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{24}}({\xi_1}^3 - 3\xi_1{\xi_2}^2)\xi, \tag{5.143}$$

$$y^{(1)}_{3,2}(\xi) = \sqrt{\frac{105}{4\pi}}\xi_1\xi_2\xi_3\xi, \tag{5.144}$$

$$y^{(1)}_{3,3}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{40}}({\xi_1}^3 + \xi_1{\xi_2}^2 - 4\xi_1{\xi_3}^2)\xi, \tag{5.145}$$

$$y^{(1)}_{3,4}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{24}}(3{\xi_1}^2\xi_2 - {\xi_2}^3)\xi, \tag{5.146}$$

$$y^{(1)}_{3,5}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{4}}({\xi_1}^2\xi_3 - {\xi_2}^2\xi_3)\xi, \tag{5.147}$$

$$y^{(1)}_{3,6}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{40}}({\xi_1}^2\xi_2 + {\xi_2}^3 - 4\xi_2\xi_3)\xi, \tag{5.148}$$

$$y^{(1)}_{3,7}(\xi) = \sqrt{\frac{105}{4\pi}}\frac{1}{\sqrt{60}}(3{\xi_1}^2\xi_3 + 3{\xi_2}^2\xi_3 - 2{\xi_3}^3)\xi, \tag{5.149}$$

and

$$y_{3,1}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{24}} \begin{pmatrix} -3{\xi_1}^2({\xi_1}^2 - 3{\xi_2}^2) + 3{\xi_1}^2 - 3{\xi_2}^2 \\ -3\xi_1\xi_2({\xi_1}^2 - 3{\xi_2}^2) - 6\xi_1\xi_2 \\ -3\xi_1\xi_3({\xi_1}^2 - 3{\xi_2}^2) \end{pmatrix}, \quad (5.150)$$

$$y_{3,2}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \begin{pmatrix} -3{\xi_1}^2\xi_2\xi_3 + \xi_2\xi_3 \\ -3\xi_1{\xi_2}^2\xi_3 + \xi_1\xi_3 \\ -3\xi_1\xi_2{\xi_3}^2 + \xi_1\xi_2 \end{pmatrix}, \quad (5.151)$$

$$y_{3,3}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{40}} \begin{pmatrix} -3{\xi_1}^2(1 - 5{\xi_3}^2) + 3{\xi_1}^2 + {\xi_2}^2 - 4{\xi_3}^2 \\ -3\xi_1\xi_2(1 - 5{\xi_3}^2) + 2\xi_1\xi_2 \\ -3\xi_1\xi_3 - 8\xi_1\xi_3 \end{pmatrix},$$

$$y_{3,4}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{24}} \begin{pmatrix} -3\xi_1\xi_2(3{\xi_1}^2 - {\xi_2}^2) + 6\xi_1\xi_2 \\ -3{\xi_2}^2(3{\xi_1}^2 - {\xi_2}^2) + 3{\xi_1}^2 - 3{\xi_2}^2 \\ -3\xi_2\xi_3(3{\xi_1}^2 - {\xi_2}^2) \end{pmatrix}, \quad (5.152)$$

$$y_{3,5}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{4}} \begin{pmatrix} -3\xi_1\xi_3({\xi_1}^2 - {\xi_2}^2) + 2\xi_1\xi_3 \\ -3\xi_2\xi_3({\xi_1}^2 - {\xi_2}^2) - 2\xi_2\xi_3 \\ -3\xi_1\xi_3({\xi_1}^2 - {\xi_2}^2) + {\xi_1}^2 - {\xi_2}^2 \end{pmatrix}, \quad (5.153)$$

$$y_{3,6}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{40}} \begin{pmatrix} -3\xi_1\xi_2(1 - 5{\xi_3}^2) + 2\xi_1\xi_2 \\ -3{\xi_2}^2(1 - 5{\xi_3}^2) + {\xi_1}^2 + 3{\xi_2}^2 - 4{\xi_3}^2 \\ -3\xi_2\xi_3(1 - 5{\xi_3}^2) - 8\xi_2\xi_3 \end{pmatrix}, \quad (5.154)$$

$$y_{3,7}^{(2)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{60}} \begin{pmatrix} -3\xi_1\xi_3(3 - 5{\xi_3}^2) + 6\xi_1\xi_3 \\ -3\xi_2\xi_3(3 - 5{\xi_3}^2) + 6\xi_2\xi_3 \\ -3{\xi_3}^2(3 - 5{\xi_3}^2) + 3(1 - 3{\xi_3}^2) \end{pmatrix}, \quad (5.155)$$

as well as

$$y_{3,1}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{24}} \xi \wedge \begin{pmatrix} 3{\xi_1}^2 - 3{\xi_2}^2 \\ -6\xi_1\xi_2 \\ 0 \end{pmatrix}, \quad (5.156)$$

$$y_{3,2}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \xi \wedge \begin{pmatrix} \xi_2\xi_3 \\ \xi_1\xi_3 \\ \xi_1\xi_2 \end{pmatrix}, \quad (5.157)$$

$$y_{3,3}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{40}} \xi \wedge \begin{pmatrix} 3{\xi_1}^2 + {\xi_2}^2 - 4{\xi_3}^2 \\ 2\xi_1\xi_2 \\ -8\xi_1\xi_3 \end{pmatrix}, \quad (5.158)$$

$$y_{3,4}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{24}} \xi \wedge \begin{pmatrix} 6\xi_1\xi_2 \\ 3{\xi_1}^2 - 3{\xi_2}^2 \\ 0 \end{pmatrix}, \quad (5.159)$$

$$y_{3,5}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{4}} \xi \wedge \begin{pmatrix} 2\xi_1\xi_3 \\ -2\xi_2\xi_3 \\ {\xi_1}^2 - {\xi_2}^2 \end{pmatrix}, \tag{5.160}$$

$$y_{3,6}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{40}} \xi \wedge \begin{pmatrix} 2\xi_1\xi_2 \\ {\xi_1}^2 + 3{\xi_2}^2 - 4{\xi_3}^2 \\ -4\xi_2 \end{pmatrix}, \tag{5.161}$$

$$y_{3,7}^{(3)}(\xi) = \sqrt{\frac{105}{28\pi}} \frac{1}{\sqrt{60}} \xi \wedge \begin{pmatrix} 6\xi_1\xi_3 \\ 6\xi_2\xi_3 \\ 3(\xi_1^2 - 2{\xi_3}^2) \end{pmatrix}. \tag{5.162}$$

5.7 Irreducibility and Orthogonal Invariance of Vector Spherical Harmonics

Another coordinate free classification of vector spherical harmonics can be given by looking at the following system of partial differential equations:

$$\xi \Delta_\xi^*(\xi \cdot f(\xi)) - (\Delta^*)^\wedge(n) f(\xi) = 0, \quad n \geq 0, \tag{5.163}$$

$$\nabla_\xi^*(\nabla_\xi^* \cdot f(\xi)) - (\Delta^*)^\wedge(n) f(\xi) = 0, \quad n \geq 1, \tag{5.164}$$

$$\mathrm{L}_\xi^*(\nabla_\xi^* \cdot (f(\xi) \wedge \xi)) - (\Delta^*)^\wedge(n) f(\xi) = 0, \quad n \geq 1, \tag{5.165}$$

where, as usual,

$$(\Delta^*)^\wedge(n) = -n(n+1). \tag{5.166}$$

Solutions $y_n^{(1)}$ of (5.163) fulfill $\xi \wedge y_n^{(1)}(\xi) = 0$ and, consequently, there exists a scalar function F such that $y_n^{(1)}(\xi) = \xi F(\xi)$. In connection with (5.163), this leads to

$$\xi(\Delta_\xi^* F(\xi) - (\Delta^*)^\wedge(n) F(\xi)) = 0, \tag{5.167}$$

which means that F_1 is a spherical harmonic of degree n, i.e., solutions of (5.163) are of the form $y_n^{(1)}(\xi) = \xi Y_n(\xi)$.

For solutions $y_n^{(2)}$ of (5.164), we get $\xi \cdot y_n^{(2)}(\xi) = 0$ and $\nabla_\xi^* \cdot (\xi \wedge y_n^{(2)}(\xi)) = 0$, such that there exists a scalar function G with $y_n^{(2)}(\xi) = \nabla_\xi^* G(\xi)$. Together with (5.164), this leads to

$$\nabla_\xi^*(\Delta_\xi^* G(\xi) - (\Delta^*)^\wedge(n) G(\xi)) = 0. \tag{5.168}$$

Consequently, we have

$$\Delta_\xi^* G(\xi) - (\Delta^*)^\wedge(n) G(\xi) = \text{const.} \tag{5.169}$$

This means that, up to a constant, G is a spherical harmonic of degree n, and solutions of (5.164) are of the form $y_n^{(2)}(\xi) = \nabla_\xi^* Y_n(\xi)$.

Analogously, solutions $y_n^{(3)}$ of (5.165) fulfill both $\xi \cdot y_n^{(3)}(\xi) = 0$ and $\nabla_\xi^* \cdot y_n^{(3)}(\xi)$ which in turn means that there exists a scalar function H such that $y_n^{(3)}(\xi) = \mathrm{L}_\xi^* H(\xi)$. Consequently,

$$\mathrm{L}_\xi^*(\Delta_\xi^* H(\xi) - (\Delta^*)^\wedge(n) H(\xi)) = 0 \tag{5.170}$$

and, therefore, $y_n^{(3)}(\xi) = \mathrm{L}_\xi^* Y_n(\xi)$.

As we have seen, the solutions of the differential equations (5.163)-(5.165) are the vector spherical harmonics (as defined in the previous section). This observation has immediate consequences for the spaces $\mathrm{harm}_n^{(i)}$ of vector spherical harmonics. In fact, they can be seen to be 'the smallest' orthogonally invariant spaces.

Theorem 5.15. *The spaces* $\mathrm{harm}_n^{(i)}$ *of vector spherical harmonics are orthogonally invariant and irreducible.*

Proof. The orthogonal invariance is a direct consequence of the invariant differential operators of (5.163, 5.164, 5.165). To be more concrete, suppose that there exists an orthogonally invariant subspace of $\mathrm{harm}_n^{(i)}$. The application of the operators $o^{(i)}$ to the respective elements would – because of Definition 5.2 – generate an orthogonally invariant subspace in the space of scalar spherical harmonics. This, however, is a contradiction to the irreducibility of the spaces Harm_n. □

Theorem 5.15 shows us that vector spherical harmonics have the same significance for spherical vector fields, as have the spherical harmonics in the theory of scalar spherical fields. In what follows, we deduce some concrete consequences for vector spherical harmonics.

Suppose that $\mathbf{t}$ is of class $O(3)$. Let, for $i \in \{1,2,3\}$, $\{y_{n,j}^{(i)}\}_{j=1,\ldots,2n+1}$ be an orthonormal system in $\mathrm{harm}_n^{(i)}$. Because of Theorem 5.15, there exist coefficients $c_{j,l}$ such that

$$R_{\mathbf{t}} y_{n,j}^{(i)} = \sum_{l=1}^{2n+1} c_{j,l} y_{n,l}^{(i)}, \quad i \in \{1,2\}, \tag{5.171}$$

$$R_{\mathbf{t}} y_{n,j}^{(i)} = \det \mathbf{t} \sum_{l=1}^{2n+1} c_{j,l} y_{n,l}^{(i)} \tag{5.172}$$

(note that, for $i = 3$, we have to take into account a minus sign for reflections because of the vector product). Consequently, for every orthogonal transformation $\mathbf{t}$ we have, on the one hand, an associated matrix $c_{j,l}$ with

$$\begin{aligned}\int_\Omega R_{\mathbf{t}} y^{(i)}_{n,j}(\xi) \cdot R_{\mathbf{t}} y^{(i)}_{n,k}(\xi)\, d\omega(\xi) &= \sum_{l=1}^{2n+1}\sum_{l'=1}^{2n+1} c_{j,l}c_{k,l'} \int_\Omega y^{(i)}_{n,l}(\xi)\cdot y^{(i)}_{n,l'}(\xi)\, d\omega(\xi) \\ &= \sum_{l=1}^{2n+1}\sum_{l'=1}^{2n+1} c_{j,l}c_{k,l'}\delta_{ll'} \\ &= \sum_{l=1}^{2n+1} c_{j,l}c_{k,l}. \end{aligned} \tag{5.173}$$

On the other hand, we may interpret $\mathbf{t}$ to be a coordinate transformation on Ω, i.e,

$$\int_\Omega R_{\mathbf{t}} y^{(i)}_{n,j}(\xi) \cdot R_{\mathbf{t}} y^{(i)}_{n,k}(\xi)\, d\omega(\xi) = \int_\Omega y^{(i)}_{n,j}(\eta)\cdot y^{(i)}_{n,k}(\eta)\, d\omega(\eta) = \delta_{jk}. \tag{5.174}$$

Comparing (5.173) and (5.174), we get

$$\sum_{l=1}^{2n+1} c_{j,l}c_{k,l} = \delta_{jk}, \tag{5.175}$$

hence, $(c_{j,l})$ is an orthogonal matrix. An analogous treatment leads to

$$\sum_{j=1}^{2n+1} c_{j,k}c_{j,l} = \delta_{kl}. \tag{5.176}$$

Now, for $\xi, \eta \in \Omega$, let

$${}^v\mathbf{p}^{(i,k)}_n(\xi,\eta) = \sum_{j=1}^{2n+1} y^{(i)}_{n,j}(\xi) \otimes y^{(k)}_{n,j}(\eta). \tag{5.177}$$

Then, every $a \in \mathbb{R}^3$, ${}^v\mathbf{p}^{(i,k)}_n(\cdot,\eta)a$ is a member of $\mathrm{harm}^{(i)}_n$. This means that (5.177) provides a mapping from $\mathrm{harm}^{(k)}_n$ onto $\mathrm{harm}^{(i)}_n$.

If $(i,k) \in \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$, then from (5.171), (5.172) and (5.176) we get

$$\begin{aligned}
\mathbf{t}^{T\, v}\mathbf{p}_n^{(i,k)}(\mathbf{t}\xi, \mathbf{t}\eta)\mathbf{t} &= \sum_{j=1}^{2n+1} \mathbf{t}^T \left[y_{n,j}^{(i)}(\mathbf{t}\xi) \otimes y_{n,j}^{(k)}(\mathbf{t}\eta) \right] \mathbf{t} \qquad (5.178)\\
&= \sum_{j=1}^{2n+1} \mathbf{t}^T y_{n,j}^{(i)}(\mathbf{t}\xi) \otimes \mathbf{t}^T y_{n,j}^{(k)}(\mathbf{t}\eta)\\
&= \sum_{j=1}^{2n+1} \left(\sum_{l=1}^{2n+1} c_{j,l} y_{n,l}^{(i)}(\xi) \otimes \sum_{m=1}^{2n+1} c_{j,m} y_{n,m}^{(k)}(\eta) \right)\\
&= \sum_{j=1}^{2n+1} \sum_{l=1}^{2n+1} \sum_{m=1}^{2n+1} c_{j,l} c_{j,m} y_{n,l}^{(i)}(\xi) \otimes y_{n,m}^{(k)}(\eta)\\
&= \sum_{l=1}^{2n+1} \sum_{m=1}^{2n+1} \delta_{lm} y_{n,l}^{(i)}(\xi) \otimes y_{n,m}^{(k)}(\eta)\\
&= {}^v\mathbf{p}_n^{(i,k)}(\xi, \eta).
\end{aligned}$$

If either $i = 3$ or $k = 3$, a similar result can be shown, taking into account a minus sign for reflections.

Summarizing our results, we are able to formulate the following lemma.

Lemma 5.16. *Let* $\mathbf{t}$ *be of class* $O(3)$. *Suppose that* $\xi, \eta \in \Omega$. *Then*

$$ {}^v\mathbf{p}_n^{(i,k)}(\mathbf{t}\xi, \mathbf{t}\eta) = \begin{cases} \mathbf{t}\, {}^v\mathbf{p}_n^{(i,k)}(\xi,\eta)\mathbf{t}^T, & \textit{else} \\ (\det \mathbf{t})\, \mathbf{t}\, {}^v\mathbf{p}_n^{(i,k)}(\xi,\eta)\mathbf{t}^T, & \textit{if either } i = 3 \textit{ or } k = 3. \end{cases} \qquad (5.179)$$

Actually, it is this lemma which makes tensors of the form (5.177) an important tool when dealing with the addition theorem for vector spherical harmonics in terms of Legendre tensors (see Section 5.9).

Let $z_n^{(i)}$ be a member of class $\mathrm{harm}_n^{(i)}$, then the span of elements of the form

$$R_{\mathbf{t}} z_n^{(i)}, \quad \mathbf{t} \in O(3), \qquad (5.180)$$

is orthogonally invariant and, according to Theorem 5.15, is $\mathrm{harm}_n^{(i)}$ itself. Hence, among the vector fields of the form (5.180) there is a basis for $\mathrm{harm}_n^{(i)}$, and we have the following representation theorem:

Theorem 5.17. *Let* $z_n^{(i)}$ *be a member of class* $\mathrm{harm}_n^{(i)}$ *with* $z_n^{(i)} \neq 0$. *Then, there exist* $2n+1$ *orthogonal transformations* $\mathbf{t}_1, \mathbf{t}_2 \ldots, \mathbf{t}_{2n+1}$ *such that every* *vector spherical harmonic* $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ *can be represented in the form*

$$y_n^{(i)} = \sum_{j=1}^{2n+1} c_j R_{\mathbf{t}_j} z_n^{(i)}, \tag{5.181}$$

with certain real numbers $c_1, c_2, \ldots, c_{2n+1}$.

If we, in particular choose, $z_n^{(i)}$ to be an element of the form ${}^v\mathbf{p}_n^{(i,i)}(\cdot, \eta)a$, with $\eta \in \Omega$ and $a \in \mathbb{R}^3$, then Lemma 5.16 allows the following reformulation.

Lemma 5.18. *There exist points* $\eta_1, \eta_2, \ldots, \eta_{2n+1} \in \Omega$ *and vectors* $a_1, a_2, \ldots, a_{2n+1} \in \mathbb{R}^3$ *such that every vector spherical harmonic* $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ *admits the representation*

$$y_n^{(i)} = \sum_{j=1}^{2n+1} c_j {}^v\mathbf{p}_n^{(i,i)}(\cdot, \eta_j)a_j, \tag{5.182}$$

with certain real numbers $c_1, c_2, \ldots, c_{2n+1}$.

Given points $\eta_1, \eta_2, \ldots, \eta_{2n+1} \in \Omega$ and given vectors $a_1, a_2, \ldots, a_{2n+1} \in \mathbb{R}^3$ can be used for a representation in the sense of Lemma 5.18 if the Gram matrix of the vectors is non-singular.

Let $j, k \in \{1, 2, \ldots, 2n+1\}$, then we have

$$\begin{aligned} \int_\Omega ({}^v\mathbf{p}_n^{(i,i)}(\xi, \eta_j)a_j)^T \quad {}^v\mathbf{p}_n^{(i,i)}\,(\xi, \eta_k)a_k \, d\omega(\xi) \\ &= a_j^T \int_\Omega ({}^v\mathbf{p}_n^{(i,i)}(\xi, \eta_j)a_j)^T \mathbf{p}_n^{(i,i)}(\xi, \eta_k)\, d\omega(\xi) a_k \\ &= a_j^{T\,v}\mathbf{p}_n^{(i,i)}(\eta_j, \eta_k)a_k. \end{aligned}$$

Lemma 5.18 is equivalent to the following statement:

Lemma 5.19. *There exist points* $\eta_1, \eta_2, \ldots, \eta_{2n+1} \in \Omega$ *and vectors* $a_1, a_2, \ldots, a_{2n+1} \in \mathbb{R}^3$ *such that the matrix*

$$\left(a_j^{T\,v}\mathbf{p}_n^{(i,i)}(\eta_j, \eta_k)a_k \right)_{\substack{j=1,\ldots,2n+1 \\ k=1,\ldots,2n+1}}, \tag{5.183}$$

$i \in \{1, 2, 3\}$, *is non-singular.*

We now turn to orthogonally invariant elements in the spaces $\mathrm{harm}_n^{(i)}$. From Lemma 2.19, we know that a vector field, which is invariant under all orthogonal transformations, i.e., $f(\mathbf{t}\xi) = \mathbf{t}f(\xi)$ for all $\mathbf{t} \in SO(3)$, is of the form $f(\xi) = C\xi$, $\xi \in \Omega$. There remains the question which elements $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ are transformed onto themselves under rotations around a fixed axis η, i.e.,

$$y_n^{(i)}(\mathbf{t}\xi) = \mathbf{t}y_n^{(i)}(\xi), \tag{5.184}$$

for $\mathbf{t} \in SO_\eta(3)$. For ${}^v\mathbf{p}_n^{(i,1)}(\xi,\eta)\eta$ as a function of ξ, we have, in connection with Lemma 5.16, the relation

$$\begin{aligned} {}^v\mathbf{p}_n^{(i,1)}(\mathbf{t}\xi,\eta)\eta &= \mathbf{t}\,{}^v\mathbf{p}_n^{(i,1)}(\xi,\mathbf{t}^T\eta)\mathbf{t}^T\mathbf{t}\eta \\ &= \mathbf{t}\,{}^v\mathbf{p}_n^{(i,1)}(\xi,\eta)\eta. \end{aligned} \tag{5.185}$$

Let $\hat{p}_n^{(i)} \in \mathrm{harm}_n^{(i)}$, $i \in \{1,2,3\}$, denote a vector field which is invariant under all transformations $\mathbf{t} \in SO_\eta(3)$. From Lemma 2.20, we know that there exist functions Φ_k, $k = 1,2,3$, such that

$$\hat{p}_n^{(i)}(\xi,\eta) = \sum_{k=1}^{3} \Phi_k(\xi\cdot\eta)\,\varepsilon_\xi^k, \tag{5.186}$$

for $\xi \neq \pm\eta$. Since $\hat{p}_n^{(i)}$ is of class $\mathrm{harm}_n^{(i)}$, we know that the functions Φ_k are arbitrarily often differentiable in $(-1,1)$. We consider the case $i=2$ (the cases $i=1,3$ can be treated analogously): Since $\hat{p}_n^{(2)} \in \mathrm{harm}_n^{(2)}$, it is clear that $\varepsilon_\xi^1 \cdot \hat{p}_n^{(2)}(\xi,\eta) = 0$ as well as $\nabla_\xi^* \cdot (\xi \wedge \hat{p}_n^{(2)}(\xi,\eta)) = 0$. Furthermore we have $\Phi_1 = \Phi_3 = 0$. Consequently, (5.186) simplifies to

$$\hat{p}_n^{(2)}(\xi,\eta) = \varepsilon_\xi^2\Phi_2(\xi\cdot\eta) = \nabla_\xi^*\Psi(\xi\cdot\eta), \tag{5.187}$$

$\xi \neq \pm\eta$, where Ψ is an antiderivative of $(1-t^2)^{-1/2}\Phi_2$ in $(-1,1)$. Since $\nabla_\xi^* \cdot \hat{p}_n^{(2)}(\xi,\eta)$ defines a spherical harmonic of degree n, this is also true for $\Delta_\xi^*\Psi(\xi\cdot\eta)$ and, consequently, for $\Psi(\xi\cdot\eta)$. Since $\Psi(\xi\cdot\eta)$ is only dependent on the scalar product $\xi\cdot\eta$ it is clear that $\Psi(\xi\cdot\eta)$ is – up to a factor – given by the Legendre polynomial $P_n(\xi\cdot\eta)$, i.e.,

$$\hat{p}_n^{(2)}(\xi,\eta) = \lambda\nabla_\xi^* P_n(\xi\cdot\eta) = \lambda\,(\eta - (\xi\cdot\eta)\xi)\,P_n'(\xi\cdot\eta), \quad \lambda \in \mathbb{R}. \tag{5.188}$$

The limit $\xi \to \pm\eta$ yields $\hat{p}_n^{(2)}(\pm\eta,\eta) = 0$. A similar argument leads to

$$\hat{p}_n^{(1)}(\xi,\eta) = \lambda\,\eta\,P_n(\xi\cdot\eta), \tag{5.189}$$

and

$$\hat{p}_n^{(3)}(\xi,\eta) = \lambda \mathrm{L}_\xi^* P_n(\xi\cdot\eta) = \lambda\,(\xi\wedge\eta)\,P_n'(\xi\cdot\eta). \tag{5.190}$$

Summarizing our results, we are led to the following representations of 'Legendre vector functions'.

Theorem 5.20. *Let $\eta \in \Omega$. In* $\mathrm{harm}_n^{(i)}$, $i = 1,2,3$, *there exists one and only one element* $p_n^{(i)}(\cdot, \eta)$, *with the following properties:*

(i)
$$p_n^{(i)}(\mathbf{t}\cdot, \eta) = \mathbf{t} p_n^{(i)}(\cdot, \eta), \; \mathbf{t} \in SO_\eta(3),$$

(ii)
$$\begin{aligned} \eta \cdot p_n^{(1)}(\eta, \eta) &= 1, \\ -(\mu_n^{(2)})^{1/2} \nabla_\xi^* \cdot p_n^{(2)}(\xi, \eta)|_{\xi=\eta} &= 1, \\ -(\mu_n^{(3)})^{1/2} \mathrm{L}_\xi^* \cdot p_n^{(3)}(\xi, \eta)|_{\xi=\eta} &= 1. \end{aligned}$$

Proof. Condition (i) defines $p_n^{(i)}(\cdot, \eta)$ up to a normalization constant. With condition (ii) of Theorem 5.20, the normalization constants can be calculated as follows:

$$p_n^{(1)}(\xi, \eta) = \xi \, P_n(\xi \cdot \eta), \tag{5.191}$$

$$p_n^{(2)}(\xi, \eta) = \frac{1}{\sqrt{n(n+1)}} \nabla_\xi^* P_n(\xi \cdot \eta), \tag{5.192}$$

$$p_n^{(3)}(\xi, \eta) = \frac{1}{\sqrt{n(n+1)}} \mathrm{L}_\xi^* P_n(\xi \cdot \eta). \tag{5.193}$$

i.e., for $i = 1,2,3$,

$$p_n^{(i)}(\xi, \eta) = (\mu_n^{(i)})^{-1/2} o_\xi^{(i)} P_n(\xi \cdot \eta). \tag{5.194}$$

□

Definition 5.21. The kernel $p_n^{(i)} : (\xi, \eta) \mapsto p_n^{(i)}(\xi, \eta), \xi, \eta \in \Omega$ (more accurately, ${}^v p_n^{(i)} : (\xi, \eta) \mapsto {}^v p_n^{(i)}(\xi, \eta), \xi, \eta \in \Omega$), is called the *(vectorial) Legendre vector kernel of degree n and type i with respect to the dual system of operators $o^{(i)}, O^{(i)}$, $i \in \{1,2,3\}$.* The kernel $p_n = \sum_{i=1}^{3} p_n^{(i)}$ is called *(vectorial) Legendre vector kernel of degree n with respect to the system $o^{(i)}, O^{(i)}$, $i = 1,2,3$..*

It should be noted that these vector fields bear a close resemblance to the Legendre vectors which are vectorial counterparts of the Legendre polynomials, and which can be used to formulate an addition theorem connecting scalar and vector spherical harmonics. This resemblance and the rotational invariance of the vector fields (5.191, 5.192, 5.193) make the Legendre vectors such important tools of vectorial theory.

According to Theorem 5.20, we can generate a basis in $\mathrm{harm}_n^{(i)}$ from any non-vanishing vector spherical harmonic using orthogonal transformations

$\mathbf{t}_1, \mathbf{t}_2 \ldots, \mathbf{t}_{2n+1}$. If, in this context, we use rotational invariant vector fields, we can find $2n+1$ linearly independent vector fields of the form $R_{\mathbf{t}_j} p_n^{(i)}(\cdot, \eta)$, $j = 1, \ldots, 2n+1$. Hence we end up with the following *representation theorem.*

Theorem 5.22. *There exist $2n+1$ points $\eta_1, \eta_2, \ldots, \eta_{2n+1} \in \Omega$ such that every vector spherical harmonic $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ can be represented in the form*

$$y_n^{(i)} = \sum_{j=1}^{2n+1} c_j p_n^{(i)}(\cdot, \eta_j) \tag{5.195}$$

with certain real constants $c_1, \ldots, c_{2n+1}$.

In what follows, we recapitulate some interesting results for rotational invariant spherical vector fields.

Lemma 5.23. *For $\xi, \eta \in \Omega$ and for $i = 1, 2, 3$*

$$\left| (\mu_n^{(i)})^{-1/2} p_n^{(i)}(\xi, \eta) \right| \leq 1.$$

Proof. The well known inequalities $|P_n(t)| \leq 1$ and $|P_n'(t)| \leq \frac{n(n+1)}{2}$, $t \in [-1, 1]$, together with (5.191, 5.192, 5.193) lead to the required result. □

An immediate consequence of the orthogonality of vector spherical harmonics is the orthogonality of rotational invariant vector fields on the sphere, i.e.,

$$\int_\Omega p_n^{(i)}(\xi, \eta_1) \cdot p_m^{(i)}(\xi, \eta_2) \, d\omega(\xi) = 0 \tag{5.196}$$

for $n \neq m$ and $\eta_1, \eta_2 \in \Omega$.

Lemma 5.24. *If $\eta_1, \eta_2 \in \Omega$, then*

$$\int_\Omega p_n^{(i)}(\xi, \eta_1) \cdot p_n^{(i)}(\xi, \eta_2) \, d\omega(\xi) = \frac{4\pi}{2n+1} P_n(\eta_1 \cdot \eta_2), \quad i = \{1, 2, 3\}.$$

Proof. We show the proof for the case $i = 2$. The other cases can be treated in analogous manner:

$$
\begin{aligned}
& \int_\Omega p_n^{(i)}(\xi,\eta_1)\cdot p_n^{(i)}(\xi,\eta_2)\; d\omega(\xi) && (5.197)\\
&= \frac{1}{n(n+1)}\int_\Omega \nabla_\xi^* P_n(\xi\cdot\eta_1)\cdot\nabla_\xi^* P_n(\xi\cdot\eta_2)\; d\omega(\xi)\\
&= \int_\Omega P_n(\xi\cdot\eta_1)P_n(\xi\cdot\eta_2)\, d\omega(\xi)\\
&= \frac{4\pi}{2n+1}P_n(\eta_1\cdot\eta_2).
\end{aligned}
$$

This is the desired result. □

5.8 Vectorial Beltrami Operator

Next, we develop a vectorial analogue of the Beltrami operator Δ^*, denoted by $\boldsymbol{\Delta}^*$. In doing so, the vector spherical harmonics of class harm_n can be recognized as eigenfunctions of the vectorial operator $\boldsymbol{\Delta}^*$. In particular , it turns out that the operator $\boldsymbol{\Delta}^*$ corresponds to the orthogonal decomposition with respect to the operators $o^{(i)}, i = 1, 2, 3$, that is to say, $\boldsymbol{\Delta}^* f \in \mathrm{c}_{(i)}(\Omega)$ for all $i = 1, 2, 3$, provided that $f \in \mathrm{c}^{(2)}_{(i)}(\Omega)$.

Our construction is based on the componentwise application of the (scalar) Beltrami operator Δ^*: The point of departure is the convention that, if $f \in \mathrm{c}^{(2)}(\Omega)$ is of the form

$$f(\xi) = \sum_{i=1}^{3} \varepsilon^i F_i(\xi), \quad \xi \in \Omega, \tag{5.198}$$

then $\Delta^* f$ is understood to be

$$\Delta^* f(\xi) = \sum_{i=1}^{3} \varepsilon^i \Delta^* F_i(\xi), \quad \xi \in \Omega. \tag{5.199}$$

Observing this setting, we are able to deduce the following identities.

Lemma 5.25. *The following statements are true.*

(i) *Let $F : \Omega \to \mathbb{R}$ be sufficiently smooth. Then*

$$
\begin{aligned}
\Delta^* o^{(1)} F &= o^{(1)}(\Delta^* - 2)F + 2o^{(2)}F,\\
\Delta^* o^{(2)} F &= -2o^{(1)}\Delta^* F + o^{(2)}\Delta^* F,\\
\Delta^* o^{(3)} F &= o^{(3)}\Delta^* F.
\end{aligned}
$$

(ii) *Let $f : \Omega \to \mathbb{R}^3$ be a sufficiently smooth vector field. Then*

$$\begin{aligned} O^{(1)}\Delta^* f &= (\Delta^* - 2)O^{(1)} f + 2O^{(2)} f, \\ O^{(2)}\Delta^* f &= -2\Delta^* O^{(1)} f + \Delta^* O^{(2)} f, \\ O^{(3)}\Delta^* f &= \Delta^* O^{(3)} f. \end{aligned}$$

Proof. We verify only the first formula of part (i), since the other assertions follow by quite similar arguments. Assume that the function under consideration is a spherical harmonic Y_n of class Harm_n. Then, it follows from (5.108) that

$$\begin{aligned} \Delta^*_\xi o^{(1)}_\xi Y_n(\xi) &= \Delta^*_\xi \frac{1}{2n+1} \left(\tilde{o}^{(1)}_n r^n Y_n(\xi)|_{r=1} + \tilde{o}^{(2)}_n r^n Y_n(\xi)|_{r=1} \right) \\ &= -\frac{(n+1)(n+2)}{2n+1} (\tilde{o}^{(1)}_n r^n Y_n(\xi))|_{r=1} \\ &\quad -\frac{n(n-1)}{2n+1} (\tilde{o}^{(2)}_n r^n Y_n(\xi))|_{r=1} \end{aligned} \tag{5.200}$$

holds for all $\xi \in \Omega$. Using (5.97), we obtain

$$\begin{aligned} \Delta^*_\xi o^{(1)}_\xi Y_n(\xi) &= -\frac{(n+1)^2(n+2)}{2n+1} o^{(1)}_\xi Y_n(\xi) + \frac{(n+1)(n+2)}{2n+1} o^{(2)}_\xi Y_n(\xi) \\ &\quad -\frac{(n-1)n^2}{2n+1} o^{(1)}_\xi Y_n(\xi) - \frac{n(n-1)}{2n+1} o^{(2)}_\xi Y_n(\xi) \\ &= (-n(n+1) - 2) o^{(1)}_\xi Y_n(\xi) + 2 o^{(2)}_\xi Y_n(\xi) \\ &= (\Delta^*_\xi - 2) o^{(1)}_\xi Y_n(\xi) + 2 o^{(2)}_\xi Y_n(\xi). \end{aligned} \tag{5.201}$$

Thus, our formula is true for every spherical harmonic. The completeness of the system of vector spherical harmonics in $\mathrm{l}^2(\Omega)$ then implies the validity for all sufficiently smooth functions. Part (ii) follows from the adjointness of the operators $o^{(i)}$ and $O^{(i)}$ and the self-adjointness of Δ^*. □

Lemma 5.25 helps us to find a vectorial Beltrami operator by observing the following identities

$$\begin{aligned} (\Delta^* + 2) o^{(1)} Y_n &= -n(n+1) o^{(1)} Y_n + 2 o^{(2)} Y_n, \\ \Delta^* o^{(2)} Y_n &= 2n(n+1) o^{(1)} Y_n - n(n+1) o^{(2)} Y_n, \\ \Delta^* o^{(3)} Y_n &= -n(n+1) o^{(3)} Y_n. \end{aligned} \tag{5.202}$$

In consequence, we have

$$\begin{aligned} p_{\mathrm{nor}}(\Delta^* + 2) o^{(1)} Y_n &= -n(n+1) o^{(1)} Y_n, \\ p_{\mathrm{tan}} \Delta^* o^{(2)} Y_n &= -n(n+1) o^{(2)} Y_n, \\ p_{\mathrm{tan}} \Delta^* o^{(3)} Y_n &= -n(n+1) o^{(3)} Y_n. \end{aligned} \tag{5.203}$$

In other words, the equations (5.203) motivate the introduction of the vectorial Beltrami operator in the following way.

Definition 5.26. The operator $\boldsymbol{\Delta}^* : \mathrm{c}^{(2)}(\Omega) \to \mathrm{c}^{(0)}(\Omega)$ given by

$$\boldsymbol{\Delta}^* = p_{\mathrm{nor}}(\Delta^* + 2)p_{\mathrm{nor}} + p_{\mathrm{tan}}\Delta^* p_{\mathrm{tan}},$$

is called *vectorial Beltrami operator*, where the application of Δ^* on vector fields is understood in the sense of (5.199).

As an immediate consequence of Lemma 5.25, we obtain the following result.

Lemma 5.27. *If $F : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}^3$ are sufficiently smooth, then, for $i \in \{1, 2, 3\}$,*

$$\begin{aligned} \boldsymbol{\Delta}^* o^{(i)} F &= o^{(i)} \Delta^* F, \\ O^{(i)} \boldsymbol{\Delta}^* f &= \Delta^* O^{(i)} f. \end{aligned}$$

Lemma 5.27 motivates us to characterize the spectrum of the operator $\boldsymbol{\Delta}^*$.

Theorem 5.28. *Any vector spherical harmonic $y_n \in \mathrm{harm}_n$ of degree n is an infinitely often differentiable eigenfunction of the operator $\boldsymbol{\Delta}^*$ with respect to the eigenvalue $(\boldsymbol{\Delta}^*)^\wedge(n) = (\Delta^*)^\wedge(n) = -(n(n+1))$. Conversely, any infinitely often differentiable eigenfunction of $\boldsymbol{\Delta}^*$ is a vector spherical harmonic.*

Remark 5.29. The vector spherical harmonics can be seen to be the eigenfunctions of an operator $\boldsymbol{\Delta}^*$ that can be introduced (without projection operators) only by use of differentiation processes in $\mathbb{R}^3$ (see T. Gervens (1989)). More explicitly, the following theorem is valid.

Theorem 5.30. *Let $\boldsymbol{\Delta}^*$ be given by*

$$\boldsymbol{\Delta}^*_\xi f(\xi) = \Delta^*_\xi f(\xi) - 2(\xi \wedge \nabla_\xi) \wedge f(\xi) - 2f(\xi), \quad f \in \mathrm{c}^{(2)}(\Omega). \tag{5.204}$$

Then

$$\boldsymbol{\Delta}^*_\xi y_n + n(n+1)y_n = 0, \quad y_n \in \mathrm{harm}_n. \tag{5.205}$$

5.9 Vectorial Addition Theorem

From the scalar theory, we know the fundamental role of the addition theorem of spherical harmonics. Our interest now is the formulation of a vectorial analogue of the addition theorem involving tensorial structure. This vectorial addition theorem assures the existence of a vectorial reproducing kernel which is a basic tool, for example, in the theory of 'vectorial zonal functions'. In addition, this addition theorem offers a better insight into orthogonal invariance within the theory of vector spherical harmonics.

Let $\{y^{(i)}_{n,j}\}_{\substack{i=1,2,3\\ j=1,\ldots,2n+1}}$ be an $\mathrm{l}^2(\Omega)$-orthonormal basis of harm_n, as defined in (5.36), corresponding to an $\mathrm{L}^2(\Omega)$-orthonormal basis $\{Y_{n,j}\}_{j=1,\ldots,2n+1}$ of Harm_n: $y^{(i)}_{n,j} = (\mu^{(i)}_n)^{-1/2} o^{(i)} Y_{n,j}$. Then, the announced vectorial analogue of the addition theorem deals with the question of determining the expression

$$ {}^v\mathbf{p}^{(i,k)}_n(\xi,\eta) = \frac{2n+1}{4\pi} \sum_{j=1}^{2n+1} y^{(i)}_{n,j}(\xi) \otimes y^{(k)}_{n,j}(\eta), \quad \xi,\eta \in \Omega. \tag{5.206} $$

Our purpose is to explain how a vectorial counterpart of tensorial nature for the Legendre polynomial comes into play. For that purpose, we first extend in canonical way the definition of the $o^{(i)}$-operators (cf. (5.17)–(5.19)) to vector fields.

Suppose that $f : \Omega \to \mathbb{R}^3$ is a sufficiently smooth vector field of the representation

$$ f(\xi) = \sum_{k=1}^{3} F_k(\xi) \varepsilon^k, \quad F_k(\xi) = f(\xi) \cdot \varepsilon^k, \quad \xi \in \Omega. \tag{5.207} $$

Then we set

$$ \begin{aligned} o^{(i)}_\xi f(\xi) &= \sum_{k=1}^{3} (o^{(i)}_\xi F_k(\xi)) \otimes \varepsilon^k \qquad (5.208) \\ &= \sum_{k=1}^{3} \left(o^{(i)} (f(\xi) \cdot \varepsilon^k) \right) \otimes \varepsilon^k, \quad i = 1,2,3. \end{aligned} $$

Thus, $o^{(i)}$ maps scalar functions to vector fields and vector fields to rank-2 tensor fields, respectively.

In accordance with this nomenclature, (5.206) can be expressed as follows:

$$ \begin{aligned} \sum_{j=1}^{2n+1} y^{(i)}_{n,j}(\xi) \otimes y^{(k)}_{n,j}(\eta) &= (\mu^{(i)}_n)^{-1/2} (\mu^{(k)}_n)^{-1/2} \sum_{j=1}^{2n+1} o^{(i)}_\xi Y_{n,j}(\xi) \otimes o^{(k)}_\eta Y_{n,j}(\eta) \\ &= (\mu^{(i)}_n)^{-1/2} (\mu^{(k)}_n)^{-1/2} o^{(i)}_\xi o^{(k)}_\eta \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) \\ &= (\mu^{(i)}_n)^{-1/2} (\mu^{(k)}_n)^{-1/2} \frac{2n+1}{4\pi} o^{(i)}_\xi o^{(k)}_\eta P_n(\xi \cdot \eta). \end{aligned} \tag{5.209} $$

In other words, ${}^v\mathbf{p}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is given in terms of the one–dimensional Legendre polynomial by

$$ {}^v\mathbf{p}_n^{(i,k)}(\xi,\eta) = (\mu_n^{(i)})^{-1/2}(\mu_n^{(k)})^{-1/2} o_\xi^{(i)} o_\eta^{(k)} P_n(\xi\cdot\eta), \quad \xi,\eta\in\Omega. \tag{5.210} $$

Altogether, this leads us to the following variant of the *addition theorem for vector spherical harmonics.*

Theorem 5.31. *Let* $\{y_{n,j}^{(i)}\}_{\substack{i=1,2,3\\ j=1,\ldots,2n+1}}$ *be an* $\mathrm{l}^2(\Omega)$*-orthonormal basis of* harm_n *as given in (5.36). Then, for* $\xi,\eta\in\Omega$*,*

$$ \begin{aligned} \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi)\otimes y_{n,j}^{(k)}(\eta) &= \frac{2n+1}{4\pi}\, {}^v\mathbf{p}_n^{(i,k)}(\xi,\eta) \\ &= (\mu_n^{(i)})^{-1/2}(\mu_n^{(k)})^{-1/2}\frac{2n+1}{4\pi} o_\xi^{(i)} o_\eta^{(k)} P_n(\xi\cdot\eta) \end{aligned} $$

holds for $i,k\in\{1,2,3\}$*.*

Definition 5.32. The kernel ${}^v\mathbf{p}_n^{(i,k)} : \Omega\times\Omega \to \mathbb{R}^3\otimes\mathbb{R}^3,\ i,k\in\{1,2,3\}$ given by

$$ (\mu_n^{(i)})^{-1/2}(\mu_n^{(k)})^{-1/2}\frac{2n+1}{4\pi} o_\xi^{(i)} o_\eta^{(k)} P_n(\xi\cdot\eta) = \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi)\otimes y_{n,j}^{(k)}(\eta) \tag{5.211} $$

is called the *(vectorial) Legendre rank-2 tensor kernel of degree* n *and type* (i,k) *with respect to the dual system of operators* $o^{(i)}, O^{(i)}$, $i\in\{1,2,3\}$. The kernel

$$ {}^v\mathbf{p}_n = \sum_{i=1}^{3}\sum_{k=1}^{3} {}^v\mathbf{p}_n^{(i,k)} \tag{5.212} $$

is called *(vectorial) Legendre rank-2 tensor kernel of degree* n *with respect to the dual system of operators* $o^{(i)}, O^{(i)}, i=1,2,3$.

The main problem to be solved is the evaluation of ${}^v\mathbf{p}_n^{(i,k)}$ as introduced by (5.210). As auxiliary results, we verify the following identities.

Lemma 5.33. *For* $\xi,\eta\in\Omega$*,*

$$ \begin{aligned} o_\xi^{(2)}(\xi-(\xi\cdot\eta)\eta) &= \mathbf{i}_{\tan}(\xi) - (\eta-(\xi\cdot\eta)\xi)\otimes\eta, \\ o_\xi^{(3)}(\xi-(\xi\cdot\eta)\eta) &= \mathbf{j}_{\tan}(\xi) - (\xi\wedge\eta)\otimes\eta, \\ o_\xi^{(2)}(\eta\wedge\xi) &= -\mathbf{j}_{\tan}(\eta) - \xi\otimes\eta\wedge\xi, \\ o_\xi^{(3)}(\eta\wedge\xi) &= (\xi\cdot\eta)\mathbf{i}_{\tan}(\xi) - (\eta-(\xi\cdot\eta)\xi)\otimes\xi. \end{aligned} $$

Proof. We prove the first and third formulae. The second and fourth formulae follow by similar arguments, since we know that $\mathrm{L}^*_\xi = \xi \wedge \nabla^*_\xi$.

First we get

$$o^{(2)}_\xi(\xi - (\xi\cdot\eta)\eta) = o^{(2)}_\xi \sum_{l=1}^{3}((\xi\cdot\varepsilon^l) - (\xi\cdot\eta)(\eta\cdot\varepsilon^l))\varepsilon^l \tag{5.213}$$
$$= \sum_{l=1}^{3}(\varepsilon^l - (\xi\cdot\varepsilon^l)\xi)\otimes\varepsilon^l - ((\eta\cdot\varepsilon^l)(\eta-(\xi\cdot\eta)\xi)\otimes\varepsilon^l)$$
$$= \mathbf{i}_{\mathrm{tan}}(\xi) - (\eta - (\xi\cdot\eta)\xi)\otimes\eta.$$

Furthermore, we have

$$o^{(2)}_\xi(\eta\wedge\xi) = o^{(2)}_\xi \sum_{l=1}^{3}((\eta\wedge\xi)\cdot\varepsilon^l)\varepsilon^l \tag{5.214}$$
$$= o^{(2)}_\xi \sum_{l=1}^{3}((\varepsilon^l\wedge\eta)\cdot\xi)\varepsilon^l$$
$$= \sum_{l=1}^{3}(\varepsilon^l\wedge\eta - ((\varepsilon^l\wedge\eta)\cdot\xi)\xi)\otimes\varepsilon^l$$
$$= \sum_{l=1}^{3}(-\eta\wedge\varepsilon^l)\otimes\varepsilon^l - ((\eta\wedge\xi)\cdot\varepsilon^l)\xi\otimes\varepsilon^l$$
$$= -\mathbf{j}_{\mathrm{tan}}(\eta) - \xi\otimes\eta\wedge\xi.$$

□

If $F : [-1,1] \to \mathbb{R}$ is sufficiently smooth, then, for $\xi,\eta\in\Omega$ we obtain from Lemma 5.33:

$$o^{(1)}_\xi o^{(1)}_\eta F(\xi\cdot\eta) = F(\xi\cdot\eta)\xi\otimes\eta, \tag{5.215}$$
$$o^{(1)}_\xi o^{(2)}_\eta F(\xi\cdot\eta) = F'(\xi\cdot\eta)\xi\otimes(\xi-(\xi\cdot\eta)\eta), \tag{5.216}$$
$$o^{(1)}_\xi o^{(3)}_\eta F(\xi\cdot\eta) = F'(\xi\cdot\eta)\xi\otimes\eta\wedge\xi. \tag{5.217}$$

Similar results hold for $o^{(2)}_\xi o^{(1)}_\eta F(\xi\cdot\eta)$ and $o^{(3)}_\xi o^{(1)}_\eta F(\xi\cdot\eta)$. Treating the tangential operators, we find for $\xi,\eta\in\Omega$

$$o^{(2)}_\xi o^{(2)}_\eta F(\xi\cdot\eta) = \nabla^*_\xi\otimes(F'(\xi\cdot\eta)(\xi-(\xi\cdot\eta)\eta)) \tag{5.218}$$
$$= (\nabla^*_\xi F'(\xi\cdot\eta))\otimes(\xi-(\xi\cdot\eta)\eta) + F'(\xi\cdot\eta)\nabla^*_\xi\otimes(\xi-(\xi\cdot\eta)\eta)$$
$$= F''(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes(\xi-(\xi\cdot\eta)\eta)$$
$$+F'(\xi\cdot\eta)(\mathbf{i}_{\mathrm{tan}}(\xi) - (\eta-(\xi\cdot\eta)\xi)\otimes\eta)$$

and

$$\begin{aligned} o^{(2)}_\xi o^{(3)}_\eta F(\xi\cdot\eta) &= \nabla^*_\xi \otimes (F'(\xi\cdot\eta)\eta\wedge\xi) \\ &= (\nabla^*_\xi F'(\xi\cdot\eta))\otimes\eta\wedge\xi + F'(\xi\cdot\eta)\nabla^*_\xi\otimes(\eta\wedge\xi) \\ &= F''(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes\eta\wedge\xi \\ &\quad +F'(\xi\cdot\eta)(-\mathbf{j}_{\tan}(\eta)-\xi\otimes\eta\wedge\xi). \end{aligned} \tag{5.219}$$

Similar calculations yield the following formulas

$$\begin{aligned} o^{(3)}_\xi o^{(2)}_\eta F(\xi\cdot\eta) &= F''(\xi\cdot\eta)\xi\wedge\eta\otimes(\xi-(\xi\cdot\eta)\eta) \\ &\quad +F'(\xi\cdot\eta)(\mathbf{j}_{\tan}(\xi)-\xi\wedge\eta\otimes\eta) \end{aligned} \tag{5.220}$$

and

$$\begin{aligned} o^{(3)}_\xi o^{(3)}_\eta F(\xi\cdot\eta) &= F''(\xi\cdot\eta)\xi\wedge\eta\otimes\eta\wedge\xi \\ &\quad +F'(\xi\cdot\eta)((\xi\cdot\eta)\mathbf{i}_{\tan}(\xi)-(\eta-(\xi\cdot\eta)\xi)\otimes\xi). \end{aligned} \tag{5.221}$$

Applying these results to the scalar Legendre polynomial P_n, we obtain the following *representation for the Legendre tensor* ${}^v\mathbf{p}_n^{(i,k)}$ *of degree* n *and type* (i,k) with respect to the dual system of operators $o^{(i)}, O^{(i)}, i,k\in\{1,2,3\}$.

Theorem 5.34. *Suppose that* n *is a non-negative integer. Then the identities*

$$\begin{aligned} {}^v\mathbf{p}_n^{(1,1)}(\xi,\eta) &= P_n(\xi\cdot\eta)\xi\otimes\eta, \\ {}^v\mathbf{p}_n^{(1,2)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\otimes(\xi-(\xi\cdot\eta)\eta), \\ {}^v\mathbf{p}_n^{(1,3)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\otimes\eta\wedge\xi, \\ {}^v\mathbf{p}_n^{(2,1)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes\eta, \\ {}^v\mathbf{p}_n^{(3,1)}(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)\xi\wedge\eta\otimes\eta, \\ {}^v\mathbf{p}_n^{(2,2)}(\xi,\eta) &= \frac{1}{n(n+1)}(P_n''(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes(\xi-(\xi\cdot\eta)\eta) \\ &\quad + P_n'(\xi\cdot\eta)(\mathbf{i}_{\tan}(\xi)-(\eta-(\xi\cdot\eta)\xi)\otimes\eta)), \\ {}^v\mathbf{p}_n^{(2,3)}(\xi,\eta) &= \frac{1}{n(n+1)}(P_n''(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes\eta\wedge\xi \\ &\quad + P_n'(\xi\cdot\eta)(-\mathbf{j}_{\tan}(\eta)-\xi\otimes\eta\wedge\xi)), \\ {}^v\mathbf{p}_n^{(3,2)}(\xi,\eta) &= \frac{1}{n(n+1)}(P_n''(\xi\cdot\eta)\xi\wedge\eta\otimes(\xi-(\xi\cdot\eta)\eta) \\ &\quad + P_n'(\xi\cdot\eta)(\mathbf{j}_{\tan}(\xi)-\xi\wedge\eta\otimes\eta)), \end{aligned}$$

$$
\begin{aligned}
{}^{v}\mathbf{p}_n^{(3,3)}(\xi,\eta) &= \frac{1}{n(n+1)}\big(P_n''(\xi\cdot\eta)\xi\wedge\eta\otimes\eta\wedge\xi \\
&\quad + P_n'(\xi\cdot\eta)((\xi\cdot\eta)\mathbf{i}_{\tan}(\xi)-(\eta-(\xi\cdot\eta)\xi)\otimes\xi)\big)
\end{aligned}
$$

hold for all $(\xi,\eta)\in\Omega\times\Omega$.

The case $\xi=\eta$ in the last theorem is of particular interest. Observing $P_n(1)=1, P_n'(1)=\frac{1}{2}n(n+1)$, and $P_n''(1)=\frac{1}{8}n(n+1)(n(n+1)-2)$ (cf. Chapter 3.5), we obtain the following corollary.

Corollary 5.35. *For* $n\in\mathbb{N}_0$ *and all* $\xi\in\Omega$

$$
\begin{aligned}
{}^{v}\mathbf{p}_n^{(1,1)}(\xi,\xi) &= \xi\otimes\xi, \\
{}^{v}\mathbf{p}_n^{(1,i)}(\xi,\xi) &= {}^{v}\mathbf{p}_n^{(i,1)}(\xi,\xi)=0, \quad i=2,3, \\
{}^{v}\mathbf{p}_n^{(2,2)}(\xi,\xi) &= {}^{v}\mathbf{p}_n^{(3,3)}(\xi,\xi)=\frac{1}{2}\mathbf{i}_{\tan}(\xi), \\
{}^{v}\mathbf{p}_n^{(2,3)}(\xi,\xi) &= -{}^{v}\mathbf{p}_n^{(3,2)}(\xi,\xi)=-\frac{1}{2}\mathbf{j}_{\tan}(\xi).
\end{aligned}
$$

It follows readily that

$$
\operatorname{trace}\,((\eta-(\xi\cdot\eta)\xi)\otimes(\xi-(\xi\cdot\eta)\eta))=-(\xi\cdot\eta)(1-(\xi\cdot\eta)^2) \tag{5.222}
$$

and

$$
\operatorname{trace}\,(\xi\wedge\eta\otimes\eta\wedge\xi)=-(1-(\xi\cdot\eta)^2). \tag{5.223}
$$

Hence, from Theorem 5.34, we get the following identities.

Lemma 5.36. *For* $n\in\mathbb{N}_0$ *and all* $\xi,\eta\in\Omega$ *we have*

$$
\begin{aligned}
\operatorname{trace}\,({}^{v}\mathbf{p}_n^{(1,1)}(\xi,\eta)) &= P_n(\xi\cdot\eta)(\xi\cdot\eta), \\
\operatorname{trace}\,({}^{v}\mathbf{p}_n^{(1,2)}(\xi,\eta)) &= \operatorname{trace}\,({}^{v}\mathbf{p}_n^{(2,1)}(\xi,\eta)) \\
&= \frac{1}{\sqrt{n(n+1)}}P_n'(\xi\cdot\eta)(1-(\xi\cdot\eta)^2), \\
\operatorname{trace}\,({}^{v}\mathbf{p}_n^{(2,2)}(\xi\cdot\eta)) &= \frac{1}{n(n+1)}(P_n''(\xi\cdot\eta)(1-(\xi\cdot\eta)^2)(\xi\cdot\eta)+2P_n'(\xi\cdot\eta)), \\
\operatorname{trace}\,({}^{v}\mathbf{p}_n^{(3,3)}(\xi,\eta)) &= P_n(\xi\cdot\eta), \\
\operatorname{trace}\,({}^{v}\mathbf{p}_n^{(1,3)}(\xi,\eta)) &= \operatorname{trace}\,({}^{v}\mathbf{p}_n^{(3,1)}(\xi,\eta)) \\
&= \operatorname{trace}\,({}^{v}\mathbf{p}_n^{(2,3)}(\xi,\eta)) \\
&= \operatorname{trace}\,({}^{v}\mathbf{p}_n^{(3,2)}(\xi,\eta)) \\
&= 0.
\end{aligned}
$$

Taking $\xi=\eta$ in the last lemma we get, in connection with Theorem 5.31, the following result.

Lemma 5.37. *Let* $n \in \mathbb{N}$ *and* $i \in \{1,2,3\}$. *If* $y^{(i)}_{n,j}$, $j = 1,\ldots,2n+1$, *forms an* $\mathrm{l}^2(\Omega)$*-orthonormal basis of* $\mathrm{harm}^{(i)}_n$, *then*

$$\sum_{j=1}^{2n+1} \left(y^{(i)}_{n,j}(\xi)\right)^2 = \frac{2n+1}{4\pi}.$$

Every vector spherical harmonic $y^{(i)}_n \in \mathrm{harm}^{(i)}_n$ of degree n and type i can be represented by its orthogonal expansion

$$y^{(i)}_n = \sum_{j=1}^{2n+1} a_{n,j} y^{(i)}_{n,j} \tag{5.224}$$

with

$$a_{n,j} = (y^{(i)}_n, y^{(i)}_{n,j})_{\mathrm{l}^2(\Omega)}, \quad j = 1,\ldots,2n+1. \tag{5.225}$$

Application of the Cauchy–Schwarz inequality in combination with Lemma 5.37 yields the estimate

$$|y^{(i)}_n(\xi)|^2 \leq \left(\sum_{j=1}^{2n+1} a^2_{n,j}\right)\left(\sum_{j=1}^{2n+1} |y^{(i)}_{n,j}(\xi)|^2\right) \tag{5.226}$$

$$= \frac{2n+1}{4\pi} \sum_{j=1}^{2n+1} a^2_{n,j} \tag{5.227}$$

for all $\xi \in \Omega$. Observing $\sum_{j=1}^{2n+1} a^2_{n,j} = \|y^{(i)}_n\|^2_{\mathrm{l}^2(\Omega)}$, we finally obtain the following lemma.

Lemma 5.38. *Suppose* $y^{(i)}_n$ *is a member of* $\mathrm{harm}^{(i)}_n$. *Then*

$$\|y^{(i)}_n\|_{\mathrm{c}(\Omega)} \leq \sqrt{\frac{2n+1}{4\pi}} \|y^{(i)}_n\|_{\mathrm{l}^2(\Omega)}. \tag{5.228}$$

In particular,

$$\|y^{(i)}_{n,j}\|_{\mathrm{c}(\Omega)} \leq \sqrt{\frac{2n+1}{4\pi}}. \tag{5.229}$$

5.10 Vectorial Funk–Hecke Formulas

Next, we deal with generalizations of the Funk–Hecke formula to the vectorial case. Two variants are considered in more detail:

(i) Let $g(\cdot,\eta) : \Omega \to \mathbb{R}^3$ be a vector field which is invariant with respect to all orthogonal transformations $\mathbf{t} \in SO(3)$ leaving $\eta \in \Omega$ fixed. Determine the integral

$$\int_\Omega g(\xi,\eta) \cdot y_n^{(i)}(\xi)\, d\omega(\xi), \tag{5.230}$$

where $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$.

(ii) Let $G \in \mathrm{L}^1[-1,1], \eta \in \Omega$ fixed. Determine the integral

$$\int_\Omega G(\xi \cdot \eta) y_n^{(i)}(\xi)\, d\omega(\xi), \tag{5.231}$$

where $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$.

Remark 5.39. Notice that the integral (5.230) is scalar-valued, while (5.231) is vector-valued. This difference causes completely different ways of establishing the Funk–Hecke formulas. The first variant uses certain properties of invariant vector fields, while the second one is based on the cartesian representation of vector spherical harmonics.

We start with the recapitulation of some topics of representation theory needed for our studies on the Funk–Hecke formula (cf. Section 3.9).

Let $\mathbf{t} \in SO(3)$. The operator $R_\mathbf{t} : \mathrm{l}^2(\Omega) \to \mathrm{l}^2(\Omega)$ has been introduced as follows: for $f \in \mathrm{l}^2(\Omega)$, $R_\mathbf{t} f(\xi) = \mathbf{t}^T f(\mathbf{t}\xi),\ \xi \in \Omega$.

Furthermore, let $S \subset SO(3)$ be a subgroup of $SO(3)$. As is well-known, a subspace $v \subset \mathrm{l}^2(\Omega)$ is called *invariant with respect to* S or simply S*-invariant* if $f \in v$ implies that $R_\mathbf{t} f \in v$ for all $\mathbf{t} \in S$. If there exists no subspace of v (other than v itself) which is S-invariant, then v is said to be *irreducible.*

Now, assume that f, g are of class $\mathrm{l}^2(\Omega), \mathbf{t} \in SO(3)$. Then, it follows that

$$(R_\mathbf{t} f, g)_{\mathrm{l}^2(\Omega)} = (f, R_{\mathbf{t}^T} g)_{\mathrm{l}^2(\Omega)}. \tag{5.232}$$

But this means that the adjoint operator of $R_\mathbf{t}$ is given by $R_{\mathbf{t}^T}$. Let $F \in \mathrm{C}^{(1)}(\Omega),\ \mathbf{t} \in SO(3)$. Then we find

$$o_\xi^{(1)} R_\mathbf{t} F(\xi) = \xi F(\mathbf{t}\xi)\ = \mathbf{t}^T \mathbf{t}\xi F(\mathbf{t}\xi)\ = R_\mathbf{t} o^{(1)} F(\xi), \tag{5.233}$$

and

$$o_\xi^{(2)} R_\mathbf{t} F(\xi) = \nabla_\xi^* F(\mathbf{t}\xi)\ = \mathbf{t}^T (\nabla^* F)(\mathbf{t}\xi)\ = R_\mathbf{t} o^{(2)} F(\xi). \tag{5.234}$$

An analogous result holds for $o^{(3)}$. Together with (5.232), we get for $F \in \mathrm{C}^{(1)}(\Omega)$ and $f \in \mathrm{c}^{(1)}(\Omega)$ and any $\mathbf{t} \in SO(3)$ that, for all $i \in \{1,2,3\}$ and $\xi \in \Omega$,

$$o_\xi^{(i)} R_{\mathbf{t}} F(\xi) = R_{\mathbf{t}} o^{(i)} F(\xi) \tag{5.235}$$

and

$$O_\xi^{(i)} R_{\mathbf{t}} f(\xi) = R_{\mathbf{t}} O^{(i)} f(\xi). \tag{5.236}$$

Therefore, we remember in connection with the results of Chapter 2.7 and Theorem 3.68 the following statements:

(i) The space $\mathrm{l}^2_{(i)}(\Omega)$ is $SO(3)$-invariant for all $i \in \{1,2,3\}$.

(ii) The set $\mathrm{harm}_n^{(i)}$ is an $SO(3)$-invariant subspace of $\mathrm{l}^2_{(i)}(\Omega)$ for all $i \in \{1,2,3\}$ and $n \geq 0_i$. Furthermore, $\mathrm{harm}_n^{(i)}$ is irreducible.

Assume that $F \in \mathrm{L}^2(\Omega)$ with $R_{\mathbf{t}} F = F$ for all $\mathbf{t} \in SO_\eta(3)$. Then we already know that there exists a function $\tilde{F} \in \mathrm{L}^2[-1,1]$ such that $F(\xi) = \tilde{F}(\xi \cdot \eta)$, $\xi \in \Omega$. Furthermore, we have shown in Theorem 3.58 that if F is, in addition, a spherical harmonic of order n, i.e., $F \in \mathrm{Harm}_n$, there exists a constant $C \in \mathbb{R}$ such that

$$F(\xi) = C\, P_n(\xi \cdot \eta), \quad \xi \in \Omega. \tag{5.237}$$

A generalization of these results to the vectorial case can be written down as follows:

(i) If $f \in \mathrm{c}^{(1)}(\Omega)$ satisfies $R_{\mathbf{t}} f = f$ for all $\mathbf{t} \in SO_\eta(3)$, η fixed, then there exist functions $F_i \in \mathrm{C}[-1,1]$, $i = 1,2,3$, such that

$$O_\xi^{(i)} f(\xi) = F_i(\xi \cdot \eta), \quad \xi \in \Omega. \tag{5.238}$$

(ii) Let $i \in \{1,2,3\}$ and $y_n^{(i)} \in \mathrm{harm}_n^{(i)}$ with $R_{\mathbf{t}} y_n^{(i)} = y_n^{(i)}$ for all $\mathbf{t} \in SO_\eta(3)$, η fixed. Then there exists a constant $C \in \mathbb{R}$ such that

$$y_n^{(i)}(\xi) = C\, o_\xi^{(i)} P_n(\xi \cdot \eta), \quad \xi \in \Omega. \tag{5.239}$$

Let $\eta \in \Omega$ be fixed. Assume that $g(\cdot,\eta) \in \mathrm{c}^{(1)}(\Omega)$ is a spherical vector field with $R_{\mathbf{t}} g(\xi,\eta) = g(\xi,\eta)$, $\xi \in \Omega$, for all $\mathbf{t} \in SO_\eta(3)$. Then it follows from the considerations above that, for $i \in \{1,2,3\}$, the functions $O_\xi^{(i)} g(\xi,\eta) = G_i(\xi \cdot \eta)$ depend only on the inner product $\xi \cdot \eta$. Thus, we may define in analogy to Theorem 3.60

$$(O^{(i)} g)^\wedge(n) = 2\pi \int_{-1}^{1} G_i(t) P_n(t)\, dt. \tag{5.240}$$

It follows from (5.23) that $y_n^{(i)} = o^{(i)} Y_n \in \mathrm{harm}_n^{(i)}$ satisfies

$$\int_\Omega g(\xi,\eta) \cdot y_n^{(i)}(\xi)\, d\omega(\xi) = \int_\Omega O_\xi^{(i)} g(\xi,\eta) Y_n(\xi)\, d\omega(\xi) = (O^{(i)} g)^\wedge(n) Y_n(\eta). \tag{5.241}$$

This leads us to the first variant of the vectorial Funk–Hecke formula.

Theorem 5.40. *Let $\eta \in \Omega$ be fixed. Assume that $g(\cdot,\eta) \in \mathrm{c}^{(1)}(\Omega)$ satisfies*

$$R_{\mathbf{t}} g(\xi,\eta) = g(\xi,\eta)$$

for all $\mathbf{t} \in SO_\eta(3)$ and all $\xi \in \Omega$. Then, for $i \in \{1,2,3\}$ and $y_n^{(i)} \in \mathrm{harm}_n^{(i)}, n \geq 0_i$,

$$\int_\Omega g(\xi,\eta) \cdot y_n^{(i)}(\xi)\, d\omega(\xi) = (\mu_n^{(i)})^{-1} (O^{(i)} g)^\wedge(n) O_\eta^{(i)} y_n^{(i)}(\eta),$$

where $(O^{(i)} g)^\wedge(n)$ is defined by (5.240).

By virtue of the addition theorem for vector spherical harmonics, we immediately obtain the following consequence.

Corollary 5.41. *Let $\eta \in \Omega$ be fixed, $g(\cdot,\eta) \in \mathrm{c}^{(1)}(\Omega)$. Assume that*

$$R_{\mathbf{t}} g(\xi,\eta) = g(\xi,\eta) \tag{5.242}$$

for all $\mathbf{t} \in SO_\eta(3)$ and all $\xi \in \Omega$. Then, for all $\zeta \in \Omega$ and $i \in \{1,2,3\}$,

$$\int_\Omega g(\xi,\eta)^{Tv} \mathbf{p}_n^{(i,i)}(\xi,\zeta)\, d\omega(\xi) = (\mu_n^{(i)})^{-1} (O^{(i)} g)^\wedge(n)\; o_\zeta^{(i)} P_n(\zeta \cdot \eta).$$

We now come to the second variant of the vectorial Funk–Hecke formula as announced in (5.231). The basic ideas to handle this problem are the representation of vector spherical harmonics by means of restrictions of homogeneous harmonic vector polynomials and the componentwise application of the (scalar) Funk–Hecke formula.

Let $Y_n \in \mathrm{Harm}_n$ be a spherical harmonic. Then, we know from Lemma 5.12 that the cartesian components of the spherical vector field

$$\xi \mapsto \tilde{o}_n^{(i)} r^n Y_n(\xi)|_{r=1},\ \xi \in \Omega, \quad i \in \{1,2,3\}, \tag{5.243}$$

are (scalar) spherical harmonics of degree $\deg^{(i)}(n)$ (cf. Lemma 5.12). Thus, we get immediately for $G \in \mathrm{L}^1[-1,1]$ and $\eta \in \Omega$

$$\int_\Omega G(\xi \cdot \eta) \tilde{o}_n^{(i)} r^n Y_n(\xi)|_{r=1} d\omega(\xi) = G^\wedge(\deg^{(i)}(n)) \tilde{o}_n^{(i)} r^n Y_n(\eta)|_{r=1}. \tag{5.244}$$

We know from the formulas (5.106, 5.107, 5.108) how a vector spherical harmonic is expressible by restrictions of homogeneous harmonic vector polynomials. Combining these results, we get the second variant of the vectorial Funk–Hecke formula.

Theorem 5.42. *Let G be of class $\mathrm{L}^1[-1,1]$. Assume that $Y_n \in \mathrm{Harm}_n$. Then, for all $\eta \in \Omega$, and for all $n = 1, 2, \ldots,$*

$$\begin{aligned}
\int_\Omega G(\xi\cdot\eta) o_\xi^{(1)} Y_n(\xi)\, d\omega(\xi) &= G^\wedge_{(1,1)}(n) o_\eta^{(1)} Y_n(\eta) + G^\wedge_{(1,2)}(n) o_\eta^{(2)} Y_n(\eta),\\
\int_\Omega G(\xi\cdot\eta) o_\xi^{(2)} Y_n(\xi)\, d\omega(\xi) &= G^\wedge_{(2,1)}(n) o_\eta^{(1)} Y_n(\eta) + G^\wedge_{(2,2)}(n) o_\eta^{(2)} Y_n(\eta),\\
\int_\Omega G(\xi\cdot\eta) o_\xi^{(3)} Y_n(\xi)\, d\omega(\xi) &= G^\wedge_{(3,3)}(n) o_\eta^{(3)} Y_n(\eta),
\end{aligned}$$

where the coefficients $G^\wedge_{(i,j)}(n)$ are given by

$$\begin{aligned}
G^\wedge_{(1,1)}(n) &= \frac{1}{2n+1}\left((n+1)G^\wedge(n+1) + nG^\wedge(n-1)\right),\\
G^\wedge_{(1,2)}(n) &= \frac{1}{2n+1}\left(G^\wedge(n-1) - G^\wedge(n+1)\right),\\
G^\wedge_{(2,1)}(n) &= \frac{n(n+1)}{2n+1}\left(G^\wedge(n-1) - G^\wedge(n+1)\right),\\
G^\wedge_{(2,2)}(n) &= \frac{1}{2n+1}\left(nG^\wedge(n+1) + (n+1)G^\wedge(n-1)\right),\\
G^\wedge_{(3,3)}(n) &= G^\wedge(n).
\end{aligned}$$

Notice that the space $\mathrm{l}^2_{(3)}(\Omega)$ is invariant with respect to the defined integral operator, while $\mathrm{l}^2_{(1)}(\Omega)$ and $\mathrm{l}^2_{(2)}(\Omega)$ are not. However, it is clear that $\mathrm{l}^2_{(1)}(\Omega) \oplus \mathrm{l}^2_{(2)}(\Omega)$ is an invariant subspace of $\mathrm{l}^2(\Omega)$.

5.11 Counterparts of the Legendre Polynomial

Next, our purpose is to extend the operators $O^{(i)}$ also to rank-2 tensor fields. For sufficiently smooth fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ of the form

$$\mathbf{f}(\xi) = \sum_{j=1}^{3}\sum_{k=1}^{3} F_{jk}(\xi)\varepsilon^j \otimes \varepsilon^k \tag{5.245}$$

we let

$$O^{(i)}_\xi \mathbf{f}(\xi) = \sum_{k=1}^{3} O^{(i)}_\xi \left(\sum_{j=1}^{3} F_{jk}(\xi)\varepsilon^j \right) \varepsilon^k. \tag{5.246}$$

According to the definition (5.246), it is clear that the (vectorial) Legendre rank–2 tensor kernel ${}^v\mathbf{p}^{(i,i)}_n(\cdot,\cdot)$ of type (i,i) is the *reproducing kernel* of harm_n in the following sense:

(i) for all $\xi \in \Omega$

$$O^{(i)}_\xi {}^v\mathbf{p}^{(i,i)}_n(\xi,\cdot) \in \mathrm{harm}^{(i)}_n \tag{5.247}$$

(ii) for every $f \in \mathrm{harm}^{(i)}_n$ and all $\xi \in \Omega$

$$O^{(i)}_\xi f(\xi) = \left(O^{(i)}_\xi {}^v\mathbf{p}^{(i,i)}_n(\xi,\cdot), f \right)_{\mathrm{l}^2(\Omega)}. \tag{5.248}$$

Moreover, let $a \in \mathbb{R}^3$ and $\eta \in \Omega$ be fixed. Then the vector field

$${}^v\mathbf{p}^{(i,k)}_n(\cdot,\eta) \cdot a = \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} (y^{(k)}_{n,j}(\eta)\cdot a) y^{(i)}_{n,j} \tag{5.249}$$

is a vector spherical harmonic of degree n and type i. Thus, we obtain from (5.227) and Lemma 5.37

$$|{}^v\mathbf{p}^{(i,k)}_n(\xi,\eta)\cdot a|^2 \leq \frac{2n+1}{4\pi} \left(\frac{4\pi}{2n+1} \right)^2 \sum_{j=1}^{2n+1} (y^{(k)}_{n,j}(\xi)\cdot a)^2 \leq |a|^2 \tag{5.250}$$

for all $\xi, \eta \in \Omega$. This gives us the following result.

Lemma 5.43. *Let $i,k,l \in \{1,2,3\}$. Then, for all $\xi,\eta \in \Omega$,*

$$|{}^v\mathbf{p}^{(i,k)}_n(\xi,\eta)\cdot \varepsilon^l| \leq 1$$

and

$$|{}^v\mathbf{p}^{(i,k)}_n(\xi,\eta)| \leq \sqrt{3}.$$

Proof. The second inequality follows directly from the fact that

$$|{}^v\mathbf{p}^{(i,k)}_n(\xi,\eta)|^2 = \sum_{l=1}^{3} |{}^v\mathbf{p}^{(i,k)}_n(\xi,\eta)\varepsilon^l|^2 . \tag{5.251}$$

□

Lemma 5.43 generalizes the estimate $|P_n(t)| \leq 1, t \in [-1,1]$ of the scalar Legendre polynomial.

Remark 5.44. The addition theorem enables us to represent a vector-valued function on the sphere Ω by use of the Legendre tensors. More explicitly, suppose that f is of class $\mathrm{l}^2(\Omega)$ with

$$f = \sum_{i=1}^{3} f^{(i)}, \quad f^{(i)} \in \mathrm{l}^2_{(i)}(\Omega). \tag{5.252}$$

Then it follows that

$$\begin{aligned} f(\xi) &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \int_{\Omega} y^{(i)}_{n,m}(\eta) \cdot f(\eta)\; d\omega(\eta)\; y^{(i)}_{n,m}(\xi) \qquad (5.253)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \int_{\Omega} (y^{(i)}_{n,m}(\xi) \otimes y^{(i)}_{n,m}(\eta)) f(\eta)\, d\omega(\eta)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \int_{\Omega} \frac{2n+1}{4\pi} {}^{v}\mathbf{p}^{(i,i)}_n(\xi,\eta) f(\eta)\, d\omega(\eta)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \int_{\Omega} \frac{2n+1}{4\pi} {}^{v}\mathbf{p}^{(i,i)}_n(\xi,\eta) f^{(i)}(\eta)\, d\omega(\eta). \end{aligned}$$

The expansion of vector fields in terms of the Legendre tensors can be regarded as a natural extension of the scalar Fourier theory to the vectorial case. In order to motivate an alternative approach based on Legendre vectors, we write down the representation of a vector-valued function on the sphere Ω in the following way:

$$\begin{aligned} & f(\xi)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \int_{\Omega} f^{(i)}(\eta) \cdot y^{(i)}_{n,m}(\eta)\; d\omega(\eta)\; y^{(i)}_{n,m}(\xi) \qquad (5.254)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \int_{\Omega} f^{(i)}(\eta) (\mu^{(i)}_n)^{-1/2} o^{(i)}_{\eta} Y_{n,m}(\eta)\, d\omega(\eta) (\mu^{(i)}_n)^{-1/2} o^{(i)}_{\xi} Y_{n,m}(\xi)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} (\mu^{(i)}_n)^{-1/2} \int_{\Omega} O^{(i)}_{\eta} f^{(i)}(\eta) (\mu^{(i)}_n)^{-1/2} o^{(i)}_{\xi} \sum_{m=1}^{2n+1} Y_{n,m}(\eta) Y_{n,m}(\xi)\, d\omega(\eta)\\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} (\mu^{(i)}_n)^{-1/2} \int_{\Omega} O^{(i)}_{\eta} f^{(i)}(\eta) (\mu^{(i)}_n)^{-1/2} o^{(i)}_{\xi} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta)\, d\omega(\eta). \end{aligned}$$

This leads us to the following definition (see Theorem 5.20).

Definition 5.45. The (vectorial) Legendre vector field $p_n^{(i)} : \Omega \times \Omega \to \mathbb{R}^3, i \in \{1,2,3\}$, of degree n and type i (with respect to the dual system of operators $o^{(i)}, O^{(i)}, i \in \{1,2,3\}$), is given by

$$p_n^{(i)}(\xi,\eta) = \left(\mu_n^{(i)}\right)^{-1/2} o_\xi^{(i)} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{5.255}$$

The (vectorial) Legendre vector field $p_n : \Omega \times \Omega \to \mathbb{R}$ of degree n with respect to the dual system of operators $o^{(i)}, O^{(i)}, i = 1,2,3$, is defined by

$$p_n(\xi,\eta) = \sum_{i=1}^{3} p_n^{(i)}(\xi,\eta). \tag{5.256}$$

Following our considerations given above, every vector-valued function $f \in \mathrm{l}^2(\Omega)$, therefore, admits an expansion

$$f(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (\mu_n^{(i)})^{-1/2} \int_\Omega p_n^{(i)}(\xi,\eta) O_\eta^{(i)} f(\eta)\, d\omega(\eta). \tag{5.257}$$

Obviously, the Legendre vectors fulfill an addition theorem, which reads as follows:

Theorem 5.46. *Let* $\{Y_{n,m}\}_{m=1,\ldots,2n+1}$ *be an* $\mathrm{L}^2(\Omega)$*–orthonormal basis of* Harm_n*. Then, for* $i \in \{1,2,3\}$ *and all* $\xi, \eta \in \Omega$,

$$\sum_{m=1}^{2n+1} y_{n,m}^{(i)}(\xi) Y_{n,m}(\eta) = \frac{2n+1}{4\pi} p_n^{(i)}(\xi,\eta). \tag{5.258}$$

The Legendre polynomials and the corresponding vectorial Legendre vectors and rank–2 tensors, as defined here, are related in the following way:

Lemma 5.47. *Let* P_n *be the Legendre polynomials of degree* n, ${}^v\mathbf{p}_n^{(i,k)}$ *the corresponding Legendre tensors of degree* n *and type* (i,k)*, and* $p_n^{(i)}$ *the corresponding Legendre vector of degree* n *and type* i*. Then, for all* $\xi, \eta \in \Omega$,

$$P_n(\xi \cdot \eta) = (\mu_n^{(i)})^{-1/2} (\mu_n^{(k)})^{-1/2} O_\xi^{(k)} O_\eta^{(i)} {}^v\mathbf{p}_n^{(i,k)}(\xi,\eta), \tag{5.259}$$

and

$$P_n(\xi \cdot \eta) = (\mu_n^{(i)})^{-1/2} O_\xi^{(i)} p_n^{(i)}(\xi,\eta). \tag{5.260}$$

Remark 5.48. Defining the system $\{y_{n,m}^{(i),R}\}_{i=1,2,3,n=0_i,\ldots,m=1,\ldots,2n+1}$ by

$$y_{n,m}^{(i),R}(x) = \left(\frac{1}{R}\right) y_{n,m}^{(i)}\left(\frac{x}{|x|}\right), \quad x \in \Omega_R, \tag{5.261}$$

we get an orthonormal basis in $l^2(\Omega_R)$, provided that the system

$$\{y^{(i)}_{n,m}\}_{i=1,2,3,n=0_i,\ldots,m=1,\ldots,2n+1} \tag{5.262}$$

constitutes an $l^2_{(i)}(\Omega)$–orthonormal basis in $l^2_{(i)}(\Omega)$, $i \in \{1,2,3\}$.

In addition, it should be noted that every vector field $f^{(i)} \in l^2_{(i)}(\Omega_R)$ can be expanded in terms of vector spherical harmonics on Ω_R as follows

$$f^{(i)} = \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} f^{(i)\wedge_R}(n,m) y^{(i),R}_{n,m}, \tag{5.263}$$

where the orthogonal (Fourier) coefficients are given by

$$f^{(i)\wedge_R}(n,m) = \left(f^{(i)}, y^{(i),R}_{n,m}\right)_{l^2(\Omega_R)} = \int_{\Omega_R} f^{(i)}(x) \cdot y^{(i);R}_{n,m}(x)\, d\omega(x). \tag{5.264}$$

Note that the identity (5.263) is understood in $\|\cdot\|_{l^2(\Omega_R)}$–sense.

5.12 Degree and Order Variances

In analogy to the scalar case, let us think of an 'output signal' g as produced by a linear operator $\boldsymbol{\lambda}$ applied to an 'input signal' f

$$\boldsymbol{\lambda} f = g, \tag{5.265}$$

where $\boldsymbol{\lambda}$ is an operator mapping of $l^2(\Omega)$ onto itself such that

$$\boldsymbol{\lambda} y^{(i)}_{n,m} = \left(\boldsymbol{\lambda}^{(i)}\right)^{\wedge} (n,m)\; y^{(i)}_{n,m}, \tag{5.266}$$

$i = 1,2,3$, $n = 0_i, 0_i+1, \ldots$, $m = 1,\ldots,2n+1$. The *symbol* $\{(\boldsymbol{\lambda}^{(i)})^{\wedge}(n,m)\}$ is supposed to be a sequence of real numbers for $i = 1,2,3$ (where, as usual, $0_1 = 0$ and $0_i = 1$, $i = 2,3$.

Remark 5.49. Note that similar arguments hold true for operators $\boldsymbol{\lambda}$ with matrical symbols $\{(\boldsymbol{\lambda}^{(i)})^{\wedge}(n,m)\}$, i.e., a sequence of matrices of class $\mathbb{R}^3 \otimes \mathbb{R}^3$, $i = 1,2,3$.

In practice, an error-affected 'output signal'

$$\tilde{g} = g + \tilde{\varepsilon}, \tag{5.267}$$

is observed, where $\tilde{\varepsilon}$ is the *observation noise.* Analogously to the scalar case, we assume that

$$\mathrm{Cov}[\tilde{g}(\xi), \tilde{g}(\eta)] = E[\tilde{\varepsilon}(\xi), \tilde{\varepsilon}(\eta)] = \mathbf{k}(\xi, \eta), \quad (\xi, \eta) \in \Omega \times \Omega, \tag{5.268}$$

is known, where the tensorial covariance kernel $\mathbf{k}(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, is explicitly given as follows.

Definition 5.50. Let $\mathbf{k}^{(i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $i \in \{1, 2, 3\}$, (more precisely, ${}^v\mathbf{k}^{(i)}$) be a tensor kernel of the form

$$\begin{aligned} \mathbf{k}^{(i)}(\xi, \eta) &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\mathbf{k}^{(i)})^\wedge(n, m)(\mu_n^{(i)})^{-1} o_\xi^{(i)} o_\eta^{(i)} Y_{n,m}(\xi) Y_{n,m}(\eta) \\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\mathbf{k}^{(i)})^\wedge(n, m) y_{n,m}^{(i)}(\xi) \otimes y_{n,m}^{(i)}(\eta), \end{aligned}$$

with the symbol $\left\{(\mathbf{k}^{(i)})^\wedge(n, m)\right\}$, $i \in \{1, 2, 3\}$, satisfying the conditions:

(i) $(\mathbf{k}^{(i)})^\wedge(n, m) \geq 0$ for $n = 0, 1, \ldots, m = 1, \ldots, 2n + 1, i \in \{1, 2, 3\}$,

(ii) $\displaystyle\sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\mathbf{k}^{(i)})^\wedge(n, m) \sup_{\eta \in \Omega} \left| y_{n,m}^{(i)}(\eta) \right|^2 < \infty$.

Then $\mathbf{k}^{(i)}$, $i \in \{1, 2, 3\}$, is called a (vectorial) covariance rank–2 tensor kernel of type i, while $\mathbf{k} = \sum_{i=1}^{3} \mathbf{k}^{(i)}$ (more precisely, ${}^v\mathbf{k}$) is called a (vectorial) covariance rank–2 tensorial kernel.

Any 'output function' (output signal) can be expanded into an orthogonal series in terms of vector spherical harmonics:

$$\begin{aligned} \tilde{g} = \boldsymbol{\lambda} \tilde{f} &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\boldsymbol{\lambda}^{(i)})^\wedge(n, m)(\tilde{f}^{(i)})^\wedge(n, m) y_{n,m}^{(i)} \\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\tilde{g}^{(i)})^\wedge(n, m) y_{n,m}^{(i)}, \end{aligned}$$

where the equality has to be understood in the sense of $\|\cdot\|_{l^2(\Omega)}$. Using this series expansion we get, for $i \in \{1, 2, 3\}$, the spectral representation

$$(\tilde{g}^{(i)})^\wedge(n, m) = (\boldsymbol{\lambda}^{(i)})^\wedge(n, m)(\tilde{f}^{(i)})^\wedge(n, m) \ . \tag{5.269}$$

This is the vectorial analogue for (3.358) and also hints at using the root-mean-square power per degree and order, respectively per degree, to characterize the vectorial signal. Motivated by the corresponding definitions for the scalar case and by Parseval's identity, we introduce the following definition (cf. W. Freeden, T. Maier (2002, 2003)).

Definition 5.51. Let g be of class $\mathrm{l}^2(\Omega)$. Let, for $i \in \{1,2,3\}, n = 0_i, 0_i + 1, \ldots,\ m = 1, \ldots, 2n+1,\ (g^{(i)})^\wedge(n,m)$ be the corresponding orthogonal coefficients. Then, the *signal degree n and order m variances of type i* of g are defined by

$$\begin{aligned} \mathrm{Var}_{n,m}^{(i)}(g) &= \int_\Omega \int_\Omega \left(y_{n,m}^{(i)}(\xi) \otimes y_{n,m}^{(i)}(\eta) \right) \cdot (g(\xi) \otimes g(\eta))\ d\omega(\xi)\ d\omega(\eta) \\ &= \int_\Omega \int_\Omega g(\xi) \cdot \left(y_{n,m}^{(i)}(\xi) \otimes y_{n,km}^{(i)}(\eta) \right) g(\eta)\ d\omega(\xi)\ d\omega(\eta) \\ &= \left(\left(g^{(i)} \right)^\wedge (n,m) \right)^2 . \end{aligned} \tag{5.270}$$

Accordingly, the signal degree n variances of type i of g are given by

$$\begin{aligned} \mathrm{Var}_n^{(i)}(g) &= \frac{2n+1}{4\pi} \int_\Omega \int_\Omega g(\xi) \cdot {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) g(\eta)\ d\omega(\eta)\ d\omega(\xi) \\ &= \sum_{m=1}^{2n+1} \left(\left(g^{(i)} \right)^\wedge (n,m) \right)^2 \\ &= \sum_{m=1}^{2n+1} \mathrm{Var}_{n,m}^{(i)}(g), \end{aligned} \tag{5.271}$$

while the signal degree variances of g read as follows:

$$\mathrm{Var}_n(\boldsymbol{\lambda}\tilde{f}) = \sum_{i=1}^{3} \mathrm{Var}_n^{(i)}(\boldsymbol{\lambda}\tilde{f}). \tag{5.272}$$

Obviously, by virtue of Parseval's identity, we obtain

$$\|\boldsymbol{\lambda}\tilde{f}\|_{\mathrm{l}^2(\Omega)} = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \mathrm{Var}_{n,m}^{(i)}(\boldsymbol{\lambda}\tilde{f}), \tag{5.273}$$

again connecting the signal degree and order variances as well as the signal degree variances with the '$\mathrm{l}^2(\Omega)$-energy' of the corresponding vectorial signal.

It is clear that the remarks concerning the frequency limiting characteristics of physical devices and the resulting bandlimited nature of the 'transmitted signals' are valid in the vectorial case as well. That is, one is usually able to consider bandlimited vector fields $\tilde{g} \in \mathrm{l}^2(\Omega)$, the signal degree variances of which satisfy $\mathrm{Var}_n(\tilde{g}) = 0$ for all $n > N$.

As an example, we consider the normal ($i = 1$) and the tangential ($i = 2$) degree variances of the EGM96–gradient field (see Fig. 5.4). Their degree variances are shown in Fig. 5.5.

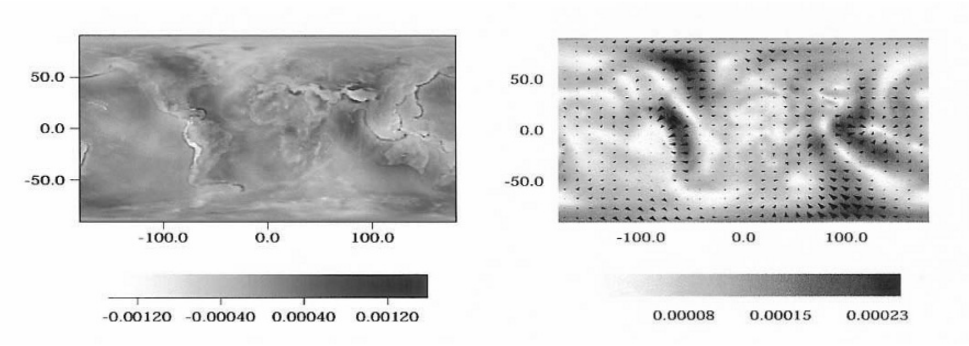


Fig. 5.4: The normal derivative [in 100 Gal] and the surface gradient illustrated for the Earth's gravitational potential model EGM96 ([in 100 Gal]).

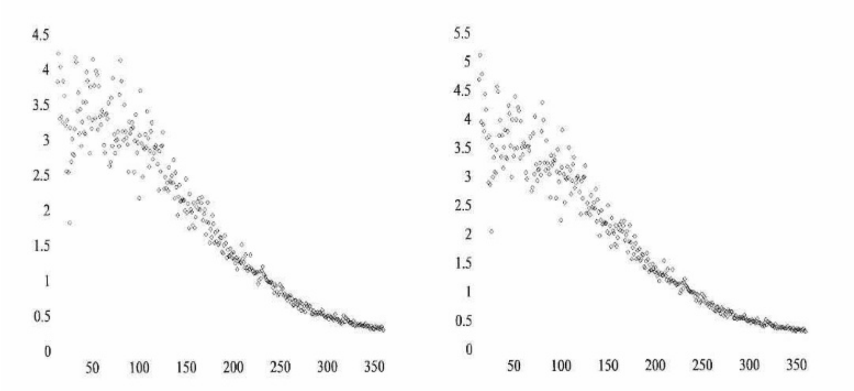


Fig. 5.5: Normal (*left*) and tangential (*right*) degree variances of the EGM96 model (from S. Beth (2000)).

In addition to the previously defined signal variances, the tensorial covariance kernel $\mathbf{k}$ is used to calculate suitable measures to characterize the noise:

Definition 5.52. In accordance with Definition 5.50, let $\left\{(\mathbf{k}^{(i)})^\wedge(n,m)\right\}$ be the symbol of a (vectorial) covariance rank–2 tensorial kernel $\mathbf{k} : \Omega \times \Omega \rightarrow$

$\mathbb{R}^{3\times 3}$. Then the *degree and order error covariance of type i* is given by

$$\begin{aligned}
&\mathrm{Cov}^{(i)}_{n,m}(\mathbf{k}) \\
&= \int_\Omega \int_\Omega \left(y^{(i)}_{n,m}(\xi) \otimes y^{(i)}_{n,m}(\eta) \right) \cdot \mathbf{k}(\xi,\eta)\ d\omega(\xi)\ d\omega(\eta) \\
&= \sum_{l=1}^{3} \sum_{n,m} \sum_{(p,q)} (\mathbf{k}^{(l)})^\wedge(p,q) \int_\Omega \int_\Omega \left(y^{(i)}_{n,m}(\eta) \cdot y^{(l)}_{p,q}(\eta) \right) \\
&\quad \left(y^{(i)}_{n,m}(\xi) \cdot y^{(l)}_{p,q}(\xi) \right) d\omega(\xi) d\omega(\eta) \\
&= (\mathbf{k}^{(i)})^\wedge(n,m).
\end{aligned}$$

Moreover, the *error covariance of type i* as well as the *error covariance* are defined by

$$\mathrm{Cov}^{(i)}_{n}(\mathbf{k}) = \sum_{m=1}^{2n+1} \mathrm{Cov}^{(i)}_{n,m}(\mathbf{k}) = \sum_{m=1}^{2n+1} (\mathbf{k}^{(i)})^\wedge(n,m) \tag{5.274}$$

and

$$\mathrm{Cov}_{n}(\mathbf{k}) = \sum_{i=1}^{3} \sum_{m=1}^{2n+1} (\mathbf{k}^{(i)})^\wedge(n,m). \tag{5.275}$$

The signal-to-noise relation is determined by the degree and order resolution set of type i.

Definition 5.53. Signal and noise spectrum intersect at the *degree and order resolution set of type i*, defined by the following relations:

(i) *Signal dominates noise*

$$\mathrm{Var}^{(i)}_{n,m}(\boldsymbol{\lambda}\tilde{f}) \geq \mathrm{Cov}^{(i)}_{n,m}(\mathbf{k}), \quad n = 0_i, 0_i+1, \ldots, m, m = 1, \ldots, 2n+1, \tag{5.276}$$

(ii) *Noise dominates signal*

$$\mathrm{Var}^{(i)}_{n,m}(\boldsymbol{\lambda}\tilde{f}) < \mathrm{Cov}^{(i)}_{n,m}(\mathbf{k}), \quad n = m, m+1, \ldots, m = 1, \ldots, 2n+1. \tag{5.277}$$

The technique of filtering the signal $\boldsymbol{\lambda}\tilde{f}$ in order to get an estimated denoised version $\widehat{\boldsymbol{\lambda} f}$ can be canonically carried over from the scalar case.

5.13 Vector Spherical Harmonics Related to Vector Homogeneous Harmonic Polynomials

Up to now, the operators $\tilde{o}_n^{(i)}$ were defined for homogeneous harmonic polynomials of degree n in $\mathbb{R}^3$ (see Section 5.5). Every $H_n \in \mathrm{Harm}_n(\mathbb{R}^3)$ can be written in the form $H_n(x) = r^n Y_n(\xi)$, $x = r\xi$, $r \geq 0$, $\xi \in \Omega$, where $Y_n \in \mathrm{Harm}_n$. It is obvious that the representation of the gradient ∇ by observing the normal and the tangential parts yields

$$\tilde{o}_n^{(1)} r^n Y_n(\xi) = (n+1) r^{n+1} o_\xi^{(1)} Y_n(\xi) - r^{n+1} o_\xi^{(2)} Y_n(\xi), \quad (5.278)$$

$$\tilde{o}_n^{(2)} r^n Y_n(\xi) = n r^{n-1} o_\xi^{(1)} Y_n(\xi) + r^{n-1} o_\xi^{(2)} Y_n(\xi), \quad (5.279)$$

$$\tilde{o}_n^{(3)} r^n Y_n(\xi) = r^n o_\xi^{(3)} Y_n(\xi). \quad (5.280)$$

Therefore, the restrictions of $r\xi \mapsto \tilde{o}_n^{(i)} r^n Y_n(\xi)$ to the unit sphere Ω can be written as linear combinations of vector spherical harmonics $o^{(i)} Y_n$.

In the sequel, we understand $\tilde{o}_n^{(i)} Y_n$ to be given by

$$\tilde{o}_n^{(i)} Y_n(\xi) = \tilde{o}_n^{(i)} H_n(x)|_{r=1}, \quad (5.281)$$

where $H_n(x) = r^n Y_n(\xi)$, $x = r\xi$, $\xi \in \Omega$. In other words, (see also (5.106)–(5.108)), we have

$$\tilde{o}_n^{(1)} Y_n(\xi) = (n+1) o_\xi^{(1)} Y_n(\xi) - o_\xi^{(2)} Y_n(\xi), \quad (5.282)$$

$$\tilde{o}_n^{(2)} Y_n(\xi) = n o_\xi^{(1)} Y_n(\xi) + o_\xi^{(2)} Y_n(\xi), \quad (5.283)$$

$$\tilde{o}_n^{(3)} Y_n(\xi) = o_\xi^{(3)} Y_n(\xi). \quad (5.284)$$

Note that $\tilde{o}_n^{(i)}$ acts on the variable ξ, as does $o^{(i)}$, but this will not be indicated in the Equations (5.282), (5.283), (5.284). It is obvious that the adjoint operators $\tilde{O}_n^{(i)}$ to $\tilde{o}_n^{(i)}$ satisfying the equations

$$(\tilde{o}_n^{(i)} G, f)_{\mathrm{l}^2(\Omega)} = (G, \tilde{O}_n^{(i)} f)_{\mathrm{L}^2(\Omega)}, \quad (5.285)$$

$f \in \mathrm{harm}_n$ and $G \in \mathrm{Harm}_n$, are given by

$$\tilde{O}_n^{(1)} f = (n+1) O^{(1)} f - O^{(2)} f, \quad (5.286)$$

$$\tilde{O}_n^{(2)} f = n O^{(1)} f + O^{(2)} f, \quad (5.287)$$

$$\tilde{O}_n^{(3)} f = O^{(3)} f. \quad (5.288)$$

Definition 5.54. Any vector field

$$\tilde{y}_n^{(i)} = \tilde{o}_n^{(i)} Y_n, \quad n \geq 0_i, \quad Y_n \in \mathrm{Harm}_n$$

is called a *vector spherical harmonic of degree n and type i* with respect to the dual system of operators $\tilde{o}_n^{(i)}, \tilde{O}_n^{(i)}, i \in \{1, 2, 3\}$.

The following lemma is easy to verify (see also W. Freeden et al. (1998)).

Lemma 5.55. *For every* $Y_n \in \mathrm{Harm}$, *we have*

$$\tilde{O}_n^{(i)} \tilde{y}_n^{(j)}(\xi) = \tilde{O}_n^{(i)} \tilde{o}_n^{(j)} Y_n(\xi) = \delta_{ij} \tilde{\mu}_n^{(i)} Y_n(\xi), \tag{5.289}$$

where the constants $\tilde{\mu}_n^{(i)}$ *are given as follows:*

$$\tilde{\mu}_n^{(i)} = \|\tilde{O}_n^{(i)} \tilde{y}_n^{(i)}\|_{\mathrm{L}^2(\Omega)} = \|\tilde{O}_n^{(i)} \tilde{o}_n^{(i)} Y_n\|_{\mathrm{L}^2(\Omega)}, \tag{5.290}$$

i.e.,

$$\tilde{\mu}_n^{(1)} = (n+1)(2n+1), \tag{5.291}$$
$$\tilde{\mu}_n^{(2)} = n(2n+1), \tag{5.292}$$
$$\tilde{\mu}_n^{(3)} = n(n+1). \tag{5.293}$$

It should be mentioned that the operators $\tilde{o}_n^{(i)} : \mathrm{Harm}_n \to \mathrm{harm}_n$, $i = 1,2,3$, admit extensions $\tilde{o}^{(i)} : \mathrm{C}^{(\infty)}(\Omega) \to \mathrm{c}^{(\infty)}(\Omega)$, $i = 1,2,3$, by using the (pseudo) differential operator $D = (-\Delta^* + \frac{1}{4})^{1/2} - \frac{1}{2}$ satisfying

$$DY_n = D^\wedge(n) Y_n = \left(\sqrt{n(n+1) + \frac{1}{4}} - \frac{1}{2} \right) Y_n = n Y_n, \tag{5.294}$$

$Y_n \in \mathrm{Harm}_n, n = 0,1,\ldots$ (for more details on the concept of spherical pseudodifferential operators, see e.g. S.L. Svensson (1983) ,W. Freeden et al. (1998)). More explicitly, we set

$$\tilde{o}^{(1)} = o^{(1)}(D+1) - o^{(2)}, \tag{5.295}$$
$$\tilde{o}^{(2)} = o^{(1)} D + o^{(2)}, \tag{5.296}$$
$$\tilde{o}^{(3)} = o^{(3)}. \tag{5.297}$$

Indeed, by observing the definition $y_n^{(i)} = o^{(i)}\ Y_n, n \geq 0_i$, we get

$$\tilde{o}^{(1)} Y_n = o^{(1)}(D+1) Y_n - o^{(2)} Y_n = (n+1) y_n^{(1)} - y_n^{(2)} = \tilde{o}_n^{(1)} Y_n = \tilde{y}_n^{(1)}, \tag{5.298}$$
$$\tilde{o}_n^{(2)} Y_n = o^{(1)} D Y_n + o^{(2)} Y_n = n y_n^{(1)} + y_n^{(2)} = \tilde{o}_n^{(2)} Y_n = \tilde{y}_n^{(2)}, \tag{5.299}$$
$$\tilde{o}_n^{(3)} Y_n = o^{(3)} Y_n = \tilde{o}_n^{(3)} Y_n = \tilde{y}_n^{(3)}. \tag{5.300}$$

Obviously, the adjoint operators $\tilde{O}^{(i)} : \mathrm{c}^{(\infty)}(\Omega) \to \mathrm{C}^{(\infty)}(\Omega)$, $i = 1,2,3$, to the operators $\tilde{o}^{(i)}$ satisfying the equation

$$(\tilde{o}^{(i)} G, f)_{\mathrm{l}^2(\Omega)} = (G, \tilde{O}^{(i)} f)_{\mathrm{L}^2(\Omega)}, \tag{5.301}$$

$f \in \mathrm{c}^{(\infty)}(\Omega), G \in \mathrm{C}^{(\infty)}(\Omega)$, are given by

$$\tilde{O}^{(1)} = O^{(1)}(D+1) - O^{(2)}, \tag{5.302}$$
$$\tilde{O}^{(2)} = O^{(1)}D + O^{(2)}, \tag{5.303}$$
$$\tilde{O}^{(3)} = O^{(3)}. \tag{5.304}$$

This consideration leads us to the introduction of the following set of vector spherical harmonics (note that our approach essentially follows the ideas of the concept as introduced by A.R. Edmonds (1957)):

Let $\{Y_{n,m}\}_{n=0,1,\ldots,n=1,\ldots,2n+1}$, be an $\mathrm{L}^2(\Omega)$-orthonormal system of spherical harmonics. Then, we let

$$\tilde{y}^{(i)}_{n,m} = \left(\tilde{\mu}^{(i)}_n\right)^{-1/2} \tilde{o}^{(i)} Y_{n,m}, \quad n = 0_i, \ldots, m = 1, \ldots, 2n+1. \tag{5.305}$$

By inverting the identities (5.282, 5.283, 5.284), we obtain the following equations for $\xi \in \Omega$:

$$o^{(1)}_\xi Y_{n,m}(\xi) = \frac{1}{2n+1}\tilde{o}^{(1)}_\xi Y_{n,m}(\xi) + \frac{1}{2n+1}\tilde{o}^{(2)}_\xi Y_{n,m}(\xi), \tag{5.306}$$
$$o^{(2)}_\xi Y_{n,m}(\xi) = \frac{-n}{2n+1}\tilde{o}^{(1)}_\xi Y_{n,m}(\xi) + \frac{n+1}{2n+1}\tilde{o}^{(2)}_\xi Y_{n,m}(\xi), \tag{5.307}$$
$$o^{(3)}_\xi Y_{n,m}(\xi) = \tilde{o}^{(3)}_\xi Y_{n,m}(\xi). \tag{5.308}$$

This provides a relation between the system $\{y^{(i)}_{n,m}\}$ and the system $\{\tilde{y}^{(i)}_{n,m}\}$ of vector spherical harmonics. More explicitly, the systems $\{y^{(i)}_{n,m}\}$ and $\{\tilde{y}^{(i)}_{n,m}\}$ are related to each other in the following way:

$$\tilde{y}^{(1)}_{n,m} = \sqrt{\frac{n+1}{2n+1}}y^{(1)}_{n,m} - \sqrt{\frac{n}{2n+1}}y^{(2)}_{n,m}, \tag{5.309}$$
$$\tilde{y}^{(2)}_{n,m} = \sqrt{\frac{n}{2n+1}}y^{(1)}_{n,m} + \sqrt{\frac{n+1}{2n+1}}y^{(2)}_{n,m}, \tag{5.310}$$
$$\tilde{y}^{(3)}_{n,m} = y^{(3)}_{n,m}. \tag{5.311}$$

Conversely

$$y^{(1)}_{n,m} = \sqrt{\frac{n+1}{2n+1}}\tilde{y}^{(1)}_{n,m} + \sqrt{\frac{n}{2n+1}}\tilde{y}^{(2)}_{n,m}, \tag{5.312}$$
$$y^{(2)}_{n,m} = -\sqrt{\frac{n}{2n+1}}\tilde{y}^{(1)}_{n,m} + \sqrt{\frac{n+1}{2n+1}}\tilde{y}^{(2)}_{n,m}, \tag{5.313}$$
$$y^{(3)}_{n,m} = \tilde{y}^{(3)}_{n,m}. \tag{5.314}$$

Our considerations enable us to formulate the following theorem.

Theorem 5.56. *Let* $\{Y_{n,m}\}_{n=0,1,\ldots,m=1,\ldots,2n+1}$ *be an* $\mathrm{L}^2(\Omega)$*–orthonormal set of scalar spherical harmonics. Then the set*

$$\{\tilde{y}^{(i)}_{n,m}\}_{i=1,2,3,n=0_i,\ldots,m=1,\ldots,2n+1} \tag{5.315}$$

as defined in (5.305) forms an $\mathrm{l}^2(\Omega)$*–orthonormal set of vector spherical harmonics which is closed in* $\mathrm{c}(\Omega)$ *with respect to* $\|\cdot\|_{\mathrm{c}(\Omega)}$ *and complete in* $\mathrm{l}^2(\Omega)$ *with respect to* $(\cdot,\cdot)_{\mathrm{l}^2(\Omega)}$. *Furthermore, for all* $\xi \in \Omega$.

$$\Delta^*_\xi \tilde{y}^{(1)}_{n,m}(\xi) = -(n+1)(n+2)\tilde{y}^{(1)}_{n,m}(\xi), \tag{5.316}$$

$$\Delta^*_\xi \tilde{y}^{(2)}_{n,m}(\xi) = -n(n-1)\tilde{y}^{(2)}_{n,m}(\xi), \tag{5.317}$$

$$\Delta^*_\xi \tilde{y}^{(3)}_{n,m}(\xi) = -n(n+1)\tilde{y}^{(3)}_{n,m}(\xi), \tag{5.318}$$

where the Beltrami operator is applied to each component of the vector fields.

In other words, Theorem 5.56 tells us that

$$\Delta^*_\xi \tilde{y}^{(1)}_{n-1,m}(\xi) = -n(n+1)\tilde{y}^{(1)}_{n-1,m}(\xi),\ n=1,2,\ldots,m=1,\ldots,2n+1, \tag{5.319}$$

$$\Delta^*_\xi \tilde{y}^{(2)}_{n+1,m}(\xi) = -n(n+1)\tilde{y}^{(2)}_{n+1,m}(\xi),\ n=0,1,\ldots,m=1,\ldots,2n+1, \tag{5.320}$$

$$\Delta^*_\xi \tilde{y}^{(3)}_{n,m}(\xi) = -n(n+1)\tilde{y}^{(3)}_{n,m}(\xi),\ n=1,2,\ldots,m=1,\ldots,2n+1. \tag{5.321}$$

On the one hand, each member of the system $\{\tilde{y}^{(i)}_{n,m}\}$ is, by definition, not decomposable into normal and tangential parts, but on the other hand, it is a set of eigenfunctions of the Beltrami operator.

5.14 Alternative Systems of Vector Spherical Harmonics

In analogy to the $\mathrm{harm}^{(i)}_n$-spaces, we introduce the following function spaces:

$$\widetilde{\mathrm{harm}}^{(i)}_n = \mathrm{span}\{\tilde{y}^{(i)}_{n,m}\}_{m=1,\ldots,2n+1}, \quad i=1,2,3, n=0_i,0_i+1,\ldots. \tag{5.322}$$

Obviously, these function spaces are characterized by the relations

$$\mathrm{harm}^{(1)}_0 = \widetilde{\mathrm{harm}}^{(1)}_0, \tag{5.323}$$

$$\mathrm{harm}^{(1)}_n \oplus \mathrm{harm}^{(2)}_n = \widetilde{\mathrm{harm}}^{(1)}_n \oplus \widetilde{\mathrm{harm}}^{(2)}_n, \quad n=1,2,\ldots, \tag{5.324}$$

$$\mathrm{harm}^{(3)}_n = \widetilde{\mathrm{harm}}^{(3)}_n, \quad n=1,2,\ldots. \tag{5.325}$$

In consequence, we have

$$\mathrm{harm}_0 = \widetilde{\mathrm{harm}}_0^{(1)}, \tag{5.326}$$

$$\mathrm{harm}_n = \bigoplus_{i=1}^{3} \widetilde{\mathrm{harm}}_n^{(i)}. \tag{5.327}$$

In what follows, we mention the relation between the system $\tilde{y}_{n,m}^{(i)}$ and the restrictions of homogeneous harmonic vector polynomials to the unit sphere Ω.

Lemma 5.57. *Suppose that H_n is of class $\mathrm{Harm}_n(\mathbb{R}^3)$. Let $\varepsilon^k H_n$ be a homogenous harmonic vector polynomial. Then*

$$\varepsilon^k H_n|\Omega = \tilde{y}_{n-1}^{(1)} + \tilde{y}_{n+1}^{(2)} + \tilde{y}_n^{(3)}, \tag{5.328}$$

where

$$\tilde{y}_{n-1}^{(1)} = \tilde{o}_{n-1}^{(1)} Y_{n-1}, \quad Y_{n-1} \in \mathrm{Harm}_{n-1}, \tag{5.329}$$

$$\tilde{y}_{n+1}^{(2)} = \tilde{o}_{n+1}^{(2)} Y_{n+1}, \quad Y_{n+1} \in \mathrm{Harm}_{n+1}, \tag{5.330}$$

$$\tilde{y}_{n}^{(3)} = \tilde{o}_{n}^{(3)} Y_{n}, \quad Y_{n} \in \mathrm{Harm}_{n}. \tag{5.331}$$

Proof. Clearly, $\varepsilon^k H_n|\Omega$ is a member of class $\mathrm{l}^2(\Omega)$ such that

$$\varepsilon^k H_n|\Omega = \sum_{i=1}^{3} \sum_{p=0_i}^{\infty} \sum_{q=1}^{2p+1} a_{p,q}^{(i)} \tilde{y}_{p,q}^{(i)}, \tag{5.332}$$

where $\{\tilde{y}_{p,q}^{(i)}\}_{p=0_i,\ldots,q=1,\ldots 2p+1}$ is an orthonormal system of vector spherical harmonics as defined in (5.305). Furthermore, it is not difficult to see that

$$\tilde{y}_{n,m}^{(1)} = \sum_{j=1}^{3} c_{j,m}^{(1)} \varepsilon^j Y_{n+1}^j, \quad Y_{n+1}^j \in \mathrm{Harm}_{n+1}, \tag{5.333}$$

$$\tilde{y}_{n,m}^{(2)} = \sum_{j=1}^{3} c_{j,m}^{(2)} \varepsilon^j Y_{n-1}^j, \quad Y_{n-1}^j \in \mathrm{Harm}_{n-1}, \tag{5.334}$$

$$\tilde{y}_{n,m}^{(3)} = \sum_{j=1}^{3} c_{j,m}^{(3)} \varepsilon^j Y_{n}^j, \quad Y_{n}^j \in \mathrm{Harm}_{n}. \tag{5.335}$$

Since $\{\tilde{y}_{p,q}^{(i)}\}_{p=0_i,\ldots,q=1,\ldots 2p+1}$ is an orthonormal basis in $\mathrm{l}^2(\Omega)$, we obtain by comparison

$$a^{(1)}_{p,q} = \begin{cases} 0, & n-1 \neq p \\ C^{(1)}_{k,p,q}, & n-1 = p \end{cases}, \tag{5.336}$$

$$a^{(2)}_{p,q} = \begin{cases} 0, & n+1 \neq p \\ C^{(2)}_{k,p,q}, & n+1 = p \end{cases}, \tag{5.337}$$

$$a^{(3)}_{p,q} = \begin{cases} 0, & n \neq p \\ C^{(3)}_{k,p,q}, & n = p \end{cases}, \tag{5.338}$$

$C^{(i)}_{k,p,q} \in \mathbb{R}$. This confirms our assertion. □

It should be mentioned that an addition theorem for the system, $\{\tilde{y}^{(i)}_{n,m}\}$ can be formulated based on that of the system $\{y^{(i)}_{n,m}\}$.

As preparation, we understand $\tilde{o}^{(i)} f$ to be defined by

$$\tilde{o}^{(i)}_{\xi} f(\xi) = \sum_{l=1}^{3} (\tilde{o}^{(i)}_{\xi} F_l(\xi)) \otimes \varepsilon^l, \quad i \in \{1,2,3\}, \tag{5.339}$$

whenever $f : \Omega \to \mathbb{R}^3$ is (a sufficiently smooth vector field) given by

$$f(\xi) = \sum_{l=1}^{3} F_l(\xi)\varepsilon^l, \; F_l = f \cdot \varepsilon^l. \tag{5.340}$$

Our point of departure is the definition of the Legendre kernel corresponding to the vector spherical harmonics $\tilde{y}^{(i)}_{n,m}$, $i = 1,2,3$, $n = 0_i, \ldots$, $m = 1, \ldots, 2n+1$ (see H. Nutz (2002)).

Definition 5.58. The kernel ${}^v\tilde{\mathbf{p}}^{(i,j)}_n : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $i, j \in \{1,2,3\}$ given by

$${}^v\tilde{\mathbf{p}}^{(i,j)}_n(\xi,\eta) = \left(\tilde{\mu}^{(i)}_n\right)^{-1/2} \left(\tilde{\mu}^{(j)}_n\right)^{-1/2} \tilde{o}^{(i)}_{\xi} \tilde{o}^{(j)}_{\eta} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{5.341}$$

is called the *(vectorial) Legendre rank-2 tensor kernel of degree* n *and type* (i,j) *with respect to* $\tilde{o}^{(i)}_n, \tilde{O}^{(i)}_n$, $i = 1,2,3$. The kernel

$${}^v\tilde{\mathbf{p}}_n = \sum_{i=1}^{3} \sum_{j=1}^{3} {}^v\tilde{\mathbf{p}}^{(i,j)}_n \tag{5.342}$$

is called *vectorial Legendre rank-2 tensor kernel of degree* n *with respect to the dual system of operators* $\tilde{o}^{(i)}, \tilde{O}^{(i)}, i = \{1,2,3\}$.

The relation between the Legendre tensor ${}^v\tilde{\mathbf{p}}_n^{(i,k)}$ and the Legendre tensor ${}^v\mathbf{p}_n^{(i,k)}$ is described by the following lemma.

Lemma 5.59. *The Legendre tensor fields* ${}^v\tilde{\mathbf{p}}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3, \ \ i,k \in \{1,2,3\}$, *as indicated above, can be expressed in terms of Legendre tensors* ${}^v\mathbf{p}_n^{(i,k)}$ *as follows:*

$$\begin{aligned}
{}^v\tilde{\mathbf{p}}_n^{(1,1)} &= (n+1)^2 (c_n^{1,1})^2 \, {}^v\mathbf{p}_n^{(1,1)} - (n+1) c_n^{1,1} c_n^{2,1} \left({}^v\mathbf{p}_n^{(1,2)} + {}^v\mathbf{p}_n^{(2,1)}\right) \\
&\quad + (c_n^{2,1})^2 \, {}^v\mathbf{p}_n^{(2,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(1,2)} &= n(n+1) c_n^{1,1} c_n^{1,2} \, {}^v\mathbf{p}_n^{(1,1)} + (n+1) c_n^{1,1} c_n^{2,2} \, {}^v\mathbf{p}_n^{(1,2)} - n c_n^{1,1} c_n^{2,2} \, {}^v\mathbf{p}_n^{(2,1)} \\
&\quad - c_n^{2,1} c_n^{2,2} \, {}^v\mathbf{p}_n^{(2,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(1,3)} &= (n+1) c_n^{1,1} c_n^{3,3} \, {}^v\mathbf{p}_n^{(1,3)} - c_n^{2,1} c_n^{3,3} \, {}^v\mathbf{p}_n^{(2,3)}, \\
{}^v\tilde{\mathbf{p}}_n^{(2,1)} &= n(n+1) c_n^{1,2} c_n^{1,1} \, {}^v\mathbf{p}_n^{(1,1)} + (n+1) c_n^{2,2} c_n^{1,1} \, {}^v\mathbf{p}_n^{(2,1)} - n c_n^{1,2} c_n^{2,1} \, {}^v\mathbf{p}_n^{(1,2)} \\
&\quad - c_n^{2,2} c_n^{2,1} \, {}^v\mathbf{p}_n^{(2,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(2,2)} &= n^2 (c_n^{1,2})^2 \, {}^v\mathbf{p}_n^{(1,1)} + n c_n^{1,2} c_n^{2,2} \left({}^v\mathbf{p}_n^{(1,2)} + {}^v\mathbf{p}_n^{(2,1)}\right) + (c_n^{2,2})^2 \, {}^v\mathbf{p}_n^{(2,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(2,3)} &= n c_n^{1,2} c_n^{3,3} \, {}^v\mathbf{p}_n^{(1,3)} + c_n^{2,2} c_n^{3,3} \, {}^v\mathbf{p}_n^{(2,3)}, \\
{}^v\tilde{\mathbf{p}}_n^{(3,1)} &= (n+1) c_n^{1,1} c_n^{3,1} \, {}^v\mathbf{p}_n^{(3,1)} - c_n^{3,3} c_n^{2,1} \, {}^v\mathbf{p}_n^{(3,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(3,2)} &= n c_n^{3,3} c_n^{1,2} \, {}^v\mathbf{p}_n^{(3,1)} + c_n^{3,3} c_n^{1,2} \, {}^v\mathbf{p}_n^{(3,2)}, \\
{}^v\tilde{\mathbf{p}}_n^{(3,3)} &= c_n^{3,3} c_n^{3,3} \, {}^v\mathbf{p}_n^{(3,3)},
\end{aligned}$$

where the constants $c_n^{i,k} \in \mathbb{R}$ *are given by*

$$c_n^{i,k} = \left(\mu_n^{(i)}\right)^{1/2} \left(\tilde{\mu}_n^{(k)}\right)^{-1/2}. \tag{5.343}$$

The addition theorem for the vector harmonics $\tilde{y}_{n,m}^{(i)}$ defined by (5.305) reads as follows (see H. Nutz (2002)).

Theorem 5.60. *Let* $\{\tilde{y}_{n,m}^{(i)}\}_{m=1,\dots,2n+1}$ *be an* $\mathrm{l}^2(\Omega)$*–orthonormal basis of* $\widetilde{\mathrm{harm}}_n^{(i)}$ *as defined by* (5.305)*. Then*

$$\sum_{m=1}^{2n+1} \tilde{y}_{n,m}^{(i)}(\xi) \otimes \tilde{y}_{n,m}^{(k)}(\eta) = \frac{2n+1}{4\pi} {}^v\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta) \tag{5.344}$$

holds for $i,k \in \{1,2,3\}$ *and* $(\xi,\eta) \in \Omega \times \Omega$.

Proof. The addition theorem follows directly from the definition of the Legendre tensor ${}^v\tilde{\mathbf{p}}_n^{(i,k)}$ and from the already known addition theorem for the Legendre tensor ${}^v\mathbf{p}_n^{(i,k)}$. We only have to observe

$$\begin{aligned}\tilde{y}^{(1)}_{n,m} \otimes \tilde{y}^{(1)}_{n,m} &= (n+1)^2 (c_n^{1,1})^2\, y^{(1)}_{n,m} \otimes y^{(1)}_{n,m} - (n+1) c_n^{1,1} c_n^{2,1}\, (y^{(1)}_{n,m} \otimes y^{(2)}_{n,m} \\ &\quad + y^{(2)}_{n,m} \otimes y^{(1)}_{n,m}) + (c_n^{2,1})^2\, y^{(2)}_{n,m} \otimes y^{(2)}_{n,m}, \end{aligned} \tag{5.345}$$

$$\begin{aligned}\tilde{y}^{(1)}_{n,m} \otimes \tilde{y}^{(2)}_{n,m} &= n(n+1) c_n^{1,1} c_n^{1,2}\, y^{(1)}_{n,m} \otimes y^{(1)}_{n,m} + (n+1) c_n^{1,1} c_n^{2,2}\, y^{(1)}_{n,m} \otimes y^{(2)}_{n,m} \\ &\quad - n c_n^{1,1} c_n^{2,2}\, y^{(2)}_{n,m} \otimes y^{(1)}_{n,m} - c_n^{2,1} c_n^{2,2}\, y^{(2)}_{n,m} \otimes y^{(2)}_{n,m}, \end{aligned} \tag{5.346}$$

$$\tilde{y}^{(1)}_{n,m} \otimes \tilde{y}^{(3)}_{n,m} = (n+1) c_n^{1,1} c_n^{3,3}\, y^{(1)}_{n,m} \otimes y^{(3)}_{n,m} - c_n^{2,1} c_n^{3,3}\, y^{(2)}_{n,m} \otimes y^{(3)}_{n,m}, \tag{5.347}$$

$$\begin{aligned}\tilde{y}^{(2)}_{n,m} \otimes \tilde{y}^{(1)}_{n,m} &= n(n+1) c_n^{1,2} c_n^{1,1}\, y^{(1)}_{n,m} \otimes y^{(1)}_{n,m} + (n+1) c_n^{2,2} c_n^{1,1}\, y^{(2)}_{n,m} \otimes y^{(1)}_{n,m} \\ &\quad - n c_n^{1,2} c_n^{2,1}\, y^{(1)}_{n,m} \otimes y^{(2)}_{n,m} - c_n^{2,2} c_n^{2,1}\, y^{(2)}_{n,m} \otimes y^{(2)}_{n,m}, \end{aligned} \tag{5.348}$$

$$\begin{aligned}\tilde{y}^{(2)}_{n,m} \otimes \tilde{y}^{(2)}_{n,m} &= n^2 (c_n^{1,2})^2\, y^{(1)}_{n,m} \otimes y^{(2)}_{n,m} + n c_n^{1,2} c_n^{2,2}\, (y^{(1)}_{n,m} \otimes y^{(2)}_{n,m} + y^{(2)}_{n,m} \otimes y^{(1)}_{n,m}) \\ &\quad + (c_n^{2,2})^2\, y^{(2)}_{n,m} \otimes y^{(2)}_{n,m}, \end{aligned} \tag{5.349}$$

$$\tilde{y}^{(2)}_{n,m} \otimes \tilde{y}^{(3)}_{n,m} = n c_n^{1,2} c_n^{3,3}\, y^{(1)}_{n,m} \otimes y^{(3)}_{n,m} + c_n^{2,2} c_n^{3,3}\, y^{(2)}_{n,m} \otimes y^{(3)}_{n,m}, \tag{5.350}$$

$$\tilde{y}^{(3)}_{n,m} \otimes \tilde{y}^{(1)}_{n,m} = (n+1) c_n^{1,1} c_n^{3,1}\, y^{(3)}_{n,m} \otimes y^{(1)}_{n,m} - c_n^{3,3} c_n^{2,1}\, y^{(3)}_{n,m} \otimes y^{(2)}_{n,m}, \tag{5.351}$$

$$\tilde{y}^{(3)}_{n,m} \otimes \tilde{y}^{(2)}_{n,m} = n c_n^{3,3} c_n^{1,2}\, y^{(3)}_{n,m} \otimes y^{(1)}_{n,m} + c_n^{3,3} c_n^{1,2}\, y^{(3)}_{n,m} \otimes y^{(2)}_{n,m}, \tag{5.352}$$

$$\tilde{y}^{(3)}_{n,m} \otimes \tilde{y}^{(3)}_{n,m} = c_n^{3,3} c_n^{3,3}\, y^{(3)}_{n,m} \otimes y^{(3)}_{n,m}. \tag{5.353}$$

□

As in the case of the Legendre tensor ${}^v\mathbf{p}_n^{(i,k)}$, we are led to an estimate for the absolute value of the tensors ${}^v\tilde{\mathbf{p}}_n^{(i,k)}$.

Lemma 5.61. *Let $i,k,l \in \{1,2,3\}$. Then, for all $\xi, \eta \in \Omega$,*

$$|{}^v\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta)\varepsilon^l| \leq 1,$$

and

$$|{}^v\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta)| \leq \sqrt{3}.$$

By virtue of the addition theorem (Theorem 5.60), we are able to conclude that $\tilde{\mathbf{k}}_{\widetilde{\mathrm{harm}}_n^{(i)}}$ is the reproducing kernel of the space $\widetilde{\mathrm{harm}}_n^{(i)}$. More explicitly, the tensor field

$$\tilde{\mathbf{k}}_{\widetilde{\mathrm{harm}}_n^{(i)}}(\xi,\eta) = \frac{2n+1}{4\pi}\, {}^v\tilde{\mathbf{p}}_n^{(i,i)}(\xi,\eta), \quad \xi,\eta \in \Omega, \tag{5.354}$$

is the reproducing kernel of $\widetilde{\mathrm{harm}}_n^{(i)}$ in the following sense:

(i) for all $\xi \in \Omega$

$$\tilde{O}_n^{(i)} \tilde{\mathbf{k}}_{\widetilde{\mathrm{harm}}_n^{(i)}}(\cdot, \xi) \in \widetilde{\mathrm{harm}}_n^{(i)}, \tag{5.355}$$

(ii) for every $f \in \widetilde{\mathrm{harm}}_n^{(i)}$ and all $\xi \in \Omega$

$$\tilde{O}_n^{(i)} f(\xi) = \left(\tilde{O}_n^{(i)} \tilde{\mathbf{k}}_{\widetilde{\mathrm{harm}}_n^{(i)}}(\cdot, \xi), f \right)_{\mathrm{l}^2(\Omega)}. \tag{5.356}$$

Note that (for sufficiently smooth) tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ of the form

$$\mathbf{f}(\xi) = \sum_{j=1}^{3} \sum_{k=1}^{3} F_{j,k}(\xi) \varepsilon^j \otimes \varepsilon^k \tag{5.357}$$

such that $\sum_{j=1}^{3} F_{j,k}(\xi)\varepsilon^j \in \widetilde{\mathrm{harm}}_n$, we understand the operators $\tilde{O}^{(i)}$ to be defined by

$$\tilde{O}_\xi^{(i)} \mathbf{f}(\xi) = \sum_{k=1}^{3} \tilde{O}_\xi^{(i)} \left(\sum_{j=1}^{3} F_{j,k}(\xi) \varepsilon^j \right) \varepsilon^k. \tag{5.358}$$

In analogy to the described way of defining the Legendre vectors based on the system $\{y_{n,m}^{(i)}\}$, we are able to define the Legendre vectors $\tilde{p}_n^{(i)}$ based on the system $\{\tilde{y}_{n,m}^{(i)}\}$.

Definition 5.62. The kernel $\tilde{p}_n^{(i)} : \Omega \times \Omega \to \mathbb{R}^3,\ \ i \in \{1, 2, 3\}$, given by

$$\tilde{p}_n^{(i)}(\xi, \eta) = \left(\tilde{\mu}_n^{(i)} \right)^{-1/2} \tilde{o}_\xi^{(i)} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{5.359}$$

is called the *(vectorial) Legendre vector kernel of degree n and type i with respect to the dual system of operators $\tilde{o}_n^{(i)}, \tilde{O}_n^{(i)}$, $i = 1, 2, 3$.* The kernel $\tilde{p}_n = \sum_{i=1}^{3} \tilde{p}_n^{(i)}$ is called *(vectorial) Legendre vector kernel of degree n with respect to $\tilde{o}_n^{(i)}, \tilde{O}_n^{(i)}$, $i \in \{1, 2, 3\}$.*

The Legendre vector kernel satisfies the following lemma.

Lemma 5.63. *Let the Legendre vectors $\tilde{p}_n^{(i)} : \Omega \times \Omega \to \mathbb{R}^3,\ \ i \in \{1, 2, 3\}$, be defined as in Definition* 5.62. *Then we have*

$$\tilde{p}_n^{(1)}(\xi, \eta) = (n+1) c_n^{1,1}\, p_n^{(1)}(\xi, \eta) - c_n^{2,1}\, p_n^{(2)}(\xi, \eta), \tag{5.360}$$
$$\tilde{p}_n^{(2)}(\xi, \eta) = n c_n^{1,2}\, p_n^{(1)}(\xi, \eta) - c_n^{2,2}\, p_n^{(2)}(\xi, \eta), \tag{5.361}$$
$$\tilde{p}_n^{(3)}(\xi, \eta) = c_n^{3,3}\, p_n^{(3)}(\xi, \eta), \tag{5.362}$$

where the constants $c_n^{i,k}$, $i, k \in \{1, 2, 3\}$, are given by (5.343)

$$c_n^{i,k} = \left(\mu_n^{(i)} \right)^{1/2} \left(\tilde{\mu}_n^{(k)} \right)^{-1/2}. \tag{5.363}$$

We are finally led to the following addition theorem for vector spherical harmonics.

Theorem 5.64. *Let* $\{Y_{n,m}\}_{m=1,\ldots,2n+1}$ *be an* $\mathrm{L}^2(\Omega)$*–orthonormal basis of* Harm_n*, and the system* $\{\tilde{y}^{(i)}_{n,m}\}$ *be given by* $\tilde{y}^{(i)}_{n,m} = \left(\tilde{\mu}^{(i)}_n\right)^{-1/2} \tilde{o}^{(i)} Y_{n,m}$*. Then*

$$\sum_{m=1}^{2n+1} \tilde{y}^{(i)}_{n,m}(\xi) Y_{n,m}(\eta) = \frac{2n+1}{4\pi} \tilde{p}^{(i)}_n(\xi,\eta), \quad \xi,\eta \in \Omega, \tag{5.364}$$

is valid for $i \in \{1,2,3\}$*.*

Remark 5.65. The extension of our results to a sphere of radius R can be achieved in canonical way as in the case of the system $\{y^{(i)}_{n,m}\}$. The details are left to the reader.

5.15 Orthogonal Expansions Using Vector Legendre Kernels

For $F \in \mathrm{L}^2(\Omega)$, we already know the orthogonal expansion

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^{\wedge}(n,j) Y_{n,j} \tag{5.365}$$

with $F^{\wedge}(n,j) = (F, Y_{n,j})_{\mathrm{L}^2(\Omega)}$. Using the addition theorem, the expansion (5.365) can be reformulated as follows:

$$\begin{aligned} F &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega F(\eta) Y_{n,j}(\eta) d\omega(\eta) Y_{n,j} \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\cdot\eta) d\omega(\eta). \end{aligned} \tag{5.366}$$

In other words, the projection of F into Harm_n, i.e., the space of all spherical harmonics with degree n, can be written as

$$\mathrm{Proj}_{\mathrm{Harm}_n}(F) = \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\cdot\eta) d\omega(\eta). \tag{5.367}$$

Our purpose is to show how these Fourier expansions look like for the vectorial case. In particular, we introduce two generalizations of the Legendre polynomial for the vectorial case, which lead to two different generalizations of (5.367).

Suppose that f is of class $l^2(\Omega)$. Letting

$$(f^{(i)})^\wedge(n,j) = \int_\Omega f(\eta) \cdot y_{n,j}^{(i)}(\eta) d\omega(\eta) \tag{5.368}$$

we have the orthogonal expansion

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) y_{n,j}^{(i)}. \tag{5.369}$$

Using the addition theorem for the vector spherical harmonics $\{y_{n,j}^{(i)}\}$, the vectorial expansion (5.369) may be rewritten in the form

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\cdot,\eta) f^{(i)}(\eta) d\omega(\eta), \tag{5.370}$$

where ${}^v\mathbf{p}_n^{(i,i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ reads as follows:

$$\sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta) = \frac{2n+1}{4\pi} {}^v\mathbf{p}_n^{(i,i)}(\xi,\eta) = (\mu_n^{(i)})^{-1} \frac{2n+1}{4\pi} o_\xi^{(i)} o_\eta^{(i)} P_n(\xi \cdot \eta), \tag{5.371}$$

for all $(\xi,\eta) \in \Omega \times \Omega$. Furthermore, it is obvious that the projection from $l^2(\Omega)$ into $\mathrm{harm}_n^{(i)}$ can be formulated as

$$\mathrm{proj}_{\mathrm{harm}_n^{(i)}}(f) = \frac{2n+1}{4\pi} \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\cdot,\eta) f(\eta) d\omega(\eta). \tag{5.372}$$

Thus, we recognize the second order tensor ${}^v\mathbf{p}_n^{(i,i)}$ as canonical generalization of the Legendre polynomial to the vector case.

In fact, there is a second way to generalize the Legendre polynomial. Let the vector spherical harmonics $y_{n,j}^{(i)}$ be constructed from an orthonormal set of scalar spherical harmonics, i.e.,

$$y_{n,j}^{(i)} = (\mu_n^{(i)})^{-1/2} o^{(i)} Y_{n,j}, \tag{5.373}$$

i=1,2,3, $n = 0_i, \ldots$, $j = 1, \ldots, 2n+1$. Assuming that $f \in l^2(\Omega)$ is in addition, sufficiently smooth, we are able to reformulate (5.369) in the following way

$$
\begin{aligned}
f &= \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\int_{\Omega} f(\eta)\cdot y^{(i)}_{n,j}(\eta) y^{(i)}_{n,j}(\cdot)d\omega(\eta) \\
&= \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\int_{\Omega} f(\eta)\cdot \frac{1}{(\mu^{(i)}_n)^{1/2}} o^{(i)}_{\eta} Y_{n,j}(\eta)d\omega(\eta) y^{(i)}_{n,j} \qquad (5.374)\\
&= \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\int_{\Omega} (O^{(i)}_{\eta} f(\eta)) \frac{1}{(\mu^{(i)}_n)^{1/2}} Y_{n,j}(\eta)d\omega(\eta) y^{(i)}_{n,j} \\
&= \sum_{i=1}^{3}\sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi}\frac{1}{(\mu^{(i)}_n)^{1/2}} \int_{\Omega} p^{(i)}_n(\cdot,\eta)\; O^{(i)}_{\eta} f^{(i)}(\eta)\; d\omega(\eta),
\end{aligned}
$$

where the *(vectorial) Legendre vector kernel* $p^{(i)}_n(\cdot,\cdot):\Omega\times\Omega\to\mathbb{R}^3\times\mathbb{R}$ (more precisely, ${}^{v}p^{(i)}_n(\cdot,\cdot)$) is given by

$$
\sum_{j=1}^{2n+1} y^{(i)}_{n,j}(\xi)Y_{n,j}(\eta) = \frac{2n+1}{4\pi} p^{(i)}_n(\xi,\eta), \qquad (\xi,\eta)\in\Omega\times\Omega \qquad (5.375)
$$

(cf. Theroem 5.46), and the operators $O^{(i)}$ which are adjoint to $o^{(i)}$ are given by

$$
\begin{aligned}
O^{(1)}_{\xi} f^{(1)}(\xi) &= \xi\cdot f^{(1)}(\xi), \quad \xi\in\Omega, && (5.376)\\
O^{(2)}_{\xi} f^{(2)}(\xi) &= -\nabla^*_{\xi}\cdot f^{(2)}(\xi), \quad \xi\in\Omega, && (5.377)\\
O^{(3)}_{\xi} f^{(3)}(\xi) &= -\mathrm{L}^*_{\xi}\cdot f^{(3)}(\xi), \quad \xi\in\Omega. && (5.378)
\end{aligned}
$$

The Legendre vectors written out read as follows:

$$
\begin{aligned}
p^{(1)}_n(\xi,\eta) &= \xi P_n(\xi\cdot\eta),\ n=0,1,\ldots, && (5.379)\\
p^{(2)}_n(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}(\eta-(\xi\cdot\eta)\,\xi)P'_n(\xi\cdot\eta),\ n=1,2,\ldots, && (5.380)\\
p^{(3)}_n(\xi,\eta) &= \frac{1}{\sqrt{n(n+1)}}(\xi\wedge\eta)P'_n(\xi\cdot\eta),\ n=1,2,\ldots && (5.381)
\end{aligned}
$$

for $(\xi,\eta)\in\Omega\times\Omega$.

Using this second generalization $p^{(i)}_n$ of the Legendre polynomials, the projection operator (5.372) can be rewritten as

$$
\mathrm{proj}_{\mathrm{harm}^{(i)}_n}(f) = \frac{2n+1}{4\pi}(\mu^{(i)}_n)^{-1/2}\int_{\Omega} O^{(i)}_{\eta} f(\eta) p^{(i)}_n(\cdot,\eta)d\omega(\eta). \qquad (5.382)
$$

For this formula to be valid, it is necessary that f is sufficiently smooth.

Even more, besides the system $\{y_{n,j}^{(i)}\}$ of vector spherical vector harmonics, the system $\{\tilde{y}_{n,j}^{(i)}\}$ (as introduced by (5.305)) can be used in orthogonal (Fourier) expansions. In more detail, suppose that f is of class $\mathrm{l}^2(\Omega)$. Letting

$$\left(\tilde{f}^{(i)}\right)^{\wedge}(n,j) = \int_{\Omega} \tilde{f}(\eta) \cdot \tilde{y}_{n,j}^{(i)}(\eta) d\omega(\eta) \tag{5.383}$$

f admits the orthogonal expansion

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (\tilde{f}^{(i)})^{\wedge}(n,j) \tilde{y}_{n,j}^{(i)}. \tag{5.384}$$

Using the addition theorem for the vector spherical harmonics $\{\tilde{y}_{n,j}^{(i)}\}$, the vectorial expansion (5.384) may be rewritten in the form

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \int_{\Omega} {}^{v}\tilde{\mathbf{p}}_n^{(i,i)}(\cdot,\eta) \tilde{f}^{(i)}(\eta) d\omega(\eta), \tag{5.385}$$

where

$$\tilde{f}^{(i)} = \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (\tilde{f}^{(i)})^{\wedge}(n,j) \tilde{y}_{n,j}^{(i)} \tag{5.386}$$

and the kernel ${}^{v}\tilde{\mathbf{p}}_n^{(i,i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ reads as follows:

$$\begin{aligned} \sum_{j=1}^{2n+1} \tilde{y}_{n,j}^{(i)}(\xi) \otimes \tilde{y}_{n,j}^{(i)}(\eta) &= \frac{2n+1}{4\pi} {}^{v}\tilde{\mathbf{p}}_n^{(i,i)}(\xi,\eta) \\ &= (\tilde{\mu}_n^{(i)})^{-1} \frac{2n+1}{4\pi} \tilde{o}_{\xi}^{(i)} \tilde{o}_{\eta}^{(i)} P_n(\xi \cdot \eta), \end{aligned} \tag{5.387}$$

for all $(\xi,\eta) \in \Omega \times \Omega$. Furthermore, it is obvious that the projection from $\mathrm{l}^2(\Omega)$ into $\widetilde{\mathrm{harm}}_n^{(i)}$ can be formulated as

$$\mathrm{proj}_{\widetilde{\mathrm{harm}}_n^{(i)}}(f) = \frac{2n+1}{4\pi} \int_{\Omega} {}^{v}\tilde{\mathbf{p}}_n^{(i,i)}(\cdot,\eta) f(\eta) d\omega(\eta). \tag{5.388}$$

Therefore, also in this case, we are able to consider the second order tensor ${}^{v}\tilde{\mathbf{p}}_n^{(i,i)}$ as canonical generalization of the Legendre polynomial to the vector case.

In fact, as for the system $\{y_{n,j}^{(i)}\}$, there is a second way to generalize the Legendre polynomial. Let the vector spherical harmonics $\tilde{y}_{n,j}^{(i)}$ be constructed from an orthonormal set of scalar spherical harmonics, i.e.,

$$\tilde{y}_{n,j}^{(i)} = (\tilde{\mu}_n^{(i)})^{-1/2} \tilde{o}^{(i)} Y_{n,j}, \tag{5.389}$$

i=1,2,3, $n = 0_i, \ldots, j = 1, \ldots, 2n+1$. Assuming that $f \in \mathrm{l}^2(\Omega)$ is, in addition, sufficiently smooth, we are able to reformulate (5.384) in the following way

$$
\begin{aligned}
f &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega \left(f(\eta) \cdot \tilde{y}^{(i)}_{n,j}(\eta) \right) d\omega(\eta) \tilde{y}^{(i)}_{n,j} \\
&= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega f(\eta) \cdot \frac{1}{(\tilde{\mu}^{(i)}_n)^{1/2}} \tilde{o}^{(i)}_\eta Y_{n,j}(\eta) d\omega(\eta) \tilde{y}^{(i)}_{n,j} \qquad (5.390) \\
&= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega \left(\tilde{O}^{(i)}_\eta f(\eta) \right) \frac{1}{(\tilde{\mu}^{(i)}_n)^{1/2}} Y_{n,j}(\eta) d\omega(\eta) \tilde{y}^{(i)}_{n,j} \\
&= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \frac{1}{(\tilde{\mu}^{(i)}_n)^{1/2}} \int_\Omega \tilde{p}^{(i)}_n(\cdot, \eta) \; \tilde{O}^{(i)}_\eta f^{(i)}(\eta) \; d\omega(\eta),
\end{aligned}
$$

where the *(vectorial) Legendre vector* $\tilde{p}^{(i)}_n(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}^3 \times \mathbb{R}$ (more precisely, ${}^v\tilde{p}^{(i)}_n(\cdot,\cdot)$) is given by

$$
\sum_{j=1}^{2n+1} \tilde{y}^{(i)}_{n,j}(\xi) Y_{n,j}(\eta) = \frac{2n+1}{4\pi} \tilde{p}^{(i)}_n(\xi,\eta), \qquad (\xi,\eta) \in \Omega \times \Omega \qquad (5.391)
$$

(cf. Theroem 5.46). Clearly, we have

$$
\tilde{p}^{(i)}_n(\xi,\eta) = (\tilde{\mu}^{(i)}_n)^{-1/2} \tilde{o}^{(i)}_\xi P_n(\xi \cdot \eta), \qquad (5.392)
$$

$(\xi,\eta) \in \Omega \times \Omega$. Using this second generalization $\tilde{p}^{(i)}_n$ of the Legendre polynomials, the projection operator (5.372) can be rewritten as

$$
\mathrm{proj}_{\widetilde{\mathrm{harm}}^{(i)}_n}(f) = \frac{2n+1}{4\pi} \frac{1}{(\tilde{\mu}^{(i)}_n)^{1/2}} \int_\Omega \left(\tilde{O}^{(i)}_\eta f(\eta) \right) \tilde{p}^{(i)}_n(\eta, \cdot) d\omega(\eta). \qquad (5.393)
$$

For this formula to be valid, it is again necessary that f is sufficiently smooth.

Remark 5.66. The second approach described above is particularly helpful for the consideration of the vector Laplace equation (see Chapter 10), since the vector fields $x \mapsto \tilde{o}^{(i)} H_n(x)$, $x \in \mathbb{R}^3$, $i = 1,2,3$, satisfy the equation

$$
\Delta \tilde{o}^{(i)} H_n = 0, \quad i = 1,2,3, \qquad (5.394)
$$

where

$$
H_n(x) = |x|^n Y_n(\xi), \; x = |x|\xi, \; \xi \in \Omega. \qquad (5.395)
$$

5.16 Bibliographical Notes

Since 1950s, a number of researchers have used spherical harmonics for a variety of vectorial problems (e.g., J. Blatt, V. Weisskopf (1952), E.H. Hill (1954), I.M. Gelfand, Z. Ya. Shapiro (1956), A.R. Edmonds (1957), H.E. Moses (1974), D.A. Varshalovich et al. (1988) and many others). Unfortunately, the vector harmonics have not had any standard form. Each major school has invented its own notation and formalism (leaving us today with a very heterogeneous legacy). Moreover, the normalizations used in some of these formalisms are not very rational. An attempt at consolidating and reviewing the literature has been made by K.S. Thorne (1980). It seems to the authors that a number of formulas that previously were derived only under restrictive assumptions have much wider realms of validity by essentially using one system, the vector harmonics of P.M. Morse, H. Feshbach (1953) (see also G.E. Backus et al. (1996) and the references therein). These vector harmonics are intimately related to the 'pure-spin vector harmonics' (see, for example, A.R. Edmonds (1957)). Despite the fact that vector spherical harmonics have long been used in physical disciplines, they are only rarely found in the mathematical literature. They have been studied by M. Lagally, W. Franz (1964). However, basic topics were unknown. In the literature, we have no knowledge about the decomposition theorem (using Green's function with respect to the Beltrami operator), the addition theorem, and the Funk–Hecke formulas developed in Section 5.10. The treatment of these results here is based on T. Gervens (1989), W. Freeden, T. Gervens (1991), W. Freeden et al. (1994), and W. Freeden, M. Gutting (2008). The alternative system of vector spherical harmonics presented at the end of the chapter has been investigated in more detail in the PhD-thesis due to H. Nutz (2002).

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

6 Tensor Spherical Harmonics

The theory of tensor spherical harmonics extends in canonical way our approach to vector spherical harmonics. A keypoint is that the tensor spherical harmonics are generated from the scalar ones by use of certain operators mapping scalar functions to tensor fields. In fact, these formulations (always being independent of any choice of spherical coordinates) offer the perspective of extending all essential results known for scalar spherical harmonics to the tensorial case, including the definition of a tensorial Beltrami operator, the addition theorem, and tensorial versions of the Funk–Hecke formula. Among other areas of application (see, for example, R. Burridge (1969), James R.W. (1976), M.N. Jones (1980), K.S. Thorne (1980), and F.J. Zerilli (1970)), tensor spherical harmonics play an important role in diverse satellite problems of physical geodesy. Of current interest (see in particular Chapter 10) is the determination of the Earth's gravitational field by satellite gravity gradiometry, where tensor valued functionals of the potential are measured at satellite height, (see, for example, R. Rummel (1986), R. Rummel, M. van Gelderen (1992), R. Rummel et al. (1993), M. Schreiner (1994), R. Rummel (1997), K.-P. Schwarz, L. Zuofa (1997), W. Freeden et al. (1998), W. Freeden (1999), W. Freeden, V. Michel (2004), K.H. Ilk et al. (2004) and the references therein).

Our approach to tensor spherical harmonics closely follows M. Schreiner (1994) and W. Freeden et al. (1998). The outline is as follows: After some nomenclature, the separation of tensor field into normal and tangential parts is discussed in Section 6.2. Integral theorems are listed within the tensorial framework on the sphere (cf. Section 6.3). In Section 6.4, we introduce tensor spherical harmonics. Based on the Green function with respect to the Beltrami operator, a decomposition theorem for spherical tensor fields is shown in Section 6.5. Orthogonal (Fourier) expansions in terms of tensor spherical harmonics are described in Section 6.6. The interrelations between tensorial homogeneous harmonic polynomials and tensor spherical harmonics are discussed in Section 6.7. The tensor spherical harmonics are characterized as eigenfunctions of a tensorial analogue of the Beltrami operator (Section 6.8). Then, tensorial versions of the addition theorem and the Funk–Hecke formula are developed in Sections 6.9 and 6.10, respectively. After the description of counterparts to the Legendre polynomial (cf. Section 6.11), we introduce tensor spherical harmonics related to tensor homogeneous harmonic polynormials in Section 6.12. Based on these results, Section 6.13 shows alternative function systems of tensor spherical harmonics. Finally, orthogonal expansions using tensor Legendre kernels are discussed in Section 6.14.

6.1 Some Nomenclature

For the convenience of the reader, we start with the repetition of some facts. As usual, a tensor of rank $k \in \mathbb{N}$ is understood to be an element of $\bigotimes_{l=1}^{k} \mathbb{R}^3$ (see, e.g., M.E. Gurtin (1971)). Using the canonical orthonormal basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ of $\mathbb{R}^3$, a tensor $\mathbf{F}$ of rank k can be written as

$$\mathbf{F} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1\ldots i_k} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k}, \quad F_{i_1\ldots i_k} \in \mathbb{R}. \tag{6.1}$$

The scalar product $\mathbf{F} \cdot \mathbf{G}$ of two rank-k tensors F, G is defined by

$$\mathbf{F} \cdot \mathbf{G} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1\ldots i_k}\ G_{i_1\ldots i_k}, \quad F_{i_1\ldots i_k},\ G_{i_1\ldots i_k} \in \mathbb{R}, \tag{6.2}$$

hence, the modulus $|\mathbf{F}|$ of a rank-k tensor is given by

$$|\mathbf{F}| = \left(\sum_{i_1,\ldots,i_k=1}^{3} |F_{i_1\ldots i_k}|^2 \right)^{1/2}. \tag{6.3}$$

If

$$\mathbf{F} = \sum_{i_1,\ldots,i_k=1}^{3} F_{i_1\ldots i_k} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k} \tag{6.4}$$

and

$$\mathbf{G} = \sum_{i_1,\ldots,i_l=1}^{3} G_{i_1\ldots i_l} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_l} \tag{6.5}$$

are a rank-k tensor and a rank-l tensor, respectively, $\mathbf{F} \otimes \mathbf{G}$ is the rank-$(k+l)$ tensor given by

$$\mathbf{F} \otimes \mathbf{G} = \sum_{i_1,\ldots,i_k=1}^{3} \sum_{j_1,\ldots,j_l=1}^{3} F_{i_1\ldots i_k}\ G_{j_1\ldots j_l} \varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_k} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_l}. \tag{6.6}$$

In what follows, we first restrict ourselves to tensor fields of rank 2. But it should be noted that most of our considerations carry over to the more general cases in an obvious way.

If $\mathbf{f}$ is a rank-2 tensor given by

$$\mathbf{f} = \sum_{i,k=1}^{3} F_{ik}\, \varepsilon^i \otimes \varepsilon^k, \tag{6.7}$$

its trace is defined by

$$\text{trace } \mathbf{f} = \sum_{i=1}^{3} F_{ii}. \tag{6.8}$$

We define the *transpose* of $\mathbf{f}$ by

$$\mathbf{f}^T = \sum_{i,k=1}^{3} F_{ik} \varepsilon^k \otimes \varepsilon^i \tag{6.9}$$

and say $\mathbf{f}$ is *symmetric* if $\mathbf{f} = \mathbf{f}^T$ and *skew-symmetric* if $\mathbf{f} = -\mathbf{f}^T$. As is well known, any tensor field $\mathbf{f}$ can be decomposed into a symmetric and skew-symmetric part: $\mathbf{f} = \text{sym } \mathbf{f} + \text{skew } \mathbf{f}$, where $\text{sym } \mathbf{f} = \frac{1}{2}(\mathbf{f} + \mathbf{f}^T)$ and $\text{skew } \mathbf{f} = \frac{1}{2}(\mathbf{f} - \mathbf{f}^T)$.

Clearly, a spherical rank-2 tensor field $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ can be represented in terms of its coordinate functions by means of functions: $F_{ik} : \Omega \to \mathbb{R}$ as follows:

$$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} F_{ik}(\xi) \varepsilon^i \otimes \varepsilon^k, \quad \xi \in \Omega. \tag{6.10}$$

If $v = \sum_{i=1}^{3} V_i \varepsilon^i$ is a vector field, then the products $v^T \mathbf{f}$ and $\mathbf{f} v$ are defined by

$$v^T \mathbf{f} = \sum_{k=1}^{3} \sum_{i=1}^{3} V_i F_{ik} \varepsilon^k \tag{6.11}$$

and

$$\mathbf{f} v = \sum_{i=1}^{3} \sum_{k=1}^{3} F_{ik} V_k \varepsilon^i, \tag{6.12}$$

respectively.

6.2 Normal and Tangential Fields

Rank-2 tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ can be separated into their tangential and normal parts. We set

$$\begin{aligned}
\mathbf{p}_{*,\mathrm{nor}}\mathbf{f}(\xi) &= (\mathbf{f}(\xi)\xi)\otimes\xi, && (6.13)\\
\mathbf{p}_{\mathrm{nor},*}\mathbf{f}(\xi) &= \xi\otimes(\xi^T\mathbf{f}(\xi)), && (6.14)\\
\mathbf{p}_{*,\tan}\mathbf{f}(\xi) &= \mathbf{f}(\xi)-\mathbf{p}_{*,\mathrm{nor}}\mathbf{f}(\xi) = \mathbf{f}(\xi)-(\mathbf{f}(\xi)\xi)\otimes\xi, && (6.15)\\
\mathbf{p}_{\tan,*}\mathbf{f}(\xi) &= \mathbf{f}(\xi)-\mathbf{p}_{\mathrm{nor},*}\mathbf{f}(\xi) = \mathbf{f}(\xi)-\xi\otimes(\xi^T\mathbf{f}(\xi)), && (6.16)\\
\mathbf{p}_{\mathrm{nor},\tan}\mathbf{f}(\xi) &= \mathbf{p}_{\mathrm{nor},*}(\mathbf{p}_{*,\tan}\mathbf{f}(\xi)) = \mathbf{p}_{*,\tan}(\mathbf{p}_{\mathrm{nor},*}\mathbf{f}(\xi)) && (6.17)\\
&= \xi\otimes(\xi^T\mathbf{f}(\xi))-(\xi^T\mathbf{f}(\xi)\cdot\xi)\xi\otimes\xi, &&\\
\mathbf{p}_{\tan,\mathrm{nor}}\mathbf{f}(\xi) &= \mathbf{p}_{\mathrm{nor},*}(\mathbf{p}_{*;\tan}\mathbf{f}(\xi)) = \mathbf{p}_{*,\tan}(\mathbf{p}_{\mathrm{nor},*}\mathbf{f}(\xi)) && (6.18)\\
&= \mathbf{f}(\xi)\xi\otimes\xi-\xi\otimes(\xi^T f(\xi)\xi\otimes\xi), && (6.19)\\
\mathbf{p}_{\mathrm{nor},\mathrm{nor}}\mathbf{f}(\xi) &= \mathbf{p}_{\mathrm{nor},*}(\mathbf{p}_{*,\mathrm{nor}}\mathbf{f}(\xi)) = \mathbf{p}_{*,\mathrm{nor}}(\mathbf{p}_{\mathrm{nor},*}\mathbf{f}(\xi)) && (6.20)\\
&= \xi\otimes(\xi^T\mathbf{f}(\xi)\xi\otimes\xi), && (6.21)\\
\mathbf{p}_{\tan,\tan}\mathbf{f}(\xi) &= \mathbf{f}(\xi)-\xi\otimes\xi^T\cdot\mathbf{f}(\xi)-\mathbf{f}(\xi)\xi\otimes\xi+(\xi^T\mathbf{f}(\xi)\cdot\xi)\xi\otimes\xi. && (6.22)
\end{aligned}$$

A tensor vector field $\mathbf{f}\in\mathbf{l}^2(\Omega)$ is called *normal* if $\mathbf{f}=\mathbf{p}_{\mathrm{nor},\mathrm{nor}}\mathbf{f}$ and *tangential* if $\mathbf{f}=\mathbf{p}_{\tan,\tan}\mathbf{f}$. It is called *left normal* if $\mathbf{f}=\mathbf{p}_{\mathrm{nor},*}\mathbf{f}$, *left normal/right tangential* if $\mathbf{f}=\mathbf{p}_{\mathrm{nor},\tan}\mathbf{f}$, and so on. Altogether, we have

$$\mathbf{f}=\mathbf{p}_{\mathrm{nor},\mathrm{nor}}\mathbf{f}+\mathbf{p}_{\mathrm{nor},\tan}\mathbf{f}+\mathbf{p}_{\tan,\mathrm{nor}}\mathbf{f}+\mathbf{p}_{\tan,\tan}\mathbf{f}. \tag{6.23}$$

The operators, as defined by (6.13) - (6.22) admit the definition of the spaces $\mathbf{l}^2_{*,\mathrm{nor}}(\Omega),\mathbf{l}^2_{*,\tan}(\Omega),\ldots,$ and $\mathbf{c}^{(p)}_{*,\mathrm{nor}}(\Omega),\mathbf{c}^{(p)}_{*,\tan}(\Omega)$, etc. We end up with the following orthogonal decompositions:

$$\begin{aligned}
\mathbf{l}^2_{*,\mathrm{nor}}(\Omega) &= \mathbf{l}^2_{\mathrm{nor},\mathrm{nor}}(\Omega)\oplus\mathbf{l}^2_{\tan,\mathrm{nor}}(\Omega), && (6.24)\\
\mathbf{l}^2_{*,\tan}(\Omega) &= \mathbf{l}^2_{\mathrm{nor},\tan}(\Omega)\oplus\mathbf{l}^2_{\tan,\tan}(\Omega), && (6.25)\\
\mathbf{l}^2_{\mathrm{nor},*}(\Omega) &= \mathbf{l}^2_{\mathrm{nor},\mathrm{nor}}(\Omega)\oplus\mathbf{l}^2_{\mathrm{nor},\tan}(\Omega), && (6.26)\\
\mathbf{l}^2_{\tan,*}(\Omega) &= \mathbf{l}^2_{\tan,\mathrm{nor}}(\Omega)\oplus\mathbf{l}^2_{\tan,\tan}(\Omega), && (6.27)\\
\mathbf{l}^2(\Omega) &= \mathbf{l}^2_{*,\mathrm{nor}}(\Omega)\oplus\mathbf{l}^2_{*,\tan}(\Omega), && (6.28)\\
\mathbf{l}^2(\Omega) &= \mathbf{l}^2_{\mathrm{nor},*}(\Omega)\oplus\mathbf{l}^2_{\tan,*}(\Omega). && (6.29)
\end{aligned}$$

It is a well known fact (see, for example, M.E. Gurtin (1971)) that every rank-2 tensor field $\mathbf{f}:\Omega\to\mathbb{R}^3\otimes\mathbb{R}^3$ can be represented as a dyadic sum

$$\mathbf{f}(\xi)=\sum_{i=1}^{p}u_i(\xi)\otimes v_i(\xi),\quad \xi\in\Omega, \tag{6.30}$$

in terms of vector fields $u_i,v_i:\Omega\to\mathbb{R}^3$ (for example, take the coordinate expression (6.10)). It should be noted that neither the upper index p nor the vector fields u_i,v_i are uniquely defined in (6.30).

A left normal tensor $\mathbf{f}$ can be expressed as

$$\mathbf{f}(\xi) = \sum_{i=1}^{3} \xi \otimes v_i(\xi), \quad \xi \in \Omega, \tag{6.31}$$

by means of vector fields v_i. For this tensor, it follows that

$$\xi^T \mathbf{f}(\xi) = \sum_{i=1}^{3} v_i(\xi), \quad \xi \in \Omega. \tag{6.32}$$

Similarly, a multiplication by ξ from the right of a right normal tensor field $\mathbf{f}$, given by

$$\mathbf{f}(\xi) = \sum_{i=1}^{3} u_i(\xi) \otimes \xi, \quad \xi \in \Omega \tag{6.33}$$

yields

$$\mathbf{f}(\xi)\xi = \sum_{i=1}^{3} u_i(\xi), \quad \xi \in \Omega. \tag{6.34}$$

As in the vectorial case, spherical tensor fields can be characterized in an elegant manner by the use of certain differential processes. Since we are mainly interested in presenting the underlying formalism, we will not spend much effort to formulate our results in their most general setting.

Let $u \in \mathrm{c}^{(1)}(\Omega)$ be a vector field given in its coordinate form by

$$u(\xi) = \sum_{i=1}^{3} U_i(\xi)\varepsilon^i, \quad \xi \in \Omega, \quad U_i \in \mathrm{C}^{(1)}(\Omega). \tag{6.35}$$

Then we define the operators $\nabla^*\otimes$ and $\mathrm{L}^*\otimes$ by

$$\begin{aligned} \nabla^*_\xi \otimes u(\xi) &= \sum_{i=1}^{3} (\nabla^*_\xi U_i(\xi)) \otimes \varepsilon^i, \quad \xi \in \Omega, &(6.36)\\ \mathrm{L}^*_\xi \otimes u(\xi) &= \sum_{i=1}^{3} (\mathrm{L}^*_\xi U_i(\xi)) \otimes \varepsilon^i, \quad \xi \in \Omega. &(6.37) \end{aligned}$$

Clearly, $\nabla^* \otimes u$ and $\mathrm{L}^* \otimes u$ are left tangential. But it is an important fact, that even if u is tangential, the tensor fields $\nabla^* \otimes u$ and $\mathrm{L}^* \otimes u$ are generally not tangential. It is obvious that the *product rule* is valid. To be specific, let $F \in \mathrm{C}^{(1)}(\Omega)$ and $u \in \mathrm{c}^{(1)}(\Omega)$, then

$$\nabla^*_\xi \otimes (F(\xi)u(\xi)) = \nabla^*_\xi F(\xi) \otimes u(\xi) + F(\xi)\nabla^*_\xi \otimes u(\xi), \quad \xi \in \Omega. \tag{6.38}$$

In order to simplify our calculations, we first apply the operators $\nabla^*\otimes$ and $L^*\otimes$ to the local orthonormal triad defined in (2.119, 2.120, 2.121). It follows that for $\varphi \in (0, 2\pi)$, $t \in (-1, 1)$

$$\nabla^* \otimes \varepsilon^r(\varphi,t) = \varepsilon^\varphi(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t) + \varepsilon^t(\varphi,t) \otimes \varepsilon^t(\varphi,t), \tag{6.39}$$

$$\nabla^* \otimes \varepsilon^\varphi(\varphi,t) = -\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^r(\varphi,t) + \frac{t}{\sqrt{1-t^2}}\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^t(\varphi,t), \tag{6.40}$$

$$\nabla^* \otimes \varepsilon^t(\varphi,t) = -\varepsilon^t(\varphi,t) \otimes \varepsilon^r(\varphi,t) - \frac{t}{\sqrt{1-t^2}}\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t), \tag{6.41}$$

and

$$L^* \otimes \varepsilon^r(\varphi,t) = \varepsilon^t(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t) - \varepsilon^\varphi(\varphi,t) \otimes \varepsilon^t(\varphi,t), \tag{6.42}$$

$$L^* \otimes \varepsilon^\varphi(\varphi,t) = -\varepsilon^t(\varphi,t) \otimes \varepsilon^r(\varphi,t) + \frac{t}{\sqrt{1-t^2}}\varepsilon^t(\varphi,t) \otimes \varepsilon^t(\varphi,t), \tag{6.43}$$

$$L^* \otimes \varepsilon^t(\varphi,t) = \varepsilon^\varphi(\varphi,t) \otimes \varepsilon^r(\varphi,t) - \frac{t}{\sqrt{1-t^2}}\varepsilon^t(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t). \tag{6.44}$$

The product rule (6.38) then yields for $u \in c^{(1)}(\Omega)$

$$\nabla^* \otimes u = \mathbf{p}_{\tan,*}\nabla^* \otimes u, \tag{6.45}$$

$$L^* \otimes u = \mathbf{p}_{\tan,*}L^* \otimes u. \tag{6.46}$$

Furthermore, we have for $u \in c^{(1)}_{\tan}(\Omega)$ and $\xi \in \Omega$

$$\mathbf{p}_{\tan,\mathrm{nor}}\nabla^*_\xi \otimes u(\xi) = -u(\xi) \otimes \xi, \tag{6.47}$$

$$\mathbf{p}_{\tan,\mathrm{nor}}L^*_\xi \otimes u(\xi) = -\xi \wedge u(\xi) \otimes \xi. \tag{6.48}$$

For the normal vector field $\xi \mapsto F(\xi)\xi$, $\xi \in \Omega$, with $F \in C^{(1)}(\Omega)$, we obtain

$$\nabla^*_\xi \otimes F(\xi)\xi = F(\xi)\mathbf{i}_{\tan}(\xi) + \nabla^*_\xi F(\xi) \otimes \xi, \tag{6.49}$$

$$L^*_\xi \otimes F(\xi)\xi = F(\xi)\mathbf{j}_{\tan}(\xi) + L^*_\xi F(\xi) \otimes \xi. \tag{6.50}$$

6.3 Integral Theorems

After the generalizations of the surface gradient operators in Section 6.2, we shall now investigate analogues to the operators, $\nabla^*\cdot$, and, $L^*\cdot$, applied to tensor fields.

Let $\mathbf{f} \in \mathbf{c}^{(1)}_{\tan,*}(\Omega)$ be a left tangential rank-2 tensor field given by

$$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} F_{ik}(\xi)\varepsilon^i \otimes \varepsilon^k, \quad F_{ik} \in \mathrm{C}^{(1)}(\Omega),\ i,k = 1,2,3. \tag{6.51}$$

Then we set

$$\begin{aligned} \nabla^*_\xi \cdot \mathbf{f}(\xi) &= \sum_{k=1}^{3} \left(\nabla^*_\xi \cdot \left(\sum_{i=1}^{3} F_{ik}(\xi)\varepsilon^i \right)\right) \varepsilon^k, \\ \mathrm{L}^*_\xi \cdot \mathbf{f}(\xi) &= \sum_{k=1}^{3} \left(\mathrm{L}^*_\xi \cdot \left(\sum_{i=1}^{3} F_{ik}(\xi)\varepsilon^i \right)\right) \varepsilon^k. \end{aligned} \tag{6.52}$$

Moreover, for $u \in \mathrm{c}^{(1)}_{\tan}(\Omega)$ and $v \in \mathrm{c}^{(1)}(\Omega)$ with

$$u = \sum_{i=1}^{3} U_i \varepsilon^i, \quad v = \sum_{i=1}^{3} V_i \varepsilon^i, \tag{6.53}$$

and $U_i, V_i \in \mathrm{C}^{(1)}(\Omega)$ for $i = 1,2,3$ we get

$$\begin{aligned} \nabla^* \cdot (u \otimes v) &= \nabla^* \cdot \left(\sum_{i,k=1}^{3} U_i V_k \varepsilon^i \otimes \varepsilon^k \right) \qquad (6.54) \\ &= \sum_{k=1}^{3} \left(\nabla^* \cdot \left(\sum_{i=1}^{3} U_i V_k \varepsilon^i \right) \varepsilon^k \right) \\ &= \sum_{k=1}^{3} \left(\nabla^* \cdot \left(\sum_{i=1}^{3} U_i \varepsilon^i \right)\right) V_k \varepsilon^k + \sum_{k=1}^{3} (u \cdot \nabla^* V_k)\varepsilon^k \\ &= (\nabla^* \cdot u)v + u \cdot (\nabla^* \otimes v). \end{aligned}$$

Thus, we end up with the *product rule*

$$\nabla^* \cdot (u \otimes v) = (\nabla^* \cdot u)v + u \cdot (\nabla^* \otimes v). \tag{6.55}$$

Analogously,

$$\mathrm{L}^* \cdot (u \otimes v) = (\mathrm{L}^* \cdot u)v + u \cdot (\mathrm{L}^* \otimes v). \tag{6.56}$$

Furthermore, for $F \in \mathrm{C}^{(1)}(\Omega)$ and $\mathbf{f} \in \mathbf{c}^{(1)}_{\tan,*}(\Omega)$, the result is

$$\nabla^*_\xi \cdot (F(\xi)\mathbf{f}(\xi)) = (\nabla^*_\xi F(\xi)) \cdot \mathbf{f}(\xi) + F(\xi)\nabla^*_\xi \cdot \mathbf{f}(\xi),\ \xi \in \Omega, \tag{6.57}$$

and a similar formula is valid for L^*.

The surface theorem of Gauß yields the following result:

Lemma 6.1. *Let* $f \in \mathrm{c}^{(1)}(\Omega)$ *and* $\mathbf{g} \in \mathbf{c}^{(1)}_{\tan,*}(\Omega)$*. Then*

$$\begin{aligned} \int_\Omega (\nabla^*_\xi \otimes f(\xi)) \cdot \mathbf{g}(\xi)\, d\omega(\xi) &= -\int_\Omega f(\xi) \cdot (\nabla^*_\xi \cdot \mathbf{g}(\xi))\, d\omega(\xi), \\ \int_\Omega (\mathrm{L}^*_\xi \otimes f(\xi)) \cdot \mathbf{g}(\xi)\, d\omega(\xi) &= -\int_\Omega f(\xi) \cdot (\mathrm{L}^*_\xi \cdot \mathbf{g}(\xi))\, d\omega(\xi). \end{aligned}$$

Proof. We only verify the first formula. Let

$$f = \sum_{i=1}^{3} F_i \varepsilon^i, \quad \mathbf{g} = \sum_{i,k=1}^{3} G_{ik} \varepsilon^i \otimes \varepsilon^k. \tag{6.58}$$

Then

$$\begin{aligned} (\nabla^* \otimes f) \cdot \mathbf{g} &= \left(\sum_{i=1}^{3} \nabla^* F_i \otimes \varepsilon^i \right) \cdot \left(\sum_{i,k=1}^{3} G_{ki} \varepsilon^k \otimes \varepsilon^i \right) \qquad (6.59) \\ &= \sum_{i=1}^{3} \left(\nabla^* F_i \right) \cdot \left(\sum_{k=1}^{3} G_{ki} \varepsilon^k \right) \end{aligned}$$

and

$$\begin{aligned} f \cdot (\nabla^* \cdot \mathbf{g}) &= \left(\sum_{i=1}^{3} F_i \varepsilon^i \right) \cdot \left(\sum_{i,k=1}^{3} \left(\nabla^* \cdot \left(\sum_{k=1}^{3} G_{ki} \varepsilon^k \right) \right) \varepsilon^i \right) \qquad (6.60) \\ &= \sum_{i=1}^{3} \left(F_i \nabla^* \right) \cdot \left(\sum_{k=1}^{3} G_{ki} \varepsilon^k \right). \end{aligned}$$

Thus, the assertion follows from the surface theorem of Gauß. □

Again, we write down the effect of the operators $\nabla^* \cdot$ and $\mathrm{L}^* \cdot$ on tensors expressed locally with the help of the triad $\varepsilon^r, \varepsilon^\varphi, \varepsilon^t$. In particular, we get for $\varphi \in (0, 2\pi)$ and $t \in (-1, 1)$

$$\nabla^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^r(\varphi,t)) = \varepsilon^\varphi(\varphi,t), \tag{6.61}$$

$$\nabla^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t)) = -\varepsilon^r(\varphi,t) + \frac{t}{\sqrt{1-t^2}} \varepsilon^t(\varphi,t), \tag{6.62}$$

$$\nabla^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^t(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}} \varepsilon^\varphi(\varphi,t), \tag{6.63}$$

$$\nabla^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^r(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}} \varepsilon^r(\varphi,t) + \varepsilon^t(\varphi,t), \tag{6.64}$$

$$\nabla^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}} \varepsilon^\varphi(\varphi,t), \tag{6.65}$$

$$\nabla^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^t(\varphi,t)) = -\varepsilon^r(\varphi,t) - \frac{t}{\sqrt{1-t^2}} \varepsilon^t, \tag{6.66}$$

and

$$\mathrm{L}^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^r(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}}\varepsilon^r(\varphi,t) - \varepsilon^t(\varphi,t), \qquad (6.67)$$

$$\mathrm{L}^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}}\varepsilon^\varphi(\varphi,t), \qquad (6.68)$$

$$\mathrm{L}^* \cdot (\varepsilon^\varphi(\varphi,t) \otimes \varepsilon^t(\varphi,t)) = -\varepsilon^r(\varphi,t) + \frac{t}{\sqrt{1-t^2}}\varepsilon^t(\varphi,t), \qquad (6.69)$$

$$\mathrm{L}^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^r(\varphi,t)) = -\varepsilon^\varphi(\varphi,t), \qquad (6.70)$$

$$\mathrm{L}^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t)) = -\varepsilon^r(\varphi,t) + \frac{t}{\sqrt{1-t^2}}\varepsilon^t(\varphi,t), \qquad (6.71)$$

$$\mathrm{L}^* \cdot (\varepsilon^t(\varphi,t) \otimes \varepsilon^t(\varphi,t)) = -\frac{t}{\sqrt{1-t^2}}\varepsilon^\varphi(\varphi,t). \qquad (6.72)$$

Since the tensors $\mathbf{i}_{\tan}$ and $\mathbf{j}_{\tan}$ are locally given by

$$\mathbf{i}_{\tan}(\varphi,t) = \varepsilon^\varphi(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t) + \varepsilon^t(\varphi,t) \otimes \varepsilon^t(\varphi,t), \qquad (6.73)$$

$$\mathbf{j}_{\tan}(\varphi,t) = \varepsilon^t(\varphi,t) \otimes \varepsilon^\varphi(\varphi,t) - \varepsilon^\varphi(\varphi,t) \otimes \varepsilon^t(\varphi,t), \qquad (6.74)$$

we obtain for $\xi \in \Omega$

$$\nabla^*_\xi \cdot \mathbf{i}_{\tan}(\xi) = -2\xi, \qquad (6.75)$$

$$\mathrm{L}^*_\xi \cdot \mathbf{i}_{\tan}(\xi) = 0, \qquad (6.76)$$

$$\nabla^*_\xi \cdot \mathbf{j}_{\tan}(\xi) = 0, \qquad (6.77)$$

$$\mathrm{L}^*_\xi \cdot \mathbf{j}_{\tan}(\xi) = -2\xi. \qquad (6.78)$$

Using the product rule (6.57) and its counterpart for $\mathrm{L}^*\cdot$, the above listed formulas help us to evaluate $\nabla^*\cdot$ and $\mathrm{L}^*\cdot$ applied to tensor fields in local representation with respect to $\varepsilon^r, \varepsilon^\varphi$, and ε^t.

It is clear from our consideration that $\xi \otimes \xi F(\xi)$, $\xi \in \Omega$, are left normal/right normal, $\xi \otimes \nabla^*_\xi F(\xi)$, $\xi \otimes \mathrm{L}^*_\xi F(\xi)$, $\xi \in \Omega$, are left normal/right tangential, $\nabla^*_\xi F(\xi) \otimes \xi$, $\mathrm{L}^*_\xi F(\xi) \otimes \xi$ are left tangential/right normal. Moreover, we have

$$p_{*,\mathrm{nor}} \nabla^*_\xi \otimes \nabla^*_\xi F(\xi) = \nabla^*_\xi \otimes \nabla^*_\xi F(\xi)\xi\xi^T = -\nabla^*_\xi F(\xi) \otimes \xi, \qquad (6.79)$$

$$p_{*,\mathrm{nor}} \nabla^*_\xi \otimes \mathrm{L}^*_\xi F(\xi) = \nabla^*_\xi \otimes \nabla^*_\xi F(\xi)\xi\xi^T = -\mathrm{L}^*_\xi F(\xi) \otimes \xi, \qquad (6.80)$$

$$p_{*,\mathrm{nor}} \mathrm{L}^*_\xi \otimes \nabla^*_\xi F(\xi) = \mathrm{L}^*_\xi \otimes \nabla^*_\xi F(\xi)\xi\xi^T = -\mathrm{L}^*_\xi F(\xi) \otimes \xi, \qquad (6.81)$$

$$p_{*,\mathrm{nor}} \mathrm{L}^*_\xi \otimes \mathrm{L}^*_\xi F(\xi) = \mathrm{L}^*_\xi \otimes \mathrm{L}^*_\xi F(\xi)\xi\xi^T = -\nabla^*_\xi F(\xi) \otimes \xi, \qquad (6.82)$$

$\xi \in \Omega$. In consequence, the expressions (6.79), (6.82) are left tangential, but, in general, not right tangential. Nevertheless, certain combinations of the four operators under consideration allow a separation into normal and tangential fields.

It is not difficult to see that

$$\begin{aligned}\text{trace } \xi \otimes \xi F(\xi) &= F(\xi), &(6.83)\\ \text{trace } \nabla^*_\xi \otimes \nabla^*_\xi F(\xi) &= \Delta^*_\xi F(\xi), &(6.84)\\ \text{trace } \mathrm{L}^*_\xi \otimes \mathrm{L}^*_\xi F(\xi) &= \Delta^*_\xi F(\xi), &(6.85)\end{aligned}$$

$\xi \in \Omega$. In all other cases, the trace vanishes.

Finally, we are able to formulate for sufficiently smooth functions $F : \Omega \to \mathbb{R}$ and tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ the following Green integral formulas:

$$\begin{aligned}\int_\Omega (\xi \otimes \xi F(\xi)) \cdot \mathbf{f}(\xi)\ d\omega(\xi) &= \int_\Omega F(\xi)\xi^T \mathbf{f}(\xi)\xi\ d\omega(\xi),\\ \int_\Omega \left(\xi \otimes \nabla^*_\xi F(\xi)\right) \cdot \mathbf{f}(\xi) d\omega(\xi) &= -\int_\Omega F(\xi)\nabla^*_\xi \cdot \left(\xi^T \mathbf{f}(\xi) - \xi \cdot (\xi^T \mathbf{f}(\xi))\xi\right)\ d\omega(\xi),\\ \int_\Omega \left(\xi \otimes \mathrm{L}^*_\xi F(\xi)\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) &= -\int_\Omega F(\xi)\mathrm{L}^*_\xi \cdot \left(\xi^T \mathbf{f}(\xi) - \xi \cdot (\xi^T \mathbf{f}(\xi))\xi\right)\ d\omega(\xi),\\ \int_\Omega \left(\nabla^*_\xi F(\xi) \otimes \xi\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) &= -\int_\Omega F(\xi)\nabla^*_\xi \cdot (\mathbf{f}(\xi)\xi - (\xi \cdot \mathbf{f}(\xi)\xi)\xi)\ d\omega(\xi),\\ \int_\Omega \left(\nabla^*_\xi F(\xi) \otimes \xi\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) &= -\int_\Omega F(\xi)\mathrm{L}^*_\xi \cdot (\mathbf{f}(\xi)\xi - (\xi \cdot \mathbf{f}(\xi)\xi)\xi)\ d\omega(\xi),\end{aligned} \tag{6.86}$$

and

$$\begin{aligned}&\int_\Omega \left(\nabla^*_\xi \otimes \nabla^*_\xi\right) F(\xi) \cdot \mathbf{f}(\xi)\ d\omega(\xi) = \int_\Omega F(\xi)\nabla^*_\xi \cdot\\ &\qquad \left\{\nabla^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))) - [\xi \cdot (\nabla_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))))]\xi\right\}\ d\omega(\xi),\\ &\int_\Omega \left((\nabla^*_\xi \otimes \mathrm{L}^*_\xi)F(\xi)\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) = \int_\Omega F(\xi)\mathrm{L}^*_\xi \cdot\\ &\qquad \left\{\nabla^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))) - \left[\xi \cdot (\nabla^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))))\right] \xi\right\}\ d\omega(\xi),\\ &\int_\Omega \left((\mathrm{L}^*_\xi \otimes \nabla^*_\xi)F(\xi)\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) = \int_\Omega F(\xi)\nabla^*_\xi \cdot\\ &\qquad \left\{\mathrm{L}^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))) - \left[\xi \cdot (\mathrm{L}^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))))\right] \xi\right\}\ d\omega(\xi),\\ &\int_\Omega \left((\mathrm{L}^*_\xi \otimes \mathrm{L}^*_\xi)F(\xi)\right) \cdot \mathbf{f}(\xi)\ d\omega(\xi) = \int_\Omega F(\xi)\mathrm{L}^*_\xi \cdot\\ &\qquad \left\{\mathrm{L}^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))) - \left[\xi \cdot (\mathrm{L}^*_\xi \cdot (\mathbf{f}(\xi) - \xi \otimes (\xi^T \mathbf{f}(\xi))))\right] \xi\right\}\ d\omega(\xi).\end{aligned} \tag{6.87}$$

6.4 Definition of Tensor Spherical Harmonics

To simplify our notation, we introduce the operators $\mathbf{q}^{(i,k)} : \mathrm{C}^{(2)}(\Omega) \to \mathbf{c}(\Omega)$ as follows:

$$\begin{aligned}
\mathbf{q}_\xi^{(1,1)} F(\xi) &= \xi \otimes \xi F(\xi), && (6.88)\\
\mathbf{q}_\xi^{(1,2)} F(\xi) &= \xi \otimes \nabla_\xi^* F(\xi), && (6.89)\\
\mathbf{q}_\xi^{(1,3)} F(\xi) &= \xi \otimes \mathrm{L}_\xi^* F(\xi), && (6.90)\\
\mathbf{q}_\xi^{(2,1)} F(\xi) &= \nabla_\xi^* F(\xi) \otimes \xi, && (6.91)\\
\mathbf{q}_\xi^{(3,1)} F(\xi) &= \mathrm{L}_\xi^* F(\xi) \otimes \xi, && (6.92)\\
\mathbf{q}_\xi^{(2,2)} F(\xi) &= \nabla_\xi^* \otimes \nabla_\xi^* F(\xi), && (6.93)\\
\mathbf{q}_\xi^{(2,3)} F(\xi) &= \nabla_\xi^* \otimes \mathrm{L}^* F(\xi), && (6.94)\\
\mathbf{q}_\xi^{(3,2)} F(\xi) &= \mathrm{L}_\xi^* \otimes \nabla_\xi^* F(\xi), && (6.95)\\
\mathbf{q}_\xi^{(3,3)} F(\xi) &= \mathrm{L}_\xi^* \otimes \mathrm{L}_\xi^* F(\xi), && (6.96)
\end{aligned}$$

$\xi \in \Omega$. It is clear from the considerations above that $\mathbf{q}^{(1,1)} F$ is left normal/right normal, $\mathbf{q}^{(1,k)} F$ are left normal/right tangential ($k = 2, 3$), and $\mathbf{q}^{(i,1)} F$ are left tangential/right normal ($i = 2, 3$). Furthermore, the tensor fields $\mathbf{q}^{(i,k)} F, i, k = 2, 3$ are left tangential, but, in general, not right tangential. In particular, (6.47) and (6.48) show that

$$\begin{aligned}
\mathbf{p}_{*,\mathrm{nor}} \nabla_\xi^* \otimes \nabla_\xi^* F(\xi) &= -\nabla_\xi^* F(\xi) \otimes \xi, && (6.97)\\
\mathbf{p}_{*,\mathrm{nor}} \nabla_\xi^* \otimes \mathrm{L}_\xi^* F(\xi) &= -\mathrm{L}_\xi^* F(\xi) \otimes \xi, && (6.98)\\
\mathbf{p}_{*,\mathrm{nor}} \mathrm{L}_\xi^* \otimes \nabla_\xi^* F(\xi) &= -\mathrm{L}_\xi^* F(\xi) \otimes \xi, && (6.99)\\
\mathbf{p}_{*,\mathrm{nor}} \mathrm{L}_\xi^* \otimes \mathrm{L}_\xi^* F(\xi) &= \nabla_\xi^* F(\xi) \otimes \xi, && (6.100)
\end{aligned}$$

$\xi \in \Omega$. Thus, certain combinations of the $\mathbf{q}^{(i,k)}$-operators allow a separation into normal and tangential tensor fields.

The trace of $\mathbf{q}^{(i,k)} F$ is given by

$$\begin{aligned}
\mathrm{trace}\ \mathbf{q}_\xi^{(1,1)} F(\xi) &= F(\xi), && (6.101)\\
\mathrm{trace}\ \mathbf{q}_\xi^{(2,2)} F(\xi) &= \mathrm{trace}\ \mathbf{q}_\xi^{(3,3)} F(\xi) = \Delta_\xi^* F(\xi), && (6.102)
\end{aligned}$$

and

$$\mathrm{trace}\ \mathbf{q}_\xi^{(i,k)} F(\xi) = 0, \qquad (6.103)$$

if $(i,k) \in \{(1,2), (1,3), (2,1), (3,1), (2,3), (3,2)\}$.

Next, we are interested in the adjoint operators of $\mathbf{q}^{(i,k)}$ (denoted by $Q^{(i,k)}$). To be more explicit, we are looking for operators $Q^{(i,k)}$ satisfying

$$\int_\Omega q^{(i,k)} F(\xi) \cdot \mathbf{f}(\xi)\ d\omega(\xi) = \int_\Omega F(\xi) Q^{(i,k)} \mathbf{f}(\xi)\ d\omega(\xi), \tag{6.104}$$

i.e.,

$$(\mathbf{q}^{(i,k)} F, \mathbf{f})_{\mathbf{l}^2(\Omega)} = (F, Q^{(i,k)} \mathbf{f})_{\mathrm{L}^2(\Omega)} \tag{6.105}$$

for all $(i,k) \in \{(1,1), \ldots, (3,3)\}$ and all sufficiently smooth functions $F : \Omega \to \mathbb{R}$ and tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$. It can be deduced from Lemma 6.1 that the operators $Q^{(i,k)}$ may be written down in the following form:

$$Q_\xi^{(1,1)} \mathbf{f}(\xi) = \xi^T \mathbf{f}(\xi) \xi, \tag{6.106}$$
$$Q_\xi^{(1,2)} \mathbf{f}(\xi) = -\nabla_\xi^* \cdot p_{\tan}(\xi^T \mathbf{f}(\xi)), \tag{6.107}$$
$$Q_\xi^{(1,3)} \mathbf{f}(\xi) = -\mathrm{L}_\xi^* \cdot p_{\tan}(\xi^T \mathbf{f}(\xi)), \tag{6.108}$$
$$Q_\xi^{(2,1)} \mathbf{f}(\xi) = -\nabla_\xi^* \cdot p_{\tan}(\mathbf{f}(\xi) \xi), \tag{6.109}$$
$$Q_\xi^{(3,1)} \mathbf{f}(\xi) = -\mathrm{L}_\xi^* \cdot p_{\tan}(\mathbf{f}(\xi) \xi), \tag{6.110}$$
$$Q_\xi^{(2,2)} \mathbf{f}(\xi) = \nabla_\xi^* \cdot p_{\tan}(\nabla_\xi^* \cdot \mathbf{p}_{\tan,*} \mathbf{f}(\xi)), \tag{6.111}$$
$$Q_\xi^{(2,3)} \mathbf{f}(\xi) = \mathrm{L}_\xi^* \cdot p_{\tan}(\nabla_\xi^* \cdot \mathbf{p}_{\tan,*} \mathbf{f}(\xi)), \tag{6.112}$$
$$Q_\xi^{(3,2)} \mathbf{f}(\xi) = \nabla_\xi^* \cdot p_{\tan}(\mathrm{L}_\xi^* \cdot \mathbf{p}_{\tan,*} \mathbf{f}(\xi)), \tag{6.113}$$
$$Q_\xi^{(3,3)} \mathbf{f}(\xi) = \mathrm{L}_\xi^* \cdot p_{\tan}(\mathrm{L}_\xi^* \cdot \mathbf{p}_{\tan,*} \mathbf{f}(\xi)), \tag{6.114}$$

provided that $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$. If $F : \Omega \to \mathbb{R}$ is sufficiently smooth, then

$$\begin{array}{rcll}
Q^{(1,1)} \mathbf{q}^{(1,1)} F &=& F, & \\
Q^{(i,k)} \mathbf{q}^{(i,k)} F &=& -\Delta^* F & \text{if } (i,k) \in \{(1,2),(1,3),(2,1),(3,1)\}, \\
Q^{(i,k)} \mathbf{q}^{(i,k)} F &=& \Delta^* \Delta^* F & \text{if } (i,k) \in \{(2,2),(2,3),(3,2),(3,3)\}, \\
Q^{(1,1)} \mathbf{q}^{(i,k)} F &=& 0 & \text{if } (i,k) \neq (1,1), \\
Q^{(1,2)} \mathbf{q}^{(i,k)} F &=& 0 & \text{if } (i,k) \neq (1,2), \\
Q^{(1,3)} \mathbf{q}^{(i,k)} F &=& 0 & \text{if } (i,k) \neq (1,3), \\
Q^{(2,1)} \mathbf{q}^{(i,k)} F &=& 0 & \text{if } (i,k) \notin \{(2,1),(2,2),(3,3)\}, \\
Q^{(2,1)} \mathbf{q}^{(2,2)} F &=& \Delta^* F, & \\
Q^{(2,1)} \mathbf{q}^{(3,3)} F &=& -\Delta^* F, &
\end{array}$$

$$\begin{array}{rcll}
Q^{(3,1)}\mathbf{q}^{(i,k)}F &=& 0 & \text{if } (i,k) \notin \{(3,1),(2,3),(3,2)\}, \\
Q^{(3,1)}\mathbf{q}^{(2,3)}F &=& \Delta^* F, & \\
Q^{(3,1)}\mathbf{q}^{(3,2)}F &=& \Delta^* F, & \\
Q^{(2,2)}\mathbf{q}^{(i,k)}F &=& 0 & \text{if } (i,k) \notin \{(2,1),(2,2)\}, \\
Q^{(2,2)}\mathbf{q}^{(2,1)}F &=& \Delta^* F, & \\
Q^{(2,3)}\mathbf{q}^{(i,k)}F &=& 0 & \text{if } (i,k) \notin \{(3,1),(2,3)\}, \\
Q^{(2,3)}\mathbf{q}^{(3,1)}F &=& \Delta^* F, & \\
Q^{(3,2)}\mathbf{q}^{(i,k)}F &=& 0 & \text{if } (i,k) \notin \{(3,1),(3,2)\}, \\
Q^{(3,2)}\mathbf{q}^{(3,1)}F &=& \Delta^* F, & \\
Q^{(3,3)}\mathbf{q}^{(i,k)}F &=& 0 & \text{if } (i,k) \notin \{(2,1),(3,3)\}, \\
Q^{(3,3)}\mathbf{q}^{(2,1)}F &=& -\Delta^* F. &
\end{array}$$

It is remarkable to mention that, contrary to the vectorial case, 'constant' tangential tensors can be detected, i.e., tensor fields $\mathbf{f} \in \mathbf{c}^{(1)}_{\mathrm{tan,tan}}(\Omega)$ satisfying $p_{\mathrm{tan}}\nabla^* \cdot \mathbf{f} = 0$ and $p_{\mathrm{tan}}\mathrm{L}^* \cdot \mathbf{f} = 0$. Indeed, both tensor fields can be characterized as linear combinations of $\mathbf{i}_{\mathrm{tan}}$ and $\mathbf{j}_{\mathrm{tan}}$.

Important relations for $F \in \mathrm{C}^{(2)}(\Omega)$ are as follows:

$$(\nabla^* \otimes \nabla^* + \mathrm{L}^* \otimes \mathrm{L}^*)F = \mathbf{i}_{\mathrm{tan}}\Delta^* F, \tag{6.115}$$

$$(\mathrm{L}^* \otimes \nabla^* - \nabla^* \otimes \mathrm{L}^*)F = \mathbf{j}_{\mathrm{tan}}\Delta^* F. \tag{6.116}$$

In view of the above equations and definitions, we finally introduce operators $\mathbf{o}^{(i,k)} : \mathrm{C}^{(2)}(\Omega) \to \mathbf{c}(\Omega)$ by

$$\mathbf{o}^{(1,1)}_\xi F(\xi) = \xi \otimes \xi F(\xi), \tag{6.117}$$

$$\mathbf{o}^{(1,2)}_\xi F(\xi) = \xi \otimes \nabla^*_\xi F(\xi), \tag{6.118}$$

$$\mathbf{o}^{(1,3)}_\xi F(\xi) = \xi \otimes \mathrm{L}^*_\xi F(\xi), \tag{6.119}$$

$$\mathbf{o}^{(2,1)}_\xi F(\xi) = \nabla^*_\xi F(\xi) \otimes \xi, \tag{6.120}$$

$$\mathbf{o}^{(3,1)}_\xi F(\xi) = \mathrm{L}^*_\xi F(\xi) \otimes \xi, \tag{6.121}$$

$$\mathbf{o}^{(2,2)}_\xi F(\xi) = \mathbf{i}_{\mathrm{tan}}(\xi) F(\xi), \tag{6.122}$$

$$\mathbf{o}^{(2,3)}_\xi F(\xi) = \left(\nabla^*_\xi \otimes \nabla^*_\xi - \mathrm{L}^*_\xi \otimes \mathrm{L}^*_\xi\right) F(\xi) + 2\nabla^*_\xi F(\xi) \otimes \xi, \tag{6.123}$$

$$\mathbf{o}^{(3,2)}_\xi F(\xi) = \left(\nabla^*_\xi \otimes \mathrm{L}^*_\xi + \mathrm{L}^*_\xi \otimes \nabla^*_\xi\right) F(\xi) + 2\mathrm{L}^*_\xi F(\xi) \otimes \xi, \tag{6.124}$$

$$\mathbf{o}^{(3,3)}_\xi F(\xi) = \mathbf{j}_{\mathrm{tan}}(\xi) F(\xi), \tag{6.125}$$

$\xi \in \Omega$.

After these considerations, it is not difficult to prove the following lemma.

Lemma 6.2. *Let $F : \Omega \to \mathbb{R}$ be sufficiently smooth. Then, the following statements are valid:*

(i) $\mathbf{o}^{(1,1)}F$ *is a normal tensor field.*

(ii) $\mathbf{o}^{(1,2)}F$ *and* $\mathbf{o}^{(1,3)}F$ *are left normal/right tangential.*

(iii) $\mathbf{o}^{(2,1)}F$ *and* $\mathbf{o}^{(3,1)}F$ *are left tangential/right normal.*

(iv) $\mathbf{o}^{(2,2)}F$, $\mathbf{o}^{(2,3)}F$, $\mathbf{o}^{(3,2)}F$ *and* $\mathbf{o}^{(3,3)}F$ *are tangential.*

(v) $\mathbf{o}^{(1,1)}F$, $\mathbf{o}^{(2,2)}F$, $\mathbf{o}^{(2,3)}F$ *and* $\mathbf{o}^{(3,2)}F$ *are symmetric.*

(vi) $\mathbf{o}^{(3,3)}F$ *is skew-symmetric.*

(vii) $\left(\mathbf{o}^{(1,2)}F\right)^T = \mathbf{o}^{(2,1)}F$ *and* $\left(\mathbf{o}^{(1,3)}F\right)^T = \mathbf{o}^{(3,1)}F$.

(viii) *For* $\xi \in \Omega$

$$\mathrm{trace}\ \mathbf{o}_\xi^{(i,k)}F(\xi) = \begin{cases} F(\xi) & \text{for} \quad (i,k) = (1,1) \\ 2F(\xi) & \text{for} \quad (i,k) = (2,2) \\ 0 & \text{for} \quad (i,k) \neq (1,1),\ (2,2). \end{cases}$$

The tangent representation theorem (cf. G.E. Backus (1966), G.E. Backus (1967)) asserts that if $\mathbf{p}_{\mathrm{tan,tan}}\mathbf{f}$ is the tangential part of a tensor field $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$, as defined above, then there exist unique scalar fields $F_{2,2}, F_{3,3}, F_{2,3}$, $F_{3,2}$ such that

$$\int_\Omega F_{2,2}(\xi)\, d\omega(\xi) = \int_\Omega F_{3,3}(\xi)\, d\omega(\xi) = 0, \tag{6.126}$$

$$\int_\Omega F_{3,2}(\xi)(\varepsilon^i \cdot \xi)\, d\omega(\xi) = \int_\Omega F_{2,3}(\xi)(\varepsilon^i \cdot \xi)\, d\omega(\xi) = 0,\ i = 1,2,3, \tag{6.127}$$

and

$$\mathbf{p}_{\mathrm{tan,tan}}\mathbf{f} = \mathbf{o}^{(2,2)}F_{2,2} + \mathbf{o}^{(2,3)}F_{2,3} + \mathbf{o}^{(3,2)}F_{3,2} + \mathbf{o}^{(3,3)}F_{3,3}. \tag{6.128}$$

Furthermore, the following orthogonality relations may be formulated.

Lemma 6.3. *Let $F, G : \Omega \to \mathbb{R}$ be sufficiently smooth. Then, the following properties hold true:*

(i) *For all* $\xi \in \Omega$, $\mathbf{o}_\xi^{(i,k)}F(\xi) \cdot \mathbf{o}_\xi^{(i',k')}F(\xi) = 0$, *whenever* $(i,k) \neq (i',k')$.

(ii) *If* $Y_n \in \mathrm{Harm}_n, Y_m \in \mathrm{Harm}_m$, $n \neq m$, *i.e.,* $(Y_n, Y_m)_{\mathrm{L}^2(\Omega)} = 0$, *then we have*

$$(\mathbf{o}^{(i,k)}Y_n, \mathbf{o}^{(i',k')}Y_m)_{\mathbf{l}^2(\Omega)} = 0 \tag{6.129}$$

for all $(i,k), (i',k') \in \{(1,1), (1,2), \ldots, (3,3)\}$.

The adjoint operators $O^{(i,k)}$ satisfying

$$\int_\Omega \mathbf{o}^{(i,k)}F(\xi)\cdot\mathbf{f}(\xi)\;d\omega(\xi) = \int_\Omega F(\xi)\;O^{(i,k)}\mathbf{f}(\xi)\;d\omega(\xi), \tag{6.130}$$

i.e., in shorthand notation

$$(\mathbf{o}^{(i,k)}F,\mathbf{f})_{\mathbf{l}^2(\Omega)} = (F, O^{(i,k)}\mathbf{f})_{\mathrm{L}^2(\Omega)} \tag{6.131}$$

for all sufficiently smooth functions $F:\Omega\to\mathbb{R}$ and tensor fields $\mathbf{f}:\Omega\to\mathbb{R}^3\otimes\mathbb{R}^3$ can be deduced from the definitions (6.106)–(6.114). In detail, for $\mathbf{f}\in\mathbf{c}^{(2)}(\Omega)$ we find

$$O_\xi^{(1,1)}\mathbf{f}(\xi) = \xi^T\mathbf{f}(\xi)\xi, \tag{6.132}$$

$$O_\xi^{(1,2)}\mathbf{f}(\xi) = -\nabla_\xi^*\cdot p_{\tan}\left(\xi^T\mathbf{f}(\xi)\right), \tag{6.133}$$

$$O_\xi^{(1,3)}\mathbf{f}(\xi) = -\mathrm{L}_\xi^*\cdot p_{\tan}\left(\xi^T\mathbf{f}(\xi)\right), \tag{6.134}$$

$$O_\xi^{(2,1)}\mathbf{f}(\xi) = -\nabla_\xi^*\cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \tag{6.135}$$

$$O_\xi^{(3,1)}\mathbf{f}(\xi) = -\mathrm{L}_\xi^*\cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \tag{6.136}$$

$$O_\xi^{(2,2)}\mathbf{f}(\xi) = \mathbf{i}_{\tan}(\xi)\cdot\mathbf{f}(\xi), \tag{6.137}$$

$$\begin{aligned} O_\xi^{(2,3)}\mathbf{f}(\xi) = {} & \nabla_\xi^*\cdot p_{\tan}\left(\nabla_\xi^*\cdot\mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) - \mathrm{L}_\xi^*\cdot p_{\tan}\left(\mathrm{L}_\xi^*\cdot\mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) \\ & -2\nabla_\xi^*\cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \end{aligned} \tag{6.138}$$

$$\begin{aligned} O_\xi^{(3,2)}\mathbf{f}(\xi) = {} & \mathrm{L}_\xi^*\cdot p_{\tan}\left(\nabla_\xi^*\cdot\mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) + \nabla_\xi^*\cdot p_{\tan}\left(\mathrm{L}_\xi^*\cdot\mathbf{p}_{\tan,*}\mathbf{f}(\xi)\right) \\ & -2\mathrm{L}_\xi^*\cdot p_{\tan}\left(\mathbf{f}(\xi)\xi\right), \end{aligned} \tag{6.139}$$

$$O_\xi^{(3,3)}\mathbf{f}(\xi) = \mathbf{j}_{\tan}(\xi)\cdot\mathbf{f}(\xi), \tag{6.140}$$

$\xi\in\Omega$. Provided that $F:\Omega\to\mathbb{R}$ is sufficiently smooth we obtain

$$O_\xi^{(i',k')}\mathbf{o}_\xi^{(i,k)}F(\xi) = 0 \;\text{ if } (i,k)\neq(i',k'), \tag{6.141}$$

whereas

$$O_\xi^{(i,k)}\mathbf{o}_\xi^{(i,k)}F(\xi) = \begin{cases} F(\xi) & \text{if } (i,k)=(1,1) \\ -\Delta^*F(\xi) & \text{if } (i,k)\in\{(1,2),(1,3) \\ & \qquad\qquad (2,1),(3,1)\} \\ 2F(\xi) & \text{if } (i,k)\in\{(2,2),(3,3)\} \\ 2\Delta^*(\Delta^*+2)F(\xi) & \text{if } (i,k)\in\{(2,3),(3,2)\} \end{cases}. \tag{6.142}$$

It can be easily deduced that for $Y_0\in\mathrm{Harm}_0$

$$\mathbf{o}^{(i,k)}Y_0 = 0 \;\text{ if } (i,k)\in\{(1,2),(1,3),(2,1),(3,1)\}, \tag{6.143}$$

while for $Y\in\mathrm{Harm}_0\cup\mathrm{Harm}_1$ we find

$$\mathbf{o}^{(i,k)}Y = 0 \;\text{ if } (i,k)\in\{(2,3),(3,2)\}. \tag{6.144}$$

Thus, for notational convenience, we let

$$0_{i,k} = \begin{cases} 0 & \text{for} \quad (i,k) \in \{(1,1),(2,2),(3,3)\} \\ 1 & \text{for} \quad (i,k) \in \{(1,2),(1,3),(2,1),(3,1)\} \\ 2 & \text{for} \quad (i,k) \in \{(2,3),(3,2)\} \end{cases} . \tag{6.145}$$

These preparations of Section 6.4 motivate us to introduce tensor spherical harmonics as follows.

Definition 6.4. Let $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$. We call $\mathbf{o}^{(i,k)}Y_n$, $n \geq 0_{i,k}$, with $Y_n \in \mathrm{Harm}_n$, a *tensor spherical harmonic of degree* n *and type* (i,k). The space of all tensor spherical harmonics of degree n and type (i,k) is denoted by $\mathbf{harm}_n^{(i,k)}$. Furthermore, we set

$$\mathbf{harm}_0 = \mathbf{harm}_0^{(1,1)} \oplus \mathbf{harm}_0^{(2,2)} \oplus \mathbf{harm}_0^{(3,3)}, \tag{6.146}$$

$$\mathbf{harm}_1 = \bigoplus_{\substack{i,k=1 \\ (i,k)\neq(2,3),(3,2)}}^{3} \mathbf{harm}_1^{(i,k)}, \tag{6.147}$$

$$\mathbf{harm}_n = \bigoplus_{i,k=1}^{3} \mathbf{harm}_n^{(i,k)}, \qquad n \geq 2. \tag{6.148}$$

An $\mathbf{l}^2(\Omega)$-orthonormal system of tensor spherical harmonics for $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$, $n \geq 0_{i,k}$, $j = 1,\ldots,2n+1$, is introduced by letting

$$\mathbf{y}_{n,j}^{(i,k)} = (\mu_n^{(i,k)})^{-1/2}\mathbf{o}^{(i,k)}Y_{n,j}, \tag{6.149}$$

where $\mu_n^{(i,k)}$ is defined by

$$\mu_n^{(i,k)} = \left\| O^{(i,k)}\mathbf{o}^{(i,k)}Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)},$$

that is

$$\mu_n^{(i,k)} = \begin{cases} 1, & (i,k) = (1,1) \\ 2, & (i,k) \in \{(2,2),(3,3)\} \\ -\Delta^{*\wedge}(n), & (i,k) \in \\ & \{(1,2),(1,3),(2,1),(3,1)\} \\ 2\Delta^{*\wedge}(n)\left(\Delta^{*\wedge}(n)+2\right), & (i,k) \in \{(2,3),(3,2)\} \end{cases} . \tag{6.150}$$

It is not difficult to see that $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}$ forms an orthonormal system in $\mathbf{l}^2(\Omega)$, i.e.,

$$\int_\Omega \mathbf{y}_{n,j}^{(i,k)}(\xi) \cdot \mathbf{y}_{m,l}^{(p,q)}(\xi)\, d\omega(\xi) = \delta_{nm}\delta_{jl}\delta_{ip}\delta_{kq}. \tag{6.151}$$

Example 6.5. Let us consider the ($L^2(\Omega)$–orthonormal) scalar spherical harmonics of degree $0, 1$,

$$Y_{0,1}(\xi) = \frac{1}{\sqrt{4\pi}}, \quad \xi \in \Omega, \tag{6.152}$$

$$Y_{1,j}(\xi) = \sqrt{\frac{3}{4\pi}}(\xi \cdot \varepsilon^j), \quad \xi \in \Omega, \quad j = 1, \ldots, 3. \tag{6.153}$$

It is not difficult to see that

$$\mathbf{y}_{0,1}^{(1,1)}(\xi) = \frac{1}{\sqrt{4\pi}}\xi \otimes \xi, \tag{6.154}$$

$$\mathbf{y}_{0,1}^{(2,2)}(\xi) = \frac{1}{\sqrt{4\pi}}(\mathbf{i} - \xi \otimes \xi), \tag{6.155}$$

$$\mathbf{y}_{0,1}^{(3,3)}(\xi) = \frac{1}{\sqrt{4\pi}}(\xi \wedge \mathbf{i}) = \frac{1}{\sqrt{4\pi}}\sum_{i=1}^{3}(\xi \wedge \varepsilon^i) \otimes \varepsilon^i, \tag{6.156}$$

$$\mathbf{y}_{1,j}^{(1,1)}(\xi) = \sqrt{\frac{3}{4\pi}}(\xi \cdot \varepsilon^j)\xi \otimes \varepsilon^j, \quad j = 1, 2, 3, \tag{6.157}$$

$$\mathbf{y}_{1,j}^{(1,2)}(\xi) = \sqrt{\frac{3}{8\pi}}\xi \otimes (\varepsilon^j - (\xi \cdot \varepsilon^j)\xi), \quad j = 1, 2, 3, \tag{6.158}$$

$$\mathbf{y}_{1,j}^{(1,3)}(\xi) = \sqrt{\frac{3}{8\pi}}\xi \otimes (\xi \wedge \varepsilon^j), \quad j = 1, 2, 3, \tag{6.159}$$

$$\mathbf{y}_{1,j}^{(2,1)}(\xi) = \sqrt{\frac{3}{8\pi}}(\varepsilon^j - (\xi \cdot \varepsilon^j)\xi) \otimes \xi, \quad j = 1, 2, 3, \tag{6.160}$$

$$\mathbf{y}_{1,j}^{(3,1)}(\xi) = \sqrt{\frac{3}{8\pi}}(\xi \wedge \varepsilon^j) \otimes \xi, \quad j = 1, 2, 3, \tag{6.161}$$

$$\mathbf{y}_{1,j}^{(3,3)}(\xi) = \sqrt{\frac{3}{8\pi}}(\xi \cdot \varepsilon^j)(\xi \wedge \mathbf{i}), \quad j = 1, 2, 3. \tag{6.162}$$

6.5 Helmholtz Decomposition Theorem

Our purpose now is to show that, if a tensor field $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ satisfies some smoothness assumptions, its decomposition into normal and tangential fields can be established by use of the Green functions with respect to the (scalar) Beltrami operator and its first iteration. Our results are based on the aforementioned tangent representation theorem for tensor fields and the already known Helmholtz decomposition theorem for spherical vector fields (as presented in Section 5.2).

Theorem 6.6. *(Helmholtz Decomposition Theorem) Let* $\mathbf{f}$ *be of class* $\mathbf{c}^{(2)}(\Omega)$. *Then there exist uniquely defined functions* $F_{i,k} \in \mathrm{C}^{(2)}(\Omega)$, $(i,k) \in \{(1,1),\ (1,2),\ldots,(3,3)\}$ *with* $(F_{i,k}, Y_0)_{\mathrm{L}^2(\Omega)} = 0$ *for all* $Y_0 \in \mathrm{Harm}_0$ *if* $(i,k) \in \{(1,2),\ (1,3),(2,1),(2,3),(3,1),(3,2)\}$ *and* $(F_{i,k}, Y_1)_{\mathrm{L}^2(\Omega)} = 0$ *for all* $Y_1 \in \mathrm{Harm}_1$ *if* $(i,k) \in \{(2,3),(3,2)\}$, *in such a way that*

$$\mathbf{f} = \sum_{i,k=1}^{3} \mathbf{o}^{(i,k)} F_{i,k}, \tag{6.163}$$

where the functions $\xi \mapsto F_{i,k}(\xi)$, $\xi \in \Omega$, *are explicitly given by*

$$F_{1,1}(\xi) = O_\xi^{(1,1)} \mathbf{f}(\xi), \tag{6.164}$$

$$F_{2,2}(\xi) = \frac{1}{2} O_\xi^{(2,2)} \mathbf{f}(\xi), \tag{6.165}$$

$$F_{3,3}(\xi) = \frac{1}{2} O_\xi^{(3,3)} \mathbf{f}(\xi), \tag{6.166}$$

$$F_{1,2}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(1,2)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{6.167}$$

$$F_{1,3}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(1,3)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{6.168}$$

$$F_{2,1}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(2,1)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{6.169}$$

$$F_{3,1}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(3,1)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{6.170}$$

$$F_{2,3}(\xi) = \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(2,3)} \mathbf{f}(\eta)\, d\omega(\eta), \tag{6.171}$$

$$F_{3,2}(\xi) = \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(3,2)} \mathbf{f}(\eta)\, d\omega(\eta). \tag{6.172}$$

Remark 6.7. In the notation of Chapter 4, we have $\partial_0 = \Delta^*$ and $\partial_0\partial_1 = \Delta^*(\Delta^*+2)$.

Proof. Using the projection operators as defined in Section 6.2, we have

$$\mathbf{f} = \mathbf{p}_{\mathrm{nor,nor}}\mathbf{f} + \mathbf{p}_{\mathrm{nor,tan}}\mathbf{f} + \mathbf{p}_{\mathrm{tan,nor}}\mathbf{f} + \mathbf{p}_{\mathrm{tan,tan}}\mathbf{f}. \tag{6.173}$$

Each term can be investigated separately. It is clear that $\mathbf{p}_{\mathrm{nor,nor}}\mathbf{f} = \mathbf{o}^{(1,1)}O^{(1,1)}\mathbf{f}$. The left normal/right tangential field $\mathbf{p}_{\mathrm{nor,tan}}\mathbf{f}$ allows a representation of the form $\mathbf{p}_{\mathrm{nor,tan}}\mathbf{f}(\xi) = \xi \otimes u(\xi)$, $\xi \in \Omega$, with $u \in c^{(2)}_{\mathrm{tan}}$. If we write u in the form $u(\xi) = \nabla^* F_{1,2}(\xi) + \mathrm{L}^* F_{1,3}(\xi)$ (as proposed in

Chapter 5.3), we find

$$\Delta_\xi^* F_{1,2}(\xi) = -O_\xi^{(1,2)}\mathbf{f}(\xi), \quad \xi \in \Omega, \tag{6.174}$$

$$\Delta_\xi^* F_{1,3}(\xi) = -O_\xi^{(1,3)}\mathbf{f}(\xi), \quad \xi \in \Omega. \tag{6.175}$$

The solution is uniquely determined by Theorem 4.27 since it can be deduced from (6.143) that $(O^{(1,2)}\mathbf{f}, Y_0)_{\mathcal{L}^2(\Omega)} = 0$ and $(O^{(1,3)}\mathbf{f}, Y_0)_{\mathcal{L}^2(\Omega)} = 0$ for all $Y_0 \in \mathrm{Harm}_0$. We have, for $\xi \in \Omega$,

$$F_{1,2}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(1,2)}\mathbf{f}(\eta)\, d\omega(\eta), \tag{6.176}$$

$$F_{1,3}(\xi) = -\int_\Omega G(\Delta^*; \xi\cdot\eta) O_\eta^{(1,3)}\mathbf{f}(\eta)\, d\omega(\eta), \tag{6.177}$$

as required. The left tangential/right normal part can be considered in a similar way.

The tensor field

$$\mathbf{p}_{\tan,\tan}\mathbf{f} - \frac{1}{2}\mathbf{o}^{(2,2)}O^{(2,2)}\mathbf{f} - \frac{1}{2}\mathbf{o}^{(3,3)}O^{(3,3)}\mathbf{f} \tag{6.178}$$

is symmetric and traceless. Based on the tangent representation theorem due to G.E. Backus (1966); G.E. Backus (1967), we obtain from the formulation

$$\mathbf{p}_{\tan,\tan}\mathbf{f} - \frac{1}{2}\mathbf{o}^{(2,2)}O^{(2,2)}\mathbf{f} - \frac{1}{2}\mathbf{o}^{(3,3)}O^{(3,3)}\mathbf{f} = \mathbf{o}^{(2,3)}F_{2,3} + \mathbf{o}^{(3,2)}F_{3,2} \tag{6.179}$$

the differential equations

$$2\Delta_\xi^*(\Delta_\xi^* + 2)F_{2,3}(\xi) = O_\xi^{(2,3)}\mathbf{f}(\xi), \tag{6.180}$$

$$2\Delta_\xi^*(\Delta_\xi^* + 2)F_{3,2}(\xi) = O_\xi^{(3,2)}\mathbf{f}(\xi) \tag{6.181}$$

for $\xi \in \Omega$ (remember that $-(\Delta^*)^\wedge(1) = 2$). By virtue of (6.141), (6.142) and Theorem 4.27, we see that the unique solutions are given by

$$F_{2,3}(\xi) = \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(2,3)}\mathbf{f}(\eta)\, d\omega(\eta), \quad \xi \in \Omega, \tag{6.182}$$

$$F_{3,2}(\xi) = \frac{1}{2}\int_\Omega G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(3,2)}\mathbf{f}(\eta)\, d\omega(\eta), \quad \xi \in \Omega. \tag{6.183}$$

Consequently, the existence is assured.

The uniqueness follows from the integral theorems for the (iterated) Beltrami differential equations. □

As an example of the above developed decomposition procedure for tensor fields, we describe how the *Hesse matrix* of a function defined in $\mathbb{R}^3 \setminus \{0\}$ can be decomposed, when restricted to the unit sphere. It turns out that this result is of particular importance in satellite gradiometry, i.e., the determination of the Earth's gravitational field from measurements of second order derivatives of the potential at satellite height. In particular, this decomposition is of advantage for the classification of those types of gradiometry measurements, which ensure existence and uniqueness in a mathematical formulation of the gradiometry problem (cf. M. Schreiner (1996)).

Suppose that the function $H : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is twice continuously differentiable. We want to show how the Hesse matrix restricted to the unit sphere Ω, i.e.,

$$\mathbf{h}(\xi) = \nabla_x \otimes \nabla_x H(x)|_{|x|=1}, \quad \xi \in \Omega, \tag{6.184}$$

can be decomposed according to the rules of Theorem 6.6. In order to evaluate

$$\nabla_x \otimes \nabla_x H(x) = \left(\xi \frac{\partial}{\partial r} + \frac{1}{r} \nabla^*_\xi \right) \otimes \left(\xi \frac{\partial}{\partial r} + \frac{1}{r} \nabla^*_\xi \right) H(r\xi), \tag{6.185}$$

we first see that

$$\xi \frac{\partial}{\partial r} \otimes \xi \frac{\partial}{\partial r} H(r\xi) = \xi \otimes \xi \left(\frac{\partial}{\partial r} \right)^2 H(r\xi), \tag{6.186}$$

$$\xi \frac{\partial}{\partial r} \otimes \frac{1}{r} \nabla^*_\xi H(r\xi) = -\frac{1}{r^2} \xi \otimes \nabla^*_\xi H(r\xi) + \frac{1}{r} \xi \otimes \nabla^*_\xi \frac{\partial}{\partial r} H(r\xi), \tag{6.187}$$

$$\frac{1}{r} \nabla^*_\xi \otimes \xi \frac{\partial}{\partial r} H(r\xi) = \frac{1}{r} \mathbf{i}_{\mathrm{tan}}(\xi) \frac{\partial}{\partial r} H(r\xi) + \frac{1}{r} \nabla^*_\xi \left(\frac{\partial}{\partial r} H(r\xi) \right) \otimes \xi, \tag{6.188}$$

$$\frac{1}{r} \nabla^*_\xi \otimes \frac{1}{r} \nabla^*_\xi H(r\xi) = \frac{1}{r^2} \nabla^*_\xi \otimes \nabla^*_\xi H(r\xi). \tag{6.189}$$

Summing up these terms, we obtain

$$\begin{aligned} \nabla_x \otimes \nabla_x H(x)|_{|x|=1} = {} & \xi \otimes \xi \left(\frac{\partial}{\partial r} \right)^2 H(r\xi)|_{r=1} \\ & + \xi \otimes \nabla^*_\xi \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi) \right) \\ & + \left(\nabla^* \frac{\partial}{\partial r} H(r\xi)|_{r=1} \right) \otimes \xi \\ & + \nabla^*_\xi \otimes \nabla^*_\xi H(\xi) \\ & + \mathbf{i}_{\mathrm{tan}}(\xi) \frac{\partial}{\partial r} H(r\xi)|_{r=1}. \end{aligned} \tag{6.190}$$

Using (6.115) and the definition of the $\mathbf{o}^{(i,k)}$-operators, we finally arrive at

$$\begin{aligned}\nabla_x \otimes \nabla_x H(x)|_{|x|=1} &= \mathbf{o}_\xi^{(1,1)} \left(\left(\frac{\partial}{\partial r} \right)^2 H(r\xi)|_{r=1} \right) \\ &+\mathbf{o}_\xi^{(1,2)} \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi) \right) \\ &+\mathbf{o}_\xi^{(2,1)} \left(\frac{\partial}{\partial r} H(r\xi)|_{r=1} - H(\xi) \right) \\ &+\mathbf{o}_\xi^{(2,2)} \left(\frac{1}{2} \Delta_\xi^* H(\xi) + \frac{\partial}{\partial r} H(r\xi)|_{r=1} \right) \\ &+\mathbf{o}_\xi^{(2,3)} \frac{1}{2} H(\xi).\end{aligned} \tag{6.191}$$

In particular, if we consider an outer (solid spherical) harmonic $H_{-n-1} : x \mapsto H_{-n-1}(x)$, $H_{-n-1}(r\xi) = r^{-(n+1)} Y_n(\xi)$, $r > 0$, $\xi \in \Omega$, we obtain the following decomposition of the Hesse matrix:

$$\begin{aligned}\nabla_x \otimes \nabla_x H_{-n-1}(x)|_{|x|=1} &= (n+1)(n+2)\mathbf{o}_\xi^{(1,1)} Y_n(\xi) \\ &- (n+2)\mathbf{o}_\xi^{(1,2)} Y_n(\xi) - (n+2)\mathbf{o}_\xi^{(2,1)} Y_n(\xi) \\ &- \frac{1}{2}(n+1)(n+2)\mathbf{o}_\xi^{(2,2)} Y_n(\xi) \\ &+ \frac{1}{2}\mathbf{o}_\xi^{(2,3)} Y_n(\xi).\end{aligned} \tag{6.192}$$

6.6 Orthogonal (Fourier) Expansions

Next, the closure and completeness of tensor spherical harmonics will be formulated intrinsically on the sphere via Bernstein summability (see W. Freeden, M. Gutting (2008)). Another proof of the closure and completeness using homogeneous harmonic tensor polynomials in three-dimensional Euclidean space $\mathbb{R}^3$ can be derived from arguments given in Section 6.7.

We begin our considerations by introducing 'Bernstein convolutions' to the nine Helmholtz functions. More explicitly, we let

$$F_{1,1}^{(n)}(\xi) = \int_\Omega B_n(\xi \cdot \alpha) O^{(1,1)} \mathbf{f}(\alpha) d\omega(\alpha), \tag{6.193}$$

$$F_{2,2}^{(n)}(\xi) = \frac{1}{2} \int_\Omega B_n(\xi \cdot \alpha) O^{(2,2)} \mathbf{f}(\alpha) d\omega(\alpha), \tag{6.194}$$

$$F^{(n)}_{3,3}(\xi) = \frac{1}{2}\int_\Omega B_n(\xi\cdot\alpha)O^{(3,3)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.195}$$

$$F^{(n)}_{1,2}(\xi) = -\int_\Omega BG_n(\xi\cdot\alpha)O^{(1,2)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.196}$$

$$F^{(n)}_{1,3}(\xi) = -\int_\Omega BG_n(\xi\cdot\alpha)O^{(1,3)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.197}$$

$$F^{(n)}_{2,1}(\xi) = -\int_\Omega BG_n(\xi\cdot\alpha)O^{(2,1)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.198}$$

$$F^{(n)}_{3,1}(\xi) = -\int_\Omega BG_n(\xi\cdot\alpha)O^{(3,1)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.199}$$

$$F^{(n)}_{2,3}(\xi) = \frac{1}{2}\int_\Omega BG^{(2)}_n(\xi\cdot\alpha)O^{(2,3)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.200}$$

$$F^{(n)}_{3,2}(\xi) = \frac{1}{2}\int_\Omega BG^{(2)}_n(\xi\cdot\alpha)O^{(3,2)}\mathbf{f}(\alpha)d\omega(\alpha), \tag{6.201}$$

where

$$\begin{aligned} BG^{(2)}_n(\xi\cdot\eta) &= \int_\Omega G(\Delta^*(\Delta^*+2);\xi\cdot\alpha)B_n(\alpha\cdot\eta)d\omega(\alpha) \qquad (6.202)\\ &= \sum_{k=2}^{n}\frac{2k+1}{4\pi}\frac{B_n^\wedge(k)}{(k-1)k(k+1)(k+2)}P_k(\xi\cdot\eta). \end{aligned}$$

Our interest is the 'Bernstein summability' of Fourier expansions in terms of tensor spherical harmonics. To this end, we need some preparatory results (viz., Lemma 6.8 and Lemma 6.9).

Lemma 6.8. *For $i,k\in\{1,2,3\}$, we have*

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|F_{i,k}(\xi)-F^{(n)}_{i,k}(\xi)\right| = 0,$$

where the functions $F_{i,k}$ are defined by (6.164)–(6.172) *and the functions $F^{(n)}_{i,k}$ are given by* (6.193)–(6.201).

Proof. Since both kernels $G(\Delta^*(\Delta^*+2);\cdot)$ and $BG^{(2)}_n$ are in $\mathrm{L}^2[-1,1]$ and the Legendre coefficients of the Bernstein kernel $B_n^\wedge(k)$ converge to 1 as $n\to\infty$ for all $k\in\mathbb{N}_0$, we obtain

$$\lim_{n\to\infty}\|G(\Delta^*(\Delta^*+2);\cdot)-BG^{(2)}_n\|_{\mathrm{L}^2[-1,1]} = 0. \tag{6.203}$$

The last limit also holds true in the L^1-metric. Consequently, we are able to deduce that $\|F_{i,k}-F^{(n)}_{i,k}\|_{\mathrm{C}(\Omega)}\longrightarrow 0$ for $i,k\in\{1,2,3\}$ as $n\longrightarrow\infty$. □

We are now prepared to verify the following result.

Lemma 6.9. *For* $i, k \in \{1, 2, 3\}$

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \|\mathbf{o}^{(i,k)} F_{i,k} - \mathbf{o}^{(i,k)} F_{i,k}^{(n)}\|_{\mathbf{c}(\Omega)} = 0.$$

Proof. For the types $(i, k) = (1, 1), (2, 2), (3, 3)$, we obtain the required convergence of $\|\mathbf{o}^{(i,k)} F_{i,k} - \mathbf{o}^{(i,k)} F_{i,k}^{(n)}\|_{\mathbf{c}(\Omega)}$ as in the scalar case, and for the types $(i, k) = (1, 2), (1, 3), (2, 1), (3, 1)$ as in the vectorial case because of the structure of the corresponding operators $\mathbf{o}^{(i,k)}$. This leaves us with the two types $(i, k) = (2, 3), (3, 2)$.

$$\begin{aligned}
&\|\mathbf{o}_\xi^{(i,k)} F^{(i,k)}(\xi) - \mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi)\|_{\mathbf{c}(\Omega)} \qquad (6.204)\\
&= \sup_{\xi\in\Omega} \left| \frac{1}{2} \left(\mathbf{o}_\xi^{(i,k)} \int_\Omega G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \right.\right.\\
&\qquad \left.\left. - \mathbf{o}_\xi^{(i,k)} \int_\Omega BG_n^{(2)}(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \right) \right|\\
&= \sup_{\xi\in\Omega} \left| \frac{1}{2} \left(\int_\Omega \mathbf{o}_\xi^{(i,k)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \right.\right.\\
&\qquad \left.\left. - \int_\Omega \mathbf{o}_\xi^{(i,k)} BG_n^{(2)}(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \right) \right|,
\end{aligned}$$

where the operator $\mathbf{o}^{(i,k)}$ can be put inside both integrals. By obvious manipulations, we now get

$$\begin{aligned}
&\sup_{\xi\in\Omega} \left| \tfrac{1}{2} \left(\int_\Omega \left(\mathbf{o}_\xi^{(i,k)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) - \mathbf{o}_\xi^{(i,k)} BG_n^{(2)}(\xi\cdot\eta) \right) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \right) \right|\\
&\le \sup_{\xi\in\Omega} \tfrac{1}{2} \int_\Omega \left| \mathbf{o}_\xi^{(i,k)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) - \mathbf{o}_\xi^{(i,k)} BG_n^{(2)}(\xi\cdot\eta) \right| \left| O_\eta^{(i,k)} \mathbf{f}(\eta) \right| d\omega(\eta)\\
&\le \tfrac{1}{2} \|O^{(i,k)} \mathbf{f}\|_{\mathrm{C}(\Omega)} \int_\Omega \left| \mathbf{o}_\xi^{(i,k)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) - \mathbf{o}_\xi^{(i,k)} BG_n^{(2)}(\xi\cdot\eta) \right| d\omega(\eta).
\end{aligned}$$

In consequence, we have to prove the convergence of the last integral, i.e., the $\mathbf{l}^1$-norm. Application of the tensorial operators $\mathbf{o}^{(2,3)}$ and $\mathbf{o}^{(3,2)}$ to the corresponding Green function results in the following identities

$$\begin{aligned}
&\mathbf{o}_\xi^{(2,3)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) \qquad (6.205)\\
&= \tfrac{1}{4\pi} G''(\xi\cdot\eta) \left[(\eta - (\xi\cdot\eta)\xi) \otimes (\eta - (\xi\cdot\eta)\xi) - (\xi\wedge\eta) \otimes (\xi\wedge\eta) \right],\\
&\mathbf{o}_\xi^{(3,2)} G(\Delta^*(\Delta^*+2); \xi\cdot\eta) \qquad (6.206)\\
&= \tfrac{1}{4\pi} G''(\xi\cdot\eta) \left[(\eta - (\xi\cdot\eta)\xi) \otimes (\xi\wedge\eta) + (\xi\wedge\eta) \otimes (\eta - (\xi\cdot\eta)\xi) \right].
\end{aligned}$$

Calculating the absolute value of the last two expressions, we find that

$$\begin{aligned}
&\left|\mathbf{o}_\xi^{(2,3)} G(\Delta^*(\Delta^*+2);\xi\cdot\eta)\right| \qquad (6.207)\\
&= \frac{1}{4\pi}\left|G''(\xi\cdot\eta)\right| \left|(\eta-(\xi\cdot\eta)\,\xi)\otimes(\eta-(\xi\cdot\eta)\,\xi)-(\xi\wedge\eta)\otimes(\xi\wedge\eta)\right|\\
&= \frac{1}{8\pi}\frac{1}{1-\xi\cdot\eta}\sqrt{2}(1-(\xi\cdot\eta)^2) = \frac{1}{4\pi}\frac{1}{\sqrt{2}}(1+\xi\cdot\eta)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbf{o}_\xi^{(3,2)} G(\Delta^*(\Delta^*+2);\xi,\eta)\right| \qquad (6.208)\\
&= \frac{1}{4\pi}\left|G''(\xi\cdot\eta)\right| \left|(\eta-(\xi\cdot\eta)\,\xi)\otimes(\xi\wedge\eta)+(\xi\wedge\eta)\otimes(\eta-(\xi\cdot\eta)\,\xi)\right|\\
&= \frac{1}{8\pi}\frac{1}{1-\xi\cdot\eta}\sqrt{2}(1-(\xi\cdot\eta)^2) = \frac{1}{4\pi}\frac{1}{\sqrt{2}}(1+\xi\cdot\eta),
\end{aligned}$$

where we used the relation

$$|x\otimes x - y\otimes y|^2 = |x|^4+|y|^4-2(x\cdot y)^2 \qquad (6.209)$$

with $x=\eta-(\xi\cdot\eta)\,\xi$ and $y=\xi\wedge\eta$ for the first operator, thereby observing that $(\eta-(\xi\cdot\eta)\,\xi)\cdot(\xi\wedge\eta)=0$ and $|\eta-(\xi\cdot\eta)\,\xi|^2=|\xi\wedge\eta|^2=1-(\xi\cdot\eta)^2$. For the second operator, a slightly different relation is required, i.e.,

$$|x\otimes y+y\otimes x|^2 = 2(x\cdot y)^2+2|x|^2|y|^2, \qquad (6.210)$$

where $x=\eta-(\xi\cdot\eta)\,\xi$ and $y=\xi\wedge\eta$.

Thus, we are able to conclude both that $\mathbf{o}^{(3,2)}G(\Delta^*(\Delta^*+2);\cdot,\eta)$ is of class $\mathbf{l}^2(\Omega)$ and $\mathbf{o}^{(2,3)}G(\Delta^*(\Delta^*+2);\cdot,\eta)$ is of class $\mathbf{l}^2(\Omega)$ for all $\eta\in\Omega$. In consequence, the desired $\mathbf{l}^1$-convergence results from the $\mathbf{l}^2$-convergence of the two kernels (both are in $\mathbf{l}^2(\Omega)$ and $B_n^\wedge(k)$ tends to 1). Thus, Lemma 6.9 is verified for all types (i,k). □

Now, we are able to formulate the *'Bernstein summability' of a Fourier series in terms of tensor spherical harmonics.*

Theorem 6.10. *For any tensor field* $\mathbf{f}\in\mathbf{c}^{(2)}(\Omega)$,

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\mathbf{f}(\xi)-\sum_{i,k=1}^{3}\sum_{m=0_{i,k}}^{n}\sum_{j=1}^{2m+1} B_n^\wedge(m)\left(\mathbf{f}^{(i,k)}\right)^\wedge(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi)\right| = 0.$$

Proof. From Lemma 6.9, we have for any tensorial field $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{f}(\xi) - \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) \right| \\
&= \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_{i,k}(\xi) - \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) \right| \\
&\leq \sum_{i,k=1}^{3} \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{o}_\xi^{(i,k)} F_{i,k}(\xi) - \mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) \right| = 0. \qquad (6.211)
\end{aligned}$$

Our aim is to consider the term $\mathbf{o}_\xi^{(1,1)} F_{1,1}^{(n)}(\xi)$ in more detail. A simple calculation yields

$$\begin{aligned}
\mathbf{o}_\xi^{(1,1)} F_{1,1}^{(n)}(\xi) &= \mathbf{o}_\xi^{(1,1)} \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(1,1)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{k=0}^{n} B_n^\wedge(k) \frac{2k+1}{4\pi} \mathbf{o}_\xi^{(1,1)} \int_\Omega P_k(\xi\cdot\eta) O_\eta^{(1,1)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{k=0}^{n} B_n^\wedge(k) \mathbf{o}_\xi^{(1,1)} \sum_{j=1}^{2k+1} \left(O^{(1,1)} \mathbf{f} \right)^\wedge (k,j) Y_{k,j}(\xi) \\
&= \sum_{k=0}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(O^{(1,1)} \mathbf{f} \right)^\wedge (k,j) \mathbf{y}_{k,j}^{(1,1)}(\xi). \qquad (6.212)
\end{aligned}$$

It should be noted that

$$\begin{aligned}
\left(O^{(1,1)} \mathbf{f} \right)^\wedge (k,j) &= \int_\Omega O_\eta^{(1,1)} \mathbf{f}(\eta) Y_{k,j}(\eta) d\omega(\eta) \\
&= \int_\Omega \mathbf{f}(\eta) \cdot \underbrace{\mathbf{o}_\eta^{(1,1)} Y_{k,j}(\eta)}_{=\mathbf{y}_{k,j}^{(1,1)}(\eta)} d\omega(\eta) = \left(\mathbf{f}^{(1,1)} \right)^\wedge (k,j) \qquad (6.213)
\end{aligned}$$

such that from (6.212) and (6.213), we are able to conclude for the type $(i,k) = (1,1)$ that

$$\mathbf{o}_\xi^{(1,1)} F_{1,1}^{(n)}(\xi) = \sum_{k=0}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(\mathbf{f}^{(1,1)} \right)^\wedge (k,j) \mathbf{y}_{k,j}^{(1,1)}(\xi). \qquad (6.214)$$

For the cases $(i,k) = (2,2), (3,3)$, we get

$$\begin{aligned}
\mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) &= \mathbf{o}_\xi^{(i,k)} \frac{1}{2} \int_\Omega B_n(\xi \cdot \eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \frac{1}{2} \sum_{m=0}^{n} B_n^\wedge(m) \frac{2m+1}{4\pi} \mathbf{o}_\xi^{(i,k)} \int_\Omega P_m(\xi \cdot \eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \frac{1}{2} \sum_{m=0}^{n} B_n^\wedge(m) O_\xi^{(i.k)} \sum_{j=1}^{2m+1} \left(O^{(i,k)} \mathbf{f} \right)^\wedge (m,j) Y_{m,j}(\xi) \\
&= \frac{1}{\sqrt{2}} \sum_{m=0}^{n} \sum_{j=1}^{2m+1} B_n^\wedge(m) \left(O^{(i,k)} \mathbf{f} \right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi). \qquad (6.215)
\end{aligned}$$

Observe that

$$\begin{aligned}
\left(O^{(i,k)} \mathbf{f} \right)^\wedge (m,j) &= \int_\Omega O_\eta^{(i,k)} \mathbf{f}(\eta) Y_{m,j}(\eta) d\omega(\eta) \\
&= \int_\Omega \mathbf{f}(\eta) \cdot \underbrace{\mathbf{o}_\eta^{(i,k)} Y_{m,j}(\eta)}_{=\sqrt{2}\mathbf{y}_{m,j}^{(i,k)}(\eta)} d\omega(\eta) = \sqrt{2} \left(\mathbf{f}^{(i,k)} \right)^\wedge (m,j). \qquad (6.216)
\end{aligned}$$

Combining (6.215) and (6.216), we get for $(i,k) = (2,2), (3,3)$

$$\mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) = \sum_{m=0}^{n} \sum_{j=1}^{2m+1} B_n^\wedge(m) \left(\mathbf{f}^{(i,k)} \right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi). \qquad (6.217)$$

For $(i,k) = (1,2), (1,3), (2,1), (3,1)$, we have

$$\begin{aligned}
\mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) &= -\mathbf{o}_\xi^{(i,k)} \frac{1}{2} \int_\Omega BG_n(\xi \cdot \eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{m=1}^{n} \frac{B_n^\wedge(m)}{m(m+1)} \frac{2m+1}{4\pi} \mathbf{o}_\xi^{(i,k)} \int_\Omega P_m(\xi \cdot \eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{m=1}^{n} \frac{B_n^\wedge(m)}{m(m+1)} \mathbf{o}_\xi^{(i,k)} \sum_{j=1}^{2m+1} \left(O^{(i,k)} \mathbf{f} \right)^\wedge (m,j) Y_{m,j}(\xi) \\
&= \sum_{m=1}^{n} \sum_{j=1}^{2m+1} \frac{B_n^\wedge(m)}{\sqrt{m(m+1)}} \left(O^{(i,k)} \mathbf{f} \right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi). \qquad (6.218)
\end{aligned}$$

Again, we have to take a look at the coefficients $\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j)$. In fact,

$$\begin{aligned}\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j) &= \int_\Omega O_\eta^{(i,k)}\mathbf{f}(\eta)Y_{m,j}(\eta)d\omega(\eta) \quad (6.219)\\ &= \int_\Omega \mathbf{f}(\eta)\cdot\mathbf{o}_\eta^{(i,k)}Y_{m,j}(\eta)d\omega(\eta),\\ &= \sqrt{m(m+1)}\int_\Omega \mathbf{f}(\eta)\cdot\mathbf{y}_{m,j}^{(i,k)}(\eta)d\omega(\eta)\\ &= \sqrt{m(m+1)}\left(\mathbf{f}^{(i,k)}\right)^\wedge(m,j).\end{aligned}$$

Putting together (6.218) and (6.220), we are able to show that, for $(i,k) = (1,2),(1,3),(2,1),(3,1)$,

$$\mathbf{o}_\xi^{(i,k)}F_{i,k}^{(n)}(\xi) = \sum_{m=1}^{n}\sum_{j=1}^{2m+1} B_n^\wedge(m)\left(\mathbf{f}^{(i,k)}\right)^\wedge(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi). \quad (6.220)$$

Finally, we treat $(i,k) = (2,3),(3,2)$. It is not hard to verify that

$$\begin{aligned}&\mathbf{o}_\xi^{(i,k)}F_{i,k}^{(n)}(\xi) \quad (6.221)\\ &= \mathbf{o}_\xi^{(i,k)}\frac{1}{2}\int_\Omega BG_n^{(2)}(\xi\cdot\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)\\ &= \sum_{m=2}^{n}\frac{B_n^\wedge(m)}{2m(m+1)(m(m+1)-2)}\frac{2m+1}{4\pi}\mathbf{o}_\xi^{(i,k)}\int_\Omega P_m(\xi\cdot\eta)O_\eta^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)\\ &= \sum_{m=2}^{n}\frac{B_n^\wedge(m)}{2m(m+1)(m(m+1)-2)}\mathbf{o}_\xi^{(i,k)}\sum_{j=1}^{2m+1}\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j)Y_{m,j}(\xi)\\ &= \sum_{m=2}^{n}\sum_{j=1}^{2m+1}\frac{B_n^\wedge(m)}{\sqrt{2m(m+1)(m(m+1)-2)}}\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi).\end{aligned}$$

This enables us to rewrite the coefficients $\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j)$ as follows

$$\begin{aligned}\left(O^{(i,k)}\mathbf{f}\right)^\wedge(m,j) &= \int_\Omega O_\eta^{(i,k)}\mathbf{f}(\eta)Y_{m,j}(\eta)d\omega(\eta) \quad (6.222)\\ &= \int_\Omega \mathbf{f}(\eta)\cdot\mathbf{o}_\eta^{(i,k)}Y_{m,j}(\eta)d\omega(\eta)\\ &= \sqrt{2m(m+1)(m(m+1)-2)}\int_\Omega \mathbf{f}(\eta)\cdot\mathbf{y}_{m,j}^{(i,k)}(\eta)d\omega(\eta)\\ &= \sqrt{2m(m+1)(m(m+1)-2)}\left(\mathbf{f}^{(i,k)}\right)^\wedge(m,j).\end{aligned}$$

Consequently, the identities (6.221) and (6.222) lead to the conclusion that for $(i,k) = (2,3),(3,2)$

$$\mathbf{o}_\xi^{(i,k)}F_{i,k}^{(n)}(\xi) = \sum_{m=2}^{n}\sum_{j=1}^{2m+1} B_n^\wedge(m)\left(\mathbf{f}^{(i,k)}\right)^\wedge(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi). \quad (6.223)$$

Altogether, the identities (6.214), (6.217), (6.220), and (6.223) in connection with (6.211) yield the desired summability of tensor spherical harmonics. More concretely,

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{f}(\xi) - \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_{i,k}^{(n)}(\xi) \right| \tag{6.224}$$

$$= \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{f}(\xi) - \sum_{i,k=1}^{3} \sum_{m=0_{i,k}}^{n} \sum_{j=1}^{2m+1} B_n^\wedge(m) \left(\mathbf{f}^{(i,k)}\right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi) \right| = 0,$$

provided that $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$. □

As in the vector case, based on a density argument, the *closure of the tensor spherical harmonics* $\left\{\mathbf{y}_{m,j}^{(i,k)}\right\}_{i,k,m,j}$ *in the space* $\mathbf{c}(\Omega)$ becomes obvious.

Theorem 6.11. *For any given* $\varepsilon > 0$ *and each* $\mathbf{f} \in \mathbf{c}(\Omega)$*, there exists a linear combination* $\sum_{i,k=1}^{3} \sum_{m=0_{i,k}}^{N} \sum_{j=1}^{2m+1} d_{m,j}^{(i,k)} \mathbf{y}_{m,j}^{(i,k)}$*, such that*

$$\left\| \mathbf{f} - \sum_{i,k=1}^{3} \sum_{m=0_{i,k}}^{N} \sum_{j=1}^{2m+1} d_{m,j}^{(i,k)} \mathbf{y}_{m,j}^{(i,k)} \right\|_{\mathbf{c}(\Omega)} \leq \varepsilon.$$

Again, standard arguments guarantee the closure in $\mathbf{c}(\Omega)$ with respect to $\|\cdot\|_{\mathbf{l}^2(\Omega)}$ as well as in $\mathbf{l}^2(\Omega)$ which in turn shows the completeness of the system $\left\{\mathbf{y}_{m,j}^{(i,k)}\right\}_{i,k,m,j}$ in $\mathbf{l}^2(\Omega)$.

Summarizing our results, we therefore obtain the following theorem.

Theorem 6.12. *Let* $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}_{\substack{i,k=1,2,3 \\ n=0_{i,k},\ldots,\ j=1,\ldots,2n+1}}$ *be defined as in (6.149). Then the following statements are valid:*

(i) *The system of tensor spherical harmonics is closed in* $\mathbf{c}(\Omega)$ *with respect to* $\|\cdot\|_{\mathbf{c}(\Omega)}$.
(ii) *The system is complete in* $\mathbf{l}^2(\Omega)$ *with respect to* $\|\cdot\|_{\mathbf{l}^2(\Omega)}$.

Once more, part (i) of this theorem justifies the approximation of continuous tensor fields on the sphere by finite sums of tensor spherical harmonics, while part (ii) is equivalent to the property that any tensor field $\mathbf{f} \in \mathbf{l}^2(\Omega)$ can be represented in $\mathbf{l}^2(\Omega)$-sense by its orthogonal expansion in terms of tensor spherical harmonics.

6.7 Homogeneous Harmonic Tensor Polynomials

Next, we want to show that the $\mathbf{l}^2(\Omega)$-orthonormal system of tensor spherical harmonics is complete in $\mathbf{l}^2(\Omega)$ with respect to $(\cdot,\cdot)_{\mathbf{l}^2(\Omega)}$ and closed in $\mathbf{c}(\Omega)$ with respect to $\|\cdot\|_{\mathbf{c}(\Omega)}$. As in the vectorial case, we start with the definition of homogeneous harmonic polynomials.

Definition 6.13. A tensor field $\mathbf{h}_n : \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $n \geq 0$, of the form

$$\mathbf{h}_n(x) = \sum_{i,k=1}^{3} H_n^{ik}(x)\varepsilon^i \otimes \varepsilon^k, \quad x \in \mathbb{R}^3, \tag{6.225}$$

is called *homogeneous harmonic tensor polynomial of degree* n, if every H_n^{ik} is a scalar homogeneous harmonic polynomial of degree n.

Using the notation

$$\mathrm{Harm}_n\varepsilon^i \otimes \varepsilon^k = \mathrm{span}\left\{Y_{n,j}\varepsilon^i \otimes \varepsilon^k\right\}_{\substack{i,k=1,2,3\\ j=1,\ldots,2n+1}}, \tag{6.226}$$

$n \in \mathbb{N}_0$, $i,k \in \{1,2,3\}$, the space of all homogeneous harmonic tensor polynomials of degree n is characterized by

$$\bigoplus_{i,k=1}^{3} \mathrm{Harm}_n(\mathbb{R}^3)\varepsilon^i \otimes \varepsilon^k. \tag{6.227}$$

As in the vector theory, the restriction of a homogeneous harmonic tensor polynomial of degree n to the unit sphere Ω does, in general, not yield a spherical harmonic of degree n. But our purpose is to show that

$$\bigoplus_{i,k=1}^{3} \mathrm{Harm}_n(\Omega)\varepsilon^i \otimes \varepsilon^k \tag{6.228}$$

is expressible as a linear combination of tensor spherical harmonic of different degrees.

As immediate consequences of the Corollaries 3.50 and 3.52, we obtain the following lemma.

Lemma 6.14. *The following statements are valid:*

(i) *The system* $\left\{Y_{n,j}\varepsilon^i \otimes \varepsilon^k\right\}_{\substack{i,k=1,2,3\\ n=0,1,\ldots,\ j=1,\ldots,2n+1}}$ *is complete in* $\mathbf{l}^2(\Omega)$.

(ii) *The system* $\{Y_{n,j}\varepsilon^i \otimes \varepsilon^k\}_{\substack{i,k=1,2,3\\ n=0,1,\ldots,\ j=1,\ldots,2n+1}}$ *is closed in* $\mathbf{c}(\Omega)$ *with respect to* $\|\cdot\|_{\mathbf{c}(\Omega)}$.

In what follows, we are interested in the relations between tensor spherical harmonics on the one hand and homogeneous harmonic tensor polynomials restricted to the unit sphere on the other hand. For that purpose, we consider operators $\tilde{\mathbf{o}}_n^{(i,k)}$, $i,k \in \{1,2,3\}$ given by

$$\tilde{\mathbf{o}}_n^{(1,1)} F(x) = \left((2n+3)x - |x|^2\nabla_x\right) \otimes \left((2n+1)x - |x|^2\nabla_x\right) F(x), \tag{6.229}$$

$$\tilde{\mathbf{o}}_n^{(1,2)} F(x) = \left((2n-1)x - |x|^2\nabla_x\right) \otimes \nabla_x F(x), \tag{6.230}$$

$$\tilde{\mathbf{o}}_n^{(1,3)} F(x) = \left((2n+1)x - |x|^2\nabla_x\right) \otimes (x \wedge \nabla_x) F(x), \tag{6.231}$$

$$\tilde{\mathbf{o}}_n^{(2,1)} F(x) = \nabla_x \otimes \left((2n+1)x - |x|^2\nabla_x\right) F(x), \tag{6.232}$$

$$\tilde{\mathbf{o}}_n^{(2,2)} F(x) = \nabla_x \otimes \nabla_x F(x), \tag{6.233}$$

$$\tilde{\mathbf{o}}_n^{(2,3)} F(x) = \nabla_x \otimes (x \wedge \nabla_x) F(x), \tag{6.234}$$

$$\tilde{\mathbf{o}}_n^{(3,1)} F(x) = (x \wedge \nabla_x) \otimes \left((2n+1)x - |x|^2\nabla_x\right) F(x), \tag{6.235}$$

$$\tilde{\mathbf{o}}_n^{(3,2)} F(x) = (x \wedge \nabla_x) \otimes \nabla_x F(x), \tag{6.236}$$

$$\tilde{\mathbf{o}}_n^{(3,3)} F(x) = (x \wedge \nabla_x) \otimes (x \wedge \nabla_x) F(x) \tag{6.237}$$

for $x \in \mathbb{R}^3$ and sufficiently smooth function $F : \mathbb{R}^3 \to \mathbb{R}$. Simple calculations in cartesian coordinates lead us in a straightforward way to the following result.

Lemma 6.15. *Let $H_n, n \in \mathbb{N}_0$, be a homogeneous harmonic polynomial of degree n. Then, $\tilde{\mathbf{o}}_n^{(i,k)} H_n$ is a homogeneous harmonic tensor polynomial of degree $\deg^{(i,k)}(n)$, where*

$$\deg^{(i,k)}(n) = \begin{cases} n-2 & \textit{for} \quad (i,k) = (2,2) \\ n-1 & \textit{for} \quad (i,k) \in \{(2,3),(3,2)\} \\ n & \textit{for} \quad (i,k) \in \{(1,2),(2,1),(3,3)\} \\ n+1 & \textit{for} \quad (i,k) \in \{(1,3),(3,1)\} \\ n+2 & \textit{for} \quad (i,k) = (1,1) \end{cases} . \tag{6.238}$$

($\deg^{(i,k)}(n) < 0$ means that $\tilde{\mathbf{o}}_n^{(i,k)} H_n = 0$).

The gradient of a sufficiently smooth function $F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is known to be equal to

$$\nabla_x F(x) = \xi \frac{\partial}{\partial r} F(r\xi) + \frac{1}{r} \nabla_\xi^* F(r\xi), \quad x = r\xi,\ r > 0,\ \xi \in \Omega. \tag{6.239}$$

Similarly, if $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ is a sufficiently smooth vector field of the form

$$f(x) = \sum_{i=1}^{3} F_i(x) \varepsilon^i, \quad |x| > 0, \tag{6.240}$$

then we have, for $r > 0$, $\xi \in \Omega$,

$$\begin{aligned}\nabla_x \otimes f(x) &= \sum_{i=1}^{3} \nabla_x F_i(x) \otimes \varepsilon^i \qquad (6.241)\\ &= \sum_{i=1}^{3} \left(\xi \frac{\partial}{\partial r} F_i(r\xi) + \frac{1}{r} \nabla_\xi^* F_i(r\xi) \right) \otimes \varepsilon^i \\ &= \xi \otimes \frac{\partial}{\partial r} f(r\xi) + \frac{1}{r} \nabla_\xi^* \otimes f(r\xi).\end{aligned}$$

Thus we get, for $Y_n \in \mathrm{Harm}_n$,

$$\begin{aligned}&\tilde{\mathbf{o}}_n^{(1,1)} r^n Y_n(\xi) \qquad (6.242)\\ &= \left((2n+3)r\xi - r^2\nabla_x\right) \otimes \left((2n+1)r\xi - r^2\nabla_x\right) r^n Y_n(\xi) \\ &= \left((2n+3)r\xi - r^2\nabla_x\right) \otimes \left((2n+1)r^{n+1}\xi Y_n(\xi) - nr^{n+1}\xi Y_n(\xi) \right.\\ &\qquad \left. -r^{n+1}\nabla_\xi^* Y_n(\xi)\right) \\ &= \left((2n+3)r\xi - r^2\nabla_x\right) \otimes \left((n+1)r^{n+1}\xi Y_n(\xi) - r^{n+1}\nabla_\xi^* Y_n(\xi)\right).\end{aligned}$$

This shows us that

$$\begin{aligned}&\tilde{\mathbf{o}}_n^{(1,1)} r^n Y_n(\xi) \qquad (6.243)\\ &= (2n+3)(n+1)r^{n+2}\xi \otimes \xi Y_n(\xi) - (2n+3)r^{n+2}\xi \otimes \nabla_\xi^* Y_n(\xi) \\ &\qquad -(n+1)r^{n+2}\nabla_\xi^* \otimes \xi Y_n(\xi) + r^{n+2}\nabla_\xi^* \otimes \nabla_\xi^* Y_n(\xi) \\ &\qquad -(n+1)r^2\xi \otimes \frac{\partial}{\partial r} r^{n+1}\xi Y_n(\xi) + r^2\xi \otimes \frac{\partial}{\partial r} r^{n+1}\nabla_\xi^* Y_n(\xi) \\ &= (n+2)(n+1)r^{n+2}\xi \otimes \xi Y_n(\xi) - (n+2)r^{n+2}\xi \otimes \nabla_\xi^* Y_n(\xi) \\ &\qquad -(n+1)r^{n+2}\nabla_\xi^* Y_n(\xi) \otimes \xi - (n+1)r^{n+2}\mathbf{i}_{\mathrm{tan}}(\xi) Y_n(\xi) \\ &\qquad +r^{n+2}\nabla_\xi^* \otimes \nabla_\xi^* Y_n(\xi),\end{aligned}$$

where we have used (6.49). By restricting this tensor field to the unit sphere Ω, we get with (6.115)

$$\begin{aligned}\tilde{\mathbf{o}}_n^{(1,1)} r^n Y_n(\xi)|_{r=1} &= (n+2)(n+1)\mathbf{o}^{(1,1)} Y_n(\xi) - (n+2)\mathbf{o}^{(1,2)} Y_n(\xi) \\ &\qquad -(n+2)\mathbf{o}^{(2,1)} Y_n(\xi) - \frac{1}{2}(n+2)(n+1)\tilde{\mathbf{o}}_n^{(2,2)} Y_n(\xi) \\ &\qquad +\frac{1}{2}\mathbf{o}^{(2,3)} Y_n(\xi).\end{aligned}$$

Similar calculations show that all restrictions of $r\xi \mapsto \tilde{\mathbf{o}}_n^{(i,k)} r^n Y_n(\xi)$ to the unit sphere Ω (i.e., $r = 1$) can be written as linear combinations of the tensor spherical harmonics $\mathbf{o}^{(i,k)}Y_n$. More explicitly, for $Y_n \in \mathrm{Harm}_n$

$$\begin{pmatrix} \tilde{\mathbf{o}}_n^{(1,1)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(1,2)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(2,1)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(2,2)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(3,3)} r^n Y_n|_{r=1} \end{pmatrix} = \mathbf{a}_n \begin{pmatrix} \mathbf{o}^{(1,1)} Y_n \\ \mathbf{o}^{(1,2)} Y_n \\ \mathbf{o}^{(2,1)} Y_n \\ \mathbf{o}^{(2,2)} Y_n \\ \mathbf{o}^{(2,3)} Y_n \end{pmatrix} \tag{6.244}$$

and

$$\begin{pmatrix} \tilde{\mathbf{o}}_n^{(1,3)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(2,3)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(3,1)} r^n Y_n|_{r=1} \\ \tilde{\mathbf{o}}_n^{(3,2)} r^n Y_n|_{r=1} \end{pmatrix} = \mathbf{b}_n \begin{pmatrix} \mathbf{o}^{(1,3)} Y_n \\ \mathbf{o}^{(3,1)} Y_n \\ \mathbf{o}^{(3,2)} Y_n \\ \mathbf{o}^{(3,3)} Y_n \end{pmatrix}, \tag{6.245}$$

where the matrices $\mathbf{a}_n$ and $\mathbf{b}_n$ are given by

$$\mathbf{a}_n = \begin{pmatrix} (n+1)(n+2) & -(n+2) & -(n+2) & -\frac{1}{2}(n+2)(n+1) & \frac{1}{2} \\ n^2 & n & -(n-1) & -\frac{1}{2}n(n-1) & -\frac{1}{2} \\ (n+1)^2 & -(n+1) & n+2 & \frac{1}{2}(n+2)(n+1) & -\frac{1}{2} \\ n(n-1) & n-1 & n-1 & -\frac{1}{2}n(n-1) & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2}n(n+1) & -\frac{1}{2} \end{pmatrix}, \tag{6.246}$$

$$\mathbf{b}_n = \begin{pmatrix} n+1 & 1 & -\frac{1}{2} & -\frac{1}{2}n(n+1) \\ n & -1 & \frac{1}{2} & \frac{1}{2}n(n+1) \\ 0 & n+2 & -\frac{1}{2} & \frac{1}{2}(n+2)(n+1) \\ 0 & n-1 & \frac{1}{2} & -\frac{1}{2}n(n-1) \end{pmatrix}. \tag{6.247}$$

Elementary calculations show that

$$\det(\mathbf{a}_n) = \frac{1}{2}n(2n+3)(2n-1)(n+1)(2n+1)^2 \tag{6.248}$$

and

$$\det(\mathbf{b}_n) = -\frac{1}{2}n(n+1)(2n+1)^2. \tag{6.249}$$

Thus, the matrices $\mathbf{a}_n$ and $\mathbf{b}_n$ are regular for $n \geq 1$. The inverse matrices read as follows:

$$\mathbf{a}_n^{-1} = \tag{6.250}$$

$$\begin{pmatrix} \frac{1}{4n^2+8n+3} & \frac{1}{4n^2-1} & \frac{1}{4n^2+8n+3} & \frac{1}{4n^2-1} & 0 \\ \frac{-n}{4n^2+8n+3} & \frac{n+1}{4n^2-1} & \frac{-n}{4n^2+8n+3} & \frac{n+1}{4n^2-1} & 0 \\ \frac{-n}{4n^2+8n+3} & \frac{1-n^2}{(4n^2-1)n} & \frac{n(2+n)}{4n^3+12n^2+11n+3} & \frac{n+1}{4n^2-1} & \frac{1}{(n+1)n} \\ \frac{-1}{4n^2+8n+3} & \frac{-1+n}{(4n^2-1)n} & \frac{2+n}{4n^3+12n^2+11n+3} & \frac{-1}{4n^2-1} & \frac{-1}{(n+1)n} \\ \frac{(n-1)n}{4n^2+8n+3} & \frac{n^3+2n^2-n-2}{(1-4n^2)n} & \frac{(2-n-n^2)n}{4n^3+12n^2+11n+3} & \frac{n^2+3n+2}{4n^2-1} & \frac{2-n-n^2}{(n+1)n} \end{pmatrix},$$

$$\mathbf{b}_n^{-1} = \begin{pmatrix} \frac{1}{2n+1} & \frac{1}{2n+1} & 0 & 0 \\ \frac{1}{2n^2+3n+1} & -\frac{1}{n(2n+1)} & \frac{n}{1+3n+2n^2} & \frac{n+1}{n(2n+1)} \\ -\frac{n^2+n-2}{1+3n+2n^2} & \frac{n^2+n-2}{n(2n+1)} & -\frac{(-1+n)n}{1+3n+2n^2} & \frac{3n+2+n^2}{n(2n+1)} \\ -\frac{1}{2n^2+3n+1} & \frac{1}{n(2n+1)} & -\frac{1}{2n^2+3n+1} & -\frac{1}{n(2n+1)} \end{pmatrix}. \tag{6.251}$$

For the special case $n = 0$, we see that $Y_0 \in \mathrm{Harm}_0$ satisfies

$$\begin{array}{rcrcl} 2\mathbf{o}^{(1,1)}Y_0 & - & \mathbf{o}^{(2,2)}Y_0 & = & \tilde{o}_0^{(1,1)}Y_0|\Omega, \\ \mathbf{o}^{(1,1)}Y_0 & + & \mathbf{o}^{(2,2)}Y_0 & = & \tilde{o}_0^{(2,1)}Y_0|\Omega, \\ & & \mathbf{o}^{(3,3)}Y_0 & = & \tilde{o}_0^{(3,1)}Y_0|\Omega. \end{array} \tag{6.252}$$

Hence, we obtain the following lemma, in view of Lemma 6.15.

Lemma 6.16. *Let $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$ be a tensor spherical harmonic of degree n and type (i,k). Then,*

$$\mathbf{y}_n^{(i,k)} \in \bigoplus_{p,q=1}^{3} \mathrm{Harm}_{n-2}\varepsilon^p \otimes \varepsilon^q \bigoplus \mathrm{Harm}_n\varepsilon^p \otimes \varepsilon^q \bigoplus \mathrm{Harm}_{n+2}\varepsilon^p \otimes \varepsilon^q, \tag{6.253}$$

if $(i,k) \in \{(1,1),(1,2),\ (2,1),\ (2,2),(2,3)\}$. Moreover,

$$\mathbf{y}_n^{(i,k)} \in \bigoplus_{p,q=1}^{3} \mathrm{Harm}_{n-1}\varepsilon^p \otimes \varepsilon^q \oplus \mathrm{Harm}_{n+1}\varepsilon^p \otimes \varepsilon^q, \tag{6.254}$$

if $(i,k) \in \{(1,3),(3,1),(3,2),(3,3)\}$.

For $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$ and $\xi \in \Omega$, it follows as an immediate consequence that

$$\mathbf{y}_n^{(i,k)}(-\xi) = (-1)^n \mathbf{y}_n^{(i,k)}(\xi) \tag{6.255}$$

if $(i,k) \in \{(1,1),(1,2),(2,1),(2,2),(2,3)\}$ and

$$\mathbf{y}_n^{(i,k)}(-\xi) = (-1)^{n+1} \mathbf{y}_n^{(i,k)}(\xi) \tag{6.256}$$

if $(i,k) \in \{(1,3),(3,1),(3,2),(3,3)\}$.

Lemma 6.16 also yields other orthogonality relations.

Lemma 6.17. *For $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$ and $Y_m \in \mathrm{Harm}_m$*

$$\int_\Omega Y_m(\xi)\mathbf{y}_n^{(i,k)}(\xi)\, d\omega(\xi) = 0 \tag{6.257}$$

if $(i,k) \in \{(1,1),(1,2),(2,1),(2,2),(2,3)\}$ and $m \notin \{n-2,n,n+2\}$ or if $(i,k) \in \{(1,3),(3,1),(3,2),(3,3)\}$ and $m \notin \{n-1,n+1\}$.

Moreover, it can be deduced that

$$\mathbf{harm}_n^{(i,k)} \subset \bigoplus_{p,q=1}^{3} \bigoplus_{l=-2}^{2} \mathrm{Harm}_l \varepsilon^p \otimes \varepsilon^q \tag{6.258}$$

holds for all $n \in \mathbb{N}_0$ and $i,k \in \{1,2,3\}$. Thus, we know that every homogeneous harmonic tensor polynomial restricted to Ω can be expressed as a finite linear combination of tensor spherical harmonics, and vice versa. Hence, closure and completeness properties also follow directly from Corollary 3.50 and Corollary 3.52.

As an immediate consequence of the completeness of the tensor spherical harmonics in $\mathbf{l}^2(\Omega)$ and the orthogonality of tensor fields $\mathbf{f}, \mathbf{g} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ of the representation $\mathbf{f} = \mathbf{o}^{(i,k)} F$ and $\mathbf{g} = \mathbf{o}^{(i',k')} G$, with $(i,k) \neq (i',k')$, we obtain an orthogonal decomposition of $\mathbf{l}^2(\Omega)$ which generalizes the relations (6.24–6.29) in canonical way. Using the notion

$$\mathbf{l}^2_{(i,k)}(\Omega) = \overline{\mathrm{span}\left\{ \mathbf{y}_{n,j}^{(i,k)} \right\}_{n=0_{i,k},\ldots,\ j=1,\ldots,2n+1}}^{\|\cdot\|_{\mathbf{l}^2(\Omega)}} \tag{6.259}$$

for $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$, we are led to an orthogonal decomposition of $\mathbf{l}^2(\Omega)$ into nine complete subspaces, namely

$$\mathbf{l}^2(\Omega) = \bigoplus_{i,k=1}^{3} \mathbf{l}^2_{(i,k)}(\Omega). \tag{6.260}$$

Obviously, we have

$$\mathbf{l}^2_{(i,k)}(\Omega) = \overline{\left\{ \mathbf{o}^{(i,k)} F \;\middle|\; F \in \mathrm{C}^{(\infty)}(\Omega) \right\}}^{\|\cdot\|_{\mathbf{l}^2(\Omega)}} \tag{6.261}$$

and

$$\mathbf{l}^2_{(i,k)}(\Omega) = \left\{ \mathbf{f} \in \mathbf{l}^2(\Omega) \;\middle|\; O^{(i',k')} \mathbf{f} = 0 \text{ for } \ (i',k') \neq (i,k) \right\}, \tag{6.262}$$

where the differentiation is understood in the weak sense. But this means that we are able to define the corresponding projection operators $\mathbf{p}_{(i,k)}$: $\mathbf{l}^2(\Omega) \longrightarrow \mathbf{l}^2_{(i,k)}(\Omega)$ in standard way.

6.8 Tensorial Beltrami Operator

The problem now is how a tensorial Beltrami operator $\boldsymbol{\Delta}^*$ can be constructed such that the tensor spherical harmonics can be characterized as eigenfunctions of this operator. Our particular aim is to define the operator $\blacktriangle^*$ in such a way that the equations

$$\blacktriangle^* \mathbf{o}^{(i,k)} F = \mathbf{o}^{(i,k)} \Delta^* F, \tag{6.263}$$
$$O^{(i,k)} \blacktriangle^* \mathbf{f} = \Delta^* O^{(i,k)} \mathbf{f}, \tag{6.264}$$

hold for all sufficiently smooth functions $F : \Omega \to \mathbb{R}$ and tensor fields $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$.

As in the vectorial case (cf. Section 5.8), we start by applying the (scalar) Beltrami operator to the cartesian components of tensor spherical harmonics. If $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$ is a spherical tensor field of the form

$$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} F_{i,k}(\xi) \varepsilon^i \otimes \varepsilon^k, \quad \xi \in \Omega, \tag{6.265}$$

we set

$$\Delta^*_\xi \mathbf{f}(\xi) = \sum_{i,k=1}^{3} \left(\Delta^*_\xi F_{i,k}(\xi) \right) \varepsilon^i \otimes \varepsilon^k, \quad \xi \in \Omega. \tag{6.266}$$

The application of the Beltrami operator to the cartesian components of a tensor spherical harmonic is easy. Using the results of Section 6.7, this can be done in the following way: Express firstly $\mathbf{o}^{(i,k)} Y_n$ as a linear combination of restrictions of homogeneous harmonic tensor polynomials to the unit sphere (cf. (6.244) and (6.245)). Since the cartesian components of these polynomials are homogeneous harmonic polynomials when restricted to Ω (of degree $\in \{n-2, n-1, \ldots, n+2\}$), the application of Δ^* yields just the eigenvalues of Δ^*. Transforming these results back, we obtain the following identities.

Lemma 6.18. *Let $Y_n \in \mathrm{Harm}_n$ be a spherical harmonic of degree n. Then we have*

$$\begin{aligned}
\Delta^* \mathbf{o}^{(1,1)} Y_n &= (-n(n+1) - 4) \mathbf{o}^{(1,1)} Y_n + 2(\mathbf{o}^{(1,2)} + \mathbf{o}^{(2,1)} + \mathbf{o}^{(2,2)}) Y_n, \\
\Delta^* \mathbf{o}^{(1,2)} Y_n &= (-n(n+1) - 2) \mathbf{o}^{(1,2)} Y_n \\
&\quad + (2n(n+1)\mathbf{o}^{(1,1)} - 2\mathbf{o}^{(2,1)} - n(n+1)\mathbf{o}^{(2,2)} + \mathbf{o}^{(2,3)}) Y_n, \\
\Delta^* \mathbf{o}^{(2,1)} Y_n &= (-n(n+1) - 2) \mathbf{o}^{(2,1)} Y_n \\
&\quad + (2n(n+1)\mathbf{o}^{(1,1)} - 2\mathbf{o}^{(1,2)} - n(n+1)\mathbf{o}^{(2,2)} + \mathbf{o}^{(2,3)}) Y_n, \\
\Delta^* \mathbf{o}^{(2,2)} Y_n &= (-n(n+1) - 2) \mathbf{o}^{(2,2)} Y_n \\
&\quad + (4\mathbf{o}^{(1,1)} - 2\mathbf{o}^{(2,1)} - 2\mathbf{o}^{(2,3)}) Y_n, \\
\Delta^* \mathbf{o}^{(2,3)} Y_n &= (-n(n+1) + 2) \mathbf{o}^{(2,3)} Y_n \\
&\quad + ((2n(n+1) - 4)\mathbf{o}^{(1,2)} + (2n(n+1) - 4)\mathbf{o}^{(2,1)}) Y_n,
\end{aligned}$$

and

$$\begin{aligned}
\Delta^*\mathbf{o}^{(1,3)}Y_n &= (-n(n+1)-2)\mathbf{o}^{(1,3)}Y_n \\
&\quad + (-2\mathbf{o}^{(3,1)} + \mathbf{o}^{(3,2)} + n(n+1)\mathbf{o}^{(3,3)})Y_n, \\
\Delta^*\mathbf{o}^{(3,1)}Y_n &= (-n(n+1)-2)\mathbf{o}^{(3,1)}Y_n \\
&\quad + (-2\mathbf{o}^{(1,3)} + \mathbf{o}^{(3,2)} - n(n+1)\mathbf{o}^{(3,3)})Y_n, \\
\Delta^*\mathbf{o}^{(3,2)}Y_n &= (-n(n+1)+2)\mathbf{o}^{(3,2)}Y_n \\
&\quad + ((2n(n+1)-4)\mathbf{o}^{(1,3)} + (2n(n+1)-4)\mathbf{o}^{(3,1)})Y_n, \\
\Delta^*\mathbf{o}^{(3,3)}Y_n &= (-n(n+1)-2)\mathbf{o}^{(3,3)}Y_n + (-2\mathbf{o}^{(1,3)} - 2\mathbf{o}^{(3,2)})Y_n.
\end{aligned}$$

An immediate consequence is that the operator $\blacktriangle^* : \mathbf{c}^{(2)}(\Omega) \longrightarrow \mathbf{c}(\Omega)$ defined by

$$\begin{aligned}
\blacktriangle^* &= \mathbf{p}_{\mathrm{nor,nor}}(\Delta^*+4)\mathbf{p}_{\mathrm{nor,nor}} + \mathbf{p}_{\mathrm{nor,tan}}(\Delta^*+2)\mathbf{p}_{\mathrm{nor,tan}} \\
&\quad + \mathbf{p}_{\mathrm{tan,nor}}(\Delta^*+2)\mathbf{p}_{\mathrm{tan,nor}} + \mathbf{p}_{(2,2)}(\Delta^*+2)\mathbf{p}_{(2,2)} \\
&\quad + \mathbf{p}_{(2,3)}(\Delta^*-2)\mathbf{p}_{(2,3)} + \mathbf{p}_{(3,2)}(\Delta^*-2)\mathbf{p}_{(3,2)} \\
&\quad + \mathbf{p}_{(3,3)}(\Delta^*+2)\mathbf{p}_{(3,3)} \qquad (6.267)
\end{aligned}$$

satisfies (6.263) and (6.264) (note that the projection operators are defined in accordance with (6.260)).

Definition 6.19. Let $\mathbf{f} \in \mathbf{c}(\Omega)$ be a tensor field of the form

$$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} F_{ik}(\xi)\varepsilon^i \otimes \varepsilon^k, \quad \xi \in \Omega. \qquad (6.268)$$

Then we define the operator $\mathbf{J} : \mathbf{c}(\Omega) \longrightarrow \mathbf{c}(\Omega)$ by

$$\mathbf{J}_\xi\mathbf{f}(\xi) = \mathbf{f}(\xi) - \sum_{i,k=1}^{3} F_{i,k}(\xi)(\xi \wedge \varepsilon^i) \otimes (\xi \wedge \varepsilon^k), \quad \xi \in \Omega. \qquad (6.269)$$

For $F \in \mathrm{C}(\Omega)$ elementary calculations yields the identities

$$\mathbf{J}\mathbf{o}^{(2,2)}F = 0, \qquad \mathbf{J}\mathbf{o}^{(2,3)}F = 2\mathbf{o}^{(2,3)}F, \qquad (6.270)$$

$$\mathbf{J}\mathbf{o}^{(3,3)}F = 0, \qquad \mathbf{J}\mathbf{o}^{(3,2)}F = 2\mathbf{o}^{(3,2)}F, \qquad (6.271)$$

and

$$\mathbf{J}\mathbf{o}^{(i,k)}F = 0, \qquad (6.272)$$

if $i = 1$ or $k = 1$. Hence, we are able to redefine the operator $\blacktriangle^*$ as follows:

$$\begin{aligned}
\blacktriangle^*\mathbf{f} &= \mathbf{p}_{\mathrm{nor,nor}}(\Delta^*+4)\mathbf{p}_{\mathrm{nor,nor}}\mathbf{f} + \mathbf{p}_{\mathrm{nor,tan}}(\Delta^*+2)\mathbf{p}_{\mathrm{nor,tan}}\mathbf{f} \qquad (6.273) \\
&\quad + \mathbf{p}_{\mathrm{tan,nor}}(\Delta^*+2)\mathbf{p}_{\mathrm{tan,nor}}\mathbf{f} + \mathbf{p}_{\mathrm{tan,tan}}(\Delta^*+2-2\mathbf{J})\mathbf{p}_{\mathrm{tan,tan}}\mathbf{f},
\end{aligned}$$

provided that $\mathbf{f} \in \mathbf{c}^{(2)}\Omega$.

Collecting our results, we obtain the following theorem.

Theorem 6.20. *The operator* $\blacktriangle^*$ *obeys the following properties:*

(i) *The operator* $\blacktriangle^* : \mathbf{c}^{(2)}(\Omega) \to \mathbf{c}(\Omega)$ *satisfies for all* $i,k \in \{1,2,3\}$

$$\begin{aligned} \blacktriangle^* \mathbf{o}^{(i,k)} &= \mathbf{o}^{(i,k)} \Delta^*, \\ O^{(i,k)} \blacktriangle^* &= \Delta^* O^{(i,k)}. \end{aligned}$$

(ii) *Any tensor spherical harmonic* $\mathbf{y}_n \in \mathbf{harm}_n$ *satisfies the relation*

$$\blacktriangle^* \mathbf{y}_n = (\Delta^*)^\wedge(n) \mathbf{y}_n.$$

(iii) *If* $\mathbf{y} \in \mathbf{c}^{(\infty)}(\Omega)$ *satisfies* $\blacktriangle^* \mathbf{y} = \lambda \mathbf{y}$ *for any* $\lambda \in \mathbb{R}$, *then* $\lambda = (\Delta^*)^\wedge(n)$, $n \in \mathbb{N}_0$, *and* $\mathbf{y} \in \mathbf{harm}_n$.

Proof. These statements follow from Lemma 3.24, Corollary 3.50, and the previous results of this chapter. □

6.9 Tensorial Addition Theorem

Next, we deal with the generalization of the addition theorem to tensor spherical harmonics. As usual, let us assume that

$$\mathbf{y}_{n,j}^{(i,k)} = \left(\mu_n^{(i,k)}\right)^{-1/2} \mathbf{o}^{(i,k)} Y_{n,j} \tag{6.274}$$

constitutes an $\mathbf{l}^2(\Omega)$-orthonormal system of tensor spherical harmonics of degree n, order j, and type (i,k). The problem is to evaluate the rank-4 tensor

$$\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(l,m)}(\eta), \quad (\xi,\eta) \in \Omega^2 = \Omega \times \Omega. \tag{6.275}$$

and to establish rank-4 tensorial versions of the Legendre polynomial for $(i,k),(l,m) \in \{(1,1),(1,2),\ldots,(3,3)\}$.

To this end, we first need an extension of the operators $\mathbf{o}^{(i,k)}$ to (sufficiently smooth) rank-2 tensor fields. More explicitly, let $\mathbf{f} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ be a smooth tensor field of the form

$$\mathbf{f}(\xi) = \sum_{i,k=1}^{3} F_{ik}(\xi) \varepsilon^i \otimes \varepsilon^k, \quad \xi \in \Omega. \tag{6.276}$$

Then, for $(l,m) \in \{(1,1),(1,2),\dots,(3,3)\}$ we set

$$\mathbf{o}_\xi^{(l,m)} \mathbf{f}(\xi) = \sum_{i,k=1}^{3} \left(\mathbf{o}_\xi^{(l,m)} F_{ik}(\xi) \right) \otimes \varepsilon^i \otimes \varepsilon^k. \tag{6.277}$$

In other words, $\mathbf{o}^{(l,m)}\mathbf{f}$ is a rank-4 tensor. Observing the setting (6.277), we are able to reformulate the expression (6.275) as follows:

$$\begin{aligned} &\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(l,m)}(\eta) \qquad (6.278)\\ &= \left(\mu_n^{(i,k)}\right)^{-(1/2)} \left(\mu_n^{(l,m)}\right)^{-(1/2)} \sum_j \mathbf{o}_\xi^{(i,k)} Y_{n,j}(\xi) \otimes \mathbf{o}_\eta^{(l,m)} Y_{n,j}(\eta) \\ &= \left(\mu_n^{(i,k)}\right)^{-(1/2)} \left(\mu_n^{(l,m)}\right)^{-(1/2)} \mathbf{o}_\xi^{(i,k)} \mathbf{o}_\eta^{(l,m)} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) \\ &= \left(\mu_n^{(i,k)}\right)^{-(1/2)} \left(\mu_n^{(l,m)}\right)^{-(1/2)} \frac{2n+1}{4\pi} \mathbf{o}_\xi^{(i,k)} \mathbf{o}_\eta^{(l,m)} P_n(\xi \cdot \eta). \end{aligned}$$

Introducing the rank-4 tensor field

$$\mathbf{P}_n^{(i,k,l,m)}(\xi,\eta) : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \tag{6.279}$$

for $(i,k),(l,m) \in \{(1,1),(1,2),\dots,(3,3)\}$, by letting

$$\mathbf{P}_n^{(i,k,l,m)}(\xi,\eta) = (\mu_n^{(i,k)})^{-1/2} (\mu_n^{(l,m)})^{-1/2} \mathbf{o}_\xi^{(i,k)} \mathbf{o}_\eta^{(l,m)} P_n(\xi \cdot \eta), \quad \xi,\eta \in \Omega, \tag{6.280}$$

we are therefore led to the following formulation of the *addition theorem for tensor spherical harmonics* involving Legendre rank-4 tensor kernels.

Theorem 6.21. *Let* $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}_{\substack{i,k=1,2,3\\ j=1,\dots,2n+1}}$ *be an* $\mathbf{l}^2(\Omega)$*-orthonormal system of tensor spherical harmonics in* $\mathbf{harm}_n$*. For index pairs* $(i,k),(l,m) \in \{(1,1),(1,2),\dots,(3,3)\}$ *and points* $\xi,\eta \in \Omega$*, we have*

$$\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(l,m)}(\eta) = \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,l,m)}(\xi,\eta). \tag{6.281}$$

Definition 6.22. The kernel $\mathbf{P}_n^{(i,k,l,m)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$, with $i,k,l,\ m \in \{1,2,3\}$, (more precisely, ${}^t\mathbf{P}_n^{(i,k,l,m)}$) given by

$$\frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,l,m)}(\xi,\eta) = \sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(l,m)}(\eta) \tag{6.282}$$

is called the *(tensorial) Legendre rank-4 tensor kernel of degree* n *and type* (i,k,l,m) *(with respect to the dual systems of operators* $\mathbf{o}^{(i,k)}, O^{(i,k)}$, $i,k \in \{1,2,3\}$ *).* The kernel

$$\mathbf{P}_n = \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \mathbf{P}_n^{(i,k,l,m)} \tag{6.283}$$

is called *(tensorial) Legendre rank-4 tensor kernel of degree* n *(with respect to the dual system of operators* $\mathbf{o}^{(i,k)}, O^{(i,k)}$, $i,k = 1,2,3$*).*

Of course, it remains to express the Legendre rank-4 tensor $\mathbf{P}_n^{(i,k,l,m)}(\xi,\eta)$ in explicit form. The essential tool is the next lemma which can be verified by use of local coordinates as introduced in (2.94).

Lemma 6.23. *Suppose that* F *is of class* $\mathrm{C}^{(2)}[-1,1]$. *Let* $\eta \in \Omega$ *fixed. Then, for* $\xi \in \Omega$,

$$\begin{aligned}
\mathbf{o}_\xi^{(1,1)} F(\xi\cdot\eta) &= F(\xi\cdot\eta)\xi\otimes\xi, & (6.284)\\
\mathbf{o}_\xi^{(1,2)} F(\xi\cdot\eta) &= F'(\xi\cdot\eta)\xi\otimes(\eta-(\xi\cdot\eta)\xi), & (6.285)\\
\mathbf{o}_\xi^{(1,3)} F(\xi\cdot\eta) &= F'(\xi\cdot\eta)\xi\otimes(\xi\wedge\eta), & (6.286)\\
\mathbf{o}_\xi^{(2,1)} F(\xi\cdot\eta) &= F'(\xi\cdot\eta)(\eta-(\xi\cdot\eta)\xi)\otimes\xi, & (6.287)\\
\mathbf{o}_\xi^{(2,2)} F(\xi\cdot\eta) &= F(\xi\cdot\eta)\mathbf{i}_{\mathrm{tan}}(\xi), & (6.288)\\
\mathbf{o}_\xi^{(2,3)} F(\xi\cdot\eta) &= F''(\xi\cdot\eta)\left[(\eta-(\xi\cdot\eta)\xi)\otimes\right.\\
&\quad \left.(\eta-(\xi\cdot\eta)\xi)-(\xi\wedge\eta)\otimes(\xi\wedge\eta)\right], & (6.289)\\
\mathbf{o}_\xi^{(3,1)} F(\xi\cdot\eta) &= F'(\xi\cdot\eta)(\xi\wedge\eta)\otimes\xi, & (6.290)\\
\mathbf{o}_\xi^{(3,2)} F(\xi\cdot\eta) &= F''(\xi\cdot\eta)\left[(\eta-(\xi\cdot\eta)\xi)\otimes\right.\\
&\quad \left.(\xi\wedge\eta)+(\xi\wedge\eta)\otimes(\eta-(\xi\cdot\eta)\xi)\right], & (6.291)\\
\mathbf{o}_\xi^{(3,3)} F(\xi\cdot\eta) &= F(\xi\cdot\eta)\mathbf{j}_{\mathrm{tan}}(\xi). & (6.292)
\end{aligned}$$

Combining Lemma 6.23 with (6.277) and (6.280), we obtain the following theorem after some lengthy calculations (see M. Schreiner (1994)).

Theorem 6.24. *Assume that* $\Phi_n^1,\ldots,\Phi_n^9$, $n\in\mathbb{N}_0$, *and* $t\in(-1,1)$ *are defined by*

$$\begin{aligned}
\Phi_n^1(t) &= P_n(t), \\
\Phi_n^2(t) &= P_n'(t), \\
\Phi_n^3(t) &= \sqrt{1-t^2}P_n'(t), \\
\Phi_n^4(t) &= -n(n+1)P_n(t) + 2tP_n'(t), \\
\Phi_n^5(t) &= -\frac{n(n+1)}{1-t^2}P_n(t) + \frac{2t}{1-t^2}P_n'(t), \\
\Phi_n^6(t) &= -n(n+1)P_n(t) + tP_n'(t), \\
\Phi_n^7(t) &= 2\frac{n(n+1)t}{\sqrt{1-t^2}}P_n(t) - \frac{2t^2+n^2t^2+nt^2-n^2-n+2}{\sqrt{1-t^2}}P_n'(t), \\
\Phi_n^8(t) &= -\frac{n(n+1)(n^2t^2+nt^2+4t^2+8-n-n^2)}{1-t^2}P_n(t) \\
&\quad + 4\frac{t(t^2+n^2t^2+nt^2+5-n^2-n)}{1-t^2}P_n'(t), \\
\Phi_n^9(t) &= 3\frac{n(n+1)t}{1-t^2}P_n(t) - \frac{4t^2+n^2t^2+nt^2+2-n^2-n}{1-t^2}P_n'(t).
\end{aligned}$$

Suppose that $\xi,\eta \in \Omega$ *with* $\xi \neq \pm\eta$*, or, equivalently,* $(\xi\cdot\eta)^2 \neq 1$*. Let us define - as usual - the orthonormal sets of vectors* $\left\{\varepsilon_\xi^1, \varepsilon_\xi^2, \varepsilon_\xi^3\right\}$ *and* $\left\{\varepsilon_\eta^1, \varepsilon_\eta^2, \varepsilon_\eta^3\right\}$ *by*

$$\begin{aligned}
\varepsilon_\xi^1 &= \xi, & \varepsilon_\eta^1 &= \eta, \\
\varepsilon_\xi^2 &= \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}(\eta - (\xi\cdot\eta)\xi), & \varepsilon_\eta^2 &= \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}(\xi - (\xi\cdot\eta)\eta), \\
\varepsilon_\xi^3 &= \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}\xi\wedge\eta, & \varepsilon_\eta^3 &= \frac{1}{\sqrt{1-(\xi\cdot\eta)^2}}\eta\wedge\xi.
\end{aligned}$$

Then, we find

$$\mathbf{P}_n^{(i,k,l,m)}(\xi,\eta) = (\mu_n^{(i,k)})^{-1/2}(\mu_n^{(l,m)})^{-1/2}\hat{\mathbf{P}}_n^{(i,k,l,m)}(\xi,\eta), \tag{6.293}$$

where

$$\begin{aligned}
\hat{\mathbf{P}}_n^{(1,1,1,1)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^1, \\
\hat{\mathbf{P}}_n^{(1,1,1,2)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^2, \\
\hat{\mathbf{P}}_n^{(1,1,1,3)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^3, \\
\hat{\mathbf{P}}_n^{(1,1,2,1)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1, \\
\hat{\mathbf{P}}_n^{(1,1,2,2)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 + \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3, \\
\hat{\mathbf{P}}_n^{(1,1,2,3)}(\xi,\eta) &= \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,
\end{aligned}$$

$$
\begin{aligned}
\hat{\mathbf{P}}_n^{(1,1,3,1)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(1,1,3,2)}(\xi,\eta) &= -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(1,1,3,3)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(1,2,1,2)}(\xi,\eta) &= \Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^2-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(1,2,1,3)}(\xi,\eta) &= -\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^3-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(1,2,2,1)}(\xi,\eta) &= \Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(1,2,2,2)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2+\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(1,2,2,3)}(\xi,\eta) &= \Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2-\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3\\
&\quad -2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3+\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2),\\
\hat{\mathbf{P}}_n^{(1,2,3,1)}(\xi,\eta) &= -\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(1,2,3,2)}(\xi,\eta) &= -\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2-\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3),\\
\hat{\mathbf{P}}_n^{(1,2,3,3)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(1,3,1,3)}(\xi,\eta) &= -\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^2+\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^1\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(1,3,2,1)}(\xi,\eta) &= -\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1-\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(1,3,2,2)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2-\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(1,3,2,3)}(\xi,\eta) &= -2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3+\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2)\\
&\quad -\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2+\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(1,3,3,1)}(\xi,\eta) &= -\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1+\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(1,3,3,2)}(\xi,\eta) &= 2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3-\varepsilon_\xi^1\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2)\\
&\quad +\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3+\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(1,3,3,3)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3+\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^1\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(2,1,2,1)}(\xi,\eta) &= \Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(2,1,2,2)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2+\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(2,1,2,3)}(\xi,\eta) &= \Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2-\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3\\
&\quad -2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3+\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2),\\
\hat{\mathbf{P}}_n^{(2,1,3,1)}(\xi,\eta) &= -\Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1-\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(2,1,3,2)}(\xi,\eta) &= -\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2-\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3),\\
\hat{\mathbf{P}}_n^{(2,1,3,3)}(\xi,\eta) &= \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3-\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(2,2,2,2)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2+\Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3\\
&\quad +\Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2+\Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathbf{P}}_n^{(2,2,2,3)}(\xi,\eta) &= \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3\\
&\quad +\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(2,2,3,1)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1 - \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(2,2,3,2)}(\xi,\eta) &= -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(2,2,3,3)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad +\Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(2,3,2,3)}(\xi,\eta) &= \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3\\
&\quad +4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2)\\
&\quad +4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2)\\
&\quad -\Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 + \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3,\\
\hat{\mathbf{P}}_n^{(2,3,3,1)}(\xi,\eta) &= -\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1 - 2\,\Phi_n^5(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1\\
&\quad -2\,\Phi_n^5(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1 + \Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(2,3,3,2)}(\xi,\eta) &= -\Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad +4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3)\\
&\quad +4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3)\\
&\quad +\Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(2,3,3,3)}(\xi,\eta) &= \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(3,1,3,1)}(\xi,\eta) &= -\Phi_n^2(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^1 + \Phi_n^6(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^1,\\
\hat{\mathbf{P}}_n^{(3,1,3,2)}(\xi,\eta) &= 2\,\Phi_n^5(\xi\cdot\eta)(\varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3 - \varepsilon_\xi^2\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2)\\
&\quad +\Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^7(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(3,1,3,3)}(\xi,\eta) &= -\Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^3(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^1\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(3,2,3,2)}(\xi,\eta) &= 4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \varepsilon_\xi^2\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3)\\
&\quad +\Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad +\Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^8(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -4\,\Phi_n^9(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^2 - \varepsilon_\xi^3\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^3),\\
\hat{\mathbf{P}}_n^{(3,2,3,3)}(\xi,\eta) &= -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad -\Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 + \Phi_n^4(\xi\cdot\eta)\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2,\\
\hat{\mathbf{P}}_n^{(3,3,3,3)}(\xi,\eta) &= \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3 - \Phi_n^1(\xi\cdot\eta)\varepsilon_\xi^2\otimes\varepsilon_\xi^3\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2\\
&\quad +\Phi_n^1(\xi\cdot\eta)(\varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^3\otimes\varepsilon_\eta^2 - \varepsilon_\xi^3\otimes\varepsilon_\xi^2\otimes\varepsilon_\eta^2\otimes\varepsilon_\eta^3).
\end{aligned}
$$

Remark 6.25. (i) The cases not listed in Theorem 6.24 can be easily realized by symmetry arguments. (ii) If $\xi = \pm\eta$, the value of $\mathbf{P}_n^{(i,k,l,m)}(\xi, \pm\xi)$ can be evaluated by taking the limit $\eta \to \xi$ and (if necessary) via the relations (6.255) and (6.256).

Since the cases $(i,k) = (l,m)$ are of particular importance, we are interested in the explicit representation of $\mathbf{P}_n^{(i,k,i,k)}(\xi, \xi)$, in addition. As preparation, we introduce two abbreviations indicating special tensors which turn out to be useful in the formulation of the next theorem, namely

$$\mathbf{A} = \sum_{i=1}^{3} \varepsilon^i \otimes \varepsilon^i \otimes \varepsilon^i \otimes \varepsilon^i - \sum_{\substack{i,k=1 \\ i \neq k}}^{3} \varepsilon^i \otimes \varepsilon^i \otimes \varepsilon^k \otimes \varepsilon^k, \tag{6.294}$$

$$\mathbf{B} = \sum_{\substack{i,k=1 \\ i \neq k}}^{3} \left(\varepsilon^i \otimes \varepsilon^k \otimes \varepsilon^i \otimes \varepsilon^k + \varepsilon^i \otimes \varepsilon^k \otimes \varepsilon^k \otimes \varepsilon^i \right), \tag{6.295}$$

($\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ is the canonical orthonormal basis in $\mathbb{R}^3$). It can be easily seen that these tensorial settings do not depend on the special choice of the orthonormal basis. Furthermore, in analogy to the definition of $\mathbf{p}_{\mathrm{tan,tan}}$ (given in Section 6.1), we introduce an operator $\mathbf{P}_{\mathrm{tan}}$, projecting a rank-4 tensor on its tangential part in $\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ by letting

$$\mathbf{A}_{\mathrm{tan}}(\xi) = \mathbf{P}_{\mathrm{tan}} \mathbf{A}(\xi), \ \xi \in \Omega, \tag{6.296}$$

$$\mathbf{B}_{\mathrm{tan}}(\xi) = \mathbf{P}_{\mathrm{tan}} \mathbf{B}(\xi), \ \xi \in \Omega. \tag{6.297}$$

The proof of the next theorem now follows by taking the limit $\eta \to \xi$ in Theorem 6.24, thereby keeping in mind that $P_n(1) = 1$ and $P_n'(1) = n(n+1)/2$.

Theorem 6.26. *Let $n \in \mathbb{N}_0$, $\xi \in \Omega$. Then the following statements are valid:*

$$\mathbf{P}_n^{(1,1,1,1)}(\xi, \xi) = \xi \otimes \xi \otimes \xi \otimes \xi,$$

$$\mathbf{P}_n^{(1,2,1,2)}(\xi, \xi) = \frac{1}{2} \left(\sum_{i=1}^{3} \xi \otimes \varepsilon^i \otimes \xi \otimes \varepsilon^i - \xi \otimes \xi \otimes \xi \otimes \xi \right),$$

$$\mathbf{P}_n^{(1,3,1,3)}(\xi, \xi) = \frac{1}{2} \left(\sum_{i=1}^{3} \xi \otimes \varepsilon^i \otimes \xi \otimes \varepsilon^i - \xi \otimes \xi \otimes \xi \otimes \xi \right),$$

$$\mathbf{P}_n^{(2,1,2,1)}(\xi, \xi) = \frac{1}{2} \left(\sum_{i=1}^{3} \varepsilon^i \otimes \xi \otimes \varepsilon^i \otimes \xi - \xi \otimes \xi \otimes \xi \otimes \xi \right),$$

$$\mathbf{P}_n^{(2,2,2,2)}(\xi, \xi) = \frac{1}{2} \mathbf{i}_{\mathrm{tan}}(\xi) \otimes \mathbf{i}_{\mathrm{tan}}(\xi),$$

$$\begin{aligned}
\mathbf{P}_n^{(2,3,2,3)}(\xi,\xi) &= -\frac{1}{2}\frac{n(n+1)+4}{n(n+1)-2}\mathbf{A}_{\tan}(\xi) + \frac{n(n+1)+1}{n(n+1)-2}\mathbf{B}_{\tan}(\xi), \\
\mathbf{P}_n^{(3,1,3,1)}(\xi,\xi) &= \frac{1}{2}n(n+1)\left(\sum_{i=1}^{3}\varepsilon^i\otimes\xi\otimes\varepsilon^i\otimes\xi - \xi\otimes\xi\otimes\xi\otimes\xi\right), \\
\mathbf{P}_n^{(3,2,3,2)}(\xi,\xi) &= \frac{n(n+1)+1}{n(n+1)-2}\mathbf{A}_{\tan}(\xi) - \frac{1}{2}\frac{n(n+1)+4}{n(n+1)-2}\mathbf{B}_{\tan}(\xi), \\
\mathbf{P}_n^{(3,3,3,3)}(\xi,\xi) &= \frac{1}{2}\mathbf{j}_{\tan}(\xi)\otimes\mathbf{j}_{\tan}(\xi).
\end{aligned}$$

Observing (6.255) and (6.256), the values of $\mathbf{P}_n^{(i,k,i,k)}(\xi,-\xi)$ can be derived immediately from this theorem.

If $\mathbf{T}$ is a rank-4 tensor of the form

$$\mathbf{T} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} T_{ijkl}\varepsilon^i\otimes\varepsilon^j\otimes\varepsilon^k\otimes\varepsilon^l, \tag{6.298}$$

then its trace is defined by (cf. M.E. Gurtin (1971))

$$\operatorname{trace}\mathbf{T} = \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ikik}. \tag{6.299}$$

In consequence, we easily see that the trace of $\mathbf{P}_n^{(i,k,i,k)}(\xi,\xi)$ is given by

$$\operatorname{trace}\mathbf{P}_n^{(i,k,i,k)}(\xi,\xi) = 1. \tag{6.300}$$

Moreover, the next result is a direct consequence of Theorem 6.21.

Lemma 6.27. *For an $\mathbf{l}^2(\Omega)$–orthonormal system $\left\{\mathbf{y}_{n,j}^{(i,k)}\right\}_{j=1,\ldots,2n+1}$ of tensor spherical harmonics of degree n and for $(i,k)\in\{(1,1),(1,2),\ldots,(3,3)\}$, $n\geq 0_{i,k}$, and $\xi\in\Omega$ we have*

$$\sum_{j=1}^{2n+1}\mathbf{y}_{n,j}^{(i,k)}(\xi)\cdot\mathbf{y}_{n,j}^{(i,k)}(\xi) = \frac{2n+1}{4\pi}. \tag{6.301}$$

Every tensor spherical harmonic $\mathbf{y}_n^{(i,k)}\in\mathbf{harm}_n^{(i,k)}$ can be written as linear combination $\mathbf{y}_n^{(i,k)} = \sum_{j=1}^{2n+1} c_j\mathbf{y}_{n,j}^{(i,k)}$ with $c_j = (\mathbf{y}_n^{(i,k)},\mathbf{y}_{n,j}^{(i,k)})_{\mathbf{l}^2(\Omega)}$. By the Cauchy–Schwarz inequality, we therefore obtain for all $\xi\in\Omega$

$$\left|\mathbf{y}_n^{(i,k)}(\xi)\right|^2 \leq \left(\sum_{j=1}^{2n+1} c_j^2\right)\left(\sum_{j=1}^{2n+1}\left|\mathbf{y}_{n,j}^{(i,k)}(\xi)\right|^2\right). \tag{6.302}$$

From Parseval's identity we are able to deduce that

$$\sum_{j=1}^{2n+1} c_j^2 = \sum_{j=1}^{2n+1} (\mathbf{y}_n^{(i,k)}, \mathbf{y}_{n,j}^{(i,k)})^2_{\mathbf{l}^2(\Omega)} = \left\| \mathbf{y}_n^{(i,k)} \right\|^2_{\mathbf{l}^2(\Omega)}. \tag{6.303}$$

Therefore we find, in connection with Lemma 6.27, the following estimates.

Lemma 6.28. *For any tensor spherical harmonic* $\mathbf{y}_n^{(i,k)}$ *of class* $\mathbf{harm}_n^{(i,k)}$ *we have*

$$\sup_{\xi\in\Omega} \left| \mathbf{y}_n^{(i,k)}(\xi) \right| \leq \sqrt{\frac{2n+1}{4\pi}} \left\| \mathbf{y}_n^{(i,k)} \right\|_{\mathbf{l}^2(\Omega)}. \tag{6.304}$$

In particular,

$$\sup_{\xi\in\Omega} \left| \mathbf{y}_{n,j}^{(i,k)}(\xi) \right| \leq \sqrt{\frac{2n+1}{4\pi}}. \tag{6.305}$$

It follows from the previous investigations that $\mathbf{P}_n^{(i,k,i,k)}(\cdot,\cdot)$ is the reproducing kernel of $\mathbf{harm}_n^{(i,k)}$ in the sense that

(i) For all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{P}_n^{(i,k,i,k)}(\xi, \cdot) \in \mathbf{harm}_n^{(i,k)} \tag{6.306}$$

(ii) For every $\mathbf{f} \in \mathbf{harm}_n^{(i,k)}$ and all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{f}(\xi) = (O_\xi^{(i,k)} \mathbf{P}_n^{(i,k,i,k)}(\xi, \cdot), \mathbf{f})_{\mathbf{l}^2(\Omega)} \tag{6.307}$$

At the end of this section, we want to mention an analogue of the estimate $|P_n(t)| \leq 1$, $t \in [-1,1]$ for the Legendre rank-4 tensor $\mathbf{P}_n^{(i,k,l,m)}(\xi,\eta)$. For that purpose, let $\mathbf{a} \in \mathbb{R}^3 \otimes \mathbb{R}^3$ and $\eta \in \Omega$ be fixed. Then

$$\mathbf{P}_n^{(i,k,l,m)}(\cdot, \eta) \cdot \mathbf{a} = \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} \left(\mathbf{y}_{n,j}^{(l,m)}(\eta) \cdot \mathbf{a} \right) \mathbf{y}_{n,j}^{(i,k)} \tag{6.308}$$

is a tensor spherical harmonic of order n and type (i,k,l,m). Hence, we obtain, from Lemma 6.28 and (6.301), that

$$\begin{aligned} \left| \mathbf{P}_n^{(i,k,l,m)}(\xi,\eta) \cdot \mathbf{a} \right|^2 &\leq \frac{2n+1}{4\pi} \left\| \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} \left(\mathbf{y}_{n,j}^{(l,m)}(\eta) \cdot \mathbf{a} \right) \right\|^2_{\mathbf{l}^2(\Omega)} \\ &= \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} \left(\mathbf{y}_{n,j}^{(l,m)}(\eta) \cdot \mathbf{a} \right)^2 \\ &\leq |\mathbf{a}|^2. \end{aligned} \tag{6.309}$$

This finally leads us to the following result.

Lemma 6.29. *Let $i, k, l, m, p, q \in \{1, 2, 3\}$. Then, for all $\xi, \eta \in \Omega$,*

$$\left| \mathbf{P}_n^{(i,k,l,m)}(\xi, \eta) \cdot (\varepsilon^p \otimes \varepsilon^q) \right| \leq 1. \tag{6.310}$$

6.10 Tensorial Funk–Hecke Formulas

As in the vectorial case, our purpose is to prove two different generalizations of the Funk–Hecke formula known from the scalar theory. In order to specify these variants, we have to discuss the following problems:

(i) Let $\eta \in \Omega$ be fixed and $\mathbf{h}(\cdot, \eta) \in C^{(2)}(\Omega)$ be invariant with respect to rotations $\mathbf{t} \in SO(3)$ satisfying $\mathbf{t}\eta = \eta$. What is the value of

$$\int_\Omega \mathbf{h}(\xi, \eta) \cdot \mathbf{y}_n^{(i,k)}(\xi) \, d\omega(\xi) \tag{6.311}$$

for $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$?

(ii) Let $H \in \mathrm{L}^1[-1, 1]$. How can the integral

$$\int_\Omega H(\xi \cdot \eta) \mathbf{y}_n^{(i,k)}(\xi) \, d\omega(\xi) \tag{6.312}$$

be determined for $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$?

Notice that the integral (6.311) is scalar-valued, while the value of (6.312) is a tensor. This difference goes along with the different methods of investigating the two formulas. For the first version of the Funk–Hecke formula, we have to discuss rotational invariant tensor fields (i.e., we have to consider representations of the group $SO(3)$); the second one can be established using the relations between tensor spherical harmonics and homogeneous harmonic tensor polynomials (as described in Section 6.7).

Let us first recapitulate the definition of the operator $R_\mathbf{t}$ to the tensorial situation (see Section 2.7): Let $\mathbf{f}$ be a tensor field of class $\mathbf{l}^2(\Omega)$. Assume that $\mathbf{t}$ is of class $SO(3)$. Then we set

$$R_\mathbf{t}\mathbf{f}(\xi) = \mathbf{t}^T \mathbf{f}(\mathbf{t}\xi)\mathbf{t}, \quad \xi \in \Omega. \tag{6.313}$$

Let $G \subset SO(3)$ be a subgroup of $SO(3)$. A subspace $\mathbf{v} \subset \mathbf{l}^2(\Omega)$ has been called *invariant with respect to* G or simply G*-invariant* if $\mathbf{f} \in \mathbf{v}$ implies that $R_\mathbf{t}\mathbf{f} \in \mathbf{v}$ for all $\mathbf{t} \in G$. If a G-invariant subspace $\mathbf{v}$ does not contain a subspace which is also G-invariant (besides $\mathbf{v}$ itself), then $\mathbf{v}$ is called *irreducible.* It is obvious that, for $\mathbf{f}, \mathbf{g} \in \mathbf{l}^2(\Omega)$, we have

$$(R_\mathbf{t}\mathbf{f}, \mathbf{g})_{\mathbf{l}^2(\Omega)} = (\mathbf{f}, R_{\mathbf{t}^T}\mathbf{g})_{\mathbf{l}^2(\Omega)}, \tag{6.314}$$

i.e., $R_{\mathbf{t}^T}$ is the adjoint operator of $R_{\mathbf{t}}$. For $F \in \mathrm{C}^{(2)}(\Omega)$ and $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$, it can be easily verified that

$$\mathbf{o}_\xi^{(i,k)} R_{\mathbf{t}} F(\xi) = R_{\mathbf{t}} \mathbf{o}_\xi^{(i,k)} F(\xi), \tag{6.315}$$

$$O_\xi^{(i,k)} R_{\mathbf{t}} \mathbf{f}(\xi) = R_{\mathbf{t}} O_\xi^{(i,k)} \mathbf{f}(\xi) \tag{6.316}$$

hold for all $\xi \in \Omega$ and $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$.

Therefore we find, in connection with the results of Section 2.7, the following properties:

(i) For all $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$, the space $\mathbf{l}^2_{(i,k)}(\Omega) \subset \mathbf{l}^2(\Omega)$ is $SO(3)$-invariant.

(ii) The space $\mathbf{harm}_n^{(i,k)}$ is an irreducible invariant subspace of $\mathbf{l}^2(\Omega)$ with respect to $SO(3)$ for all $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$ and all $n \geq 0_{i,k}$.

Furthermore, we are led to the following statements:

Lemma 6.30. *Let $\eta \in \Omega$ be fixed. Then, the following statements are valid:*

(i) *If $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$ with $R_{\mathbf{t}}\mathbf{f} = \mathbf{f}$ for all $\mathbf{t} \in SO_\eta(3)$, then there exist $F_{i,k} \in \mathrm{C}[-1,1]$, $i,k \in \{1,2,3\}$, such that*

$$O^{(i,k)} \mathbf{f}(\xi) = F_{i,k}(\xi \cdot \eta), \quad \xi \in \Omega. \tag{6.317}$$

(ii) *Let $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$ and $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$ such that $R_{\mathbf{t}} \mathbf{y}_n^{(i,k)} = \mathbf{y}_n^{(i,k)}$ for all $\mathbf{t} \in SO_\eta(3)$. Then, there exists a constant $C \in \mathbb{R}$ such that*

$$\mathbf{y}_n^{(i,k)}(\xi) = C\, \mathbf{o}_\xi^{(i,k)} P_n(\xi \cdot \eta), \quad \xi \in \Omega. \tag{6.318}$$

Suppose that $\eta \in \Omega$ is fixed. Assume that $\mathbf{h}(\cdot,\eta) \in \mathbf{c}^{(2)}(\Omega)$ with $R_{\mathbf{t}}\mathbf{h}(\xi,\eta) = \mathbf{h}(\xi,\eta)$ for all $\mathbf{t} \in SO_\eta(3)$. Then, for $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$, the function $O_\xi^{(i,k)} \mathbf{h}(\xi,\eta) = H_{i,k}(\xi \cdot \eta)$ depends only on the inner product $\xi \cdot \eta$. Thus, we are allowed to define

$$\left(O^{(i,k)}\mathbf{h}\right)^\wedge(n) = 2\pi \int_{-1}^{1} H_{i,k}(t) P_n(t)\, dt. \tag{6.319}$$

In fact, for $\mathbf{y}_n^{(i,k)} = \mathbf{o}^{(i,k)} Y_n \in \mathbf{harm}_n^{(i,k)}$, we find

$$\begin{aligned} \int_\Omega \mathbf{h}(\xi,\eta) \cdot \mathbf{y}_n^{(i,k)}(\xi)\, d\omega(\xi) &= \int_\Omega O_\xi^{(i,k)} \mathbf{h}(\xi,\eta) Y_n(\xi)\, d\omega(\xi) \qquad (6.320) \\ &= \left(O^{(i,k)}\mathbf{h}\right)^\wedge(n) Y_n(\eta). \end{aligned}$$

This assures the *first version of the tensorial Funk–Hecke formulas*:

Theorem 6.31. *Let $\eta \in \Omega$ be fixed. Assume that $\mathbf{h}(\cdot,\eta) \in \mathbf{c}^{(2)}(\Omega)$ satisfies*

$$R_{\mathbf{t}}\mathbf{h}(\xi,\eta) = \mathbf{h}(\xi,\eta) \tag{6.321}$$

for all $\mathbf{t} \in SO_\eta(3)$ and $\xi \in \Omega$. Then, for $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$ and $\mathbf{y}_n^{(i,k)} \in \mathbf{harm}_n^{(i,k)}$, $n \geq 0_{i,k}$,

$$\int_\Omega \mathbf{h}(\xi,\eta)\cdot\mathbf{y}_n^{(i,k)}(\xi)\, d\omega(\xi) = (\mu_n^{(i,k)})^{-1}\left(O^{(i,k)}\mathbf{h}\right)^\wedge(n)\; O_\eta^{(i,k)}\mathbf{y}_n^{(i,k)}(\eta), \tag{6.322}$$

where $\left(O^{(i,k)}\mathbf{h}\right)^\wedge(n)$ is given by (6.319)*.*

Next, we are concerned with the second tensorial version of the Funk–Hecke formula, as already anounced in (6.312). Let $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$. Consider $\mathbf{k}_n^{(i,k)} r^n Y_n|_{r=1}$, $n \in \mathbb{N}_0$, $Y_n \in \mathrm{Harm}_n$, which are the restrictions of homogeneous harmonic tensor polynomials to the unit sphere Ω. The cartesian components of $\mathbf{k}_n^{(i,k)} r^n Y_n|_{r=1}$ are spherical harmonics of degree $\deg^{(i,k)}(n)$ (cf. Lemma 6.15). Hence, it follows for $H \in \mathrm{L}^1[-1,1]$ and all $\eta \in \Omega$ that

$$\int_\Omega H(\xi\cdot\eta)\mathbf{k}_n^{(i,k)} r^n Y_n(\xi)|_{r=1}\, d\omega(\xi) = H^\wedge(\deg^{(i,k)}(n))\mathbf{k}_n^{(i,k)} r^n Y_n(\eta)|_{r=1}. \tag{6.323}$$

It is known from Section 6.7 that every tensor spherical harmonic $\mathbf{y}_n \in \mathbf{harm}_n$ of degree n can be expressed as linear combination of restrictions of homogeneous harmonic tensor polynomials of degrees $n-2,\ldots,n+2$ (cf. (6.244) and (6.245)). Therefore, by virtue of Theorem 3.60, the transformation matrices (6.246), (6.247), and their inverses (6.250), (6.251), we arrive at the following result which provides the *second tensorial version of the Funk–Hecke formula.*

Theorem 6.32. *Let $Y_n \in \mathrm{Harm}_n$ be a spherical harmonic of degree n. Moreover, suppose that H is a member of class $\mathrm{L}^1[-1,1]$. Furthermore, let $\eta \in \Omega$ be fixed.*

If $n = 0$, then

$$\int_\Omega H(\xi\cdot\eta)\begin{pmatrix} \mathbf{o}_\xi^{(1,1)}Y_0(\xi) \\ \mathbf{o}_\xi^{(2,2)}Y_0(\xi) \\ \mathbf{o}_\xi^{(3,3)}Y_0(\xi)\end{pmatrix} d\omega(\xi)$$

$$= \begin{pmatrix} \frac{1}{3}\left(H^\wedge(0)+2H^\wedge(2)\right)\mathbf{o}_\eta^{(1,1)}Y_0(\eta) + \frac{1}{3}\left(H^\wedge(0)-H^\wedge(2)\right)\mathbf{o}_\eta^{(2,2)}Y_0(\eta) \\ \frac{2}{3}\left(H^\wedge(0)-H^\wedge(2)\right)\mathbf{o}_\eta^{(1,1)}Y_0(\eta) + \frac{1}{3}\left(2H^\wedge(0)+H^\wedge(2)\right)\mathbf{o}_\eta^{(2,2)}Y_0(\eta) \\ H^\wedge(1)\mathbf{o}_\eta^{(3,3)}Y_0(\eta)\end{pmatrix}.$$

If $n \geq 1$, then

$$\int_\Omega H(\xi\cdot\eta)\begin{pmatrix}\mathbf{o}_\xi^{(1,1)}Y_n(\xi)\\ \mathbf{o}_\xi^{(1,2)}Y_n(\xi)\\ \mathbf{o}_\xi^{(2,1)}Y_n(\xi)\\ \mathbf{o}_\xi^{(2,2)}Y_n(\xi)\\ \mathbf{o}_\xi^{(2,3)}Y_n(\xi)\end{pmatrix} d\omega(\xi)$$

$$= \left(H^\wedge(n-2)\mathbf{m}_{n-2}+H^\wedge(n)\mathbf{m}_n+H^\wedge(n+2)\mathbf{m}_{n+2}\right)\begin{pmatrix}\mathbf{o}_\eta^{(1,1)}Y_n(\eta)\\ \mathbf{o}_\eta^{(1,2)}Y_n(\eta)\\ \mathbf{o}_\eta^{(2,1)}Y_n(\eta)\\ \mathbf{o}_\eta^{(2,2)}Y_n(\eta)\\ \mathbf{o}_\eta^{(2,3)}Y_n(\eta)\end{pmatrix},$$

and

$$\int_\Omega H(\xi\cdot\eta)\begin{pmatrix}\mathbf{o}_\xi^{(1,3)}Y_n(\xi)\\ \mathbf{o}_\xi^{(3,1)}Y_n(\xi)\\ \mathbf{o}_\xi^{(3,2)}Y_n(\xi)\\ \mathbf{o}_\xi^{(3,3)}Y_n(\xi)\end{pmatrix} d\omega(\xi)$$

$$= \left(H^\wedge(n-1)\mathbf{m}_{n-1}+H^\wedge(n+1)\mathbf{m}_{n+1}\right)\begin{pmatrix}\mathbf{o}_\eta^{(1,3)}Y_n(\eta)\\ \mathbf{o}_\eta^{(3,1)}Y_n(\eta)\\ \mathbf{o}_\eta^{(3,2)}Y_n(\eta)\\ \mathbf{o}_\eta^{(3,3)}Y_n(\eta)\end{pmatrix},$$

where the matrices $\mathbf{m}_{n-2},\ldots\mathbf{m}_{n+2}$, respectively, are given by

$$(2n+1)\mathbf{m}_{n-2} =$$
$$\begin{pmatrix}
\frac{(2+n)(n+1)}{4n+6} & \frac{(2+n)(n+1)(n-1)}{(4n-2)n} & \frac{-(2+n)^2}{4n+6} & \frac{(2+n)(n+1)}{4n-2} & \frac{(2+n)(2n+1)}{2n}\\
\frac{(1-n)n}{4n+6} & \frac{(n-1)^2}{4n-2} & \frac{(n-1)n(2+n)}{(4n+6)(n+1)} & \frac{(1-n)n}{4n-2} & \frac{(1-n)(2n+1)}{2n+2}\\
\frac{(2+n)(n+1)}{-(4n+6)} & \frac{(2+n)(n+1)(n-1)}{(4n-2)n} & \frac{(2+n)^2}{4n+6} & \frac{(2+n)(n+1)}{2-4n} & \frac{(n+2)(2n+1)}{-2n}\\
\frac{(n-1)n}{4n+6} & \frac{(n-1)^2}{2-4n} & \frac{(1-n)n(2+n)}{(4n+6)(n+1)} & \frac{(n-1)n}{4n-2} & \frac{(n-1)(2n+1)}{2n+2}\\
\frac{(n+1)n}{4n+6} & \frac{(1-n)(n+1)}{4n-2} & \frac{-(2+n)n}{4n+6} & \frac{(n+1)n}{(4n-2)} & \frac{2n+1}{2}
\end{pmatrix},$$

$$(2n+1)\mathbf{m}_n = \begin{pmatrix} \frac{n(7+5n)}{4n+6} & \frac{(2+n)(n+1)^2}{(2-4n)n} & \frac{(2+n)n}{-(4n+6)} & \frac{(6+3n)(n+1)}{2-4n} & \frac{(2+n)(2n+1)}{-2n} \\ \frac{(n+1)n}{-(4n+6)} & \frac{(n+1)(5n-3)}{4n-2} & \frac{\left(2-3n^2-3n\right)n}{(4n+6)(n+1)} & \frac{(n+1)n}{2-4n} & \frac{(n-1)2n+1)}{2n+2} \\ \frac{(n+1)n}{-(4n+6)} & \frac{(n+1)\left(2-3n^2-3n\right)}{(4n-2)n} & \frac{n(5n+8)}{4n+6} & \frac{(n+1)n}{2-4n} & \frac{(2+n)(2n+1)}{2n} \\ \frac{(3-3n)n}{4n+6} & \frac{(1-n)(n+1)}{4n-2} & \frac{(1-n)n^2}{(4n+6)(n+1)} & \frac{(n+1)(5n-2)}{4n-2} & \frac{(1-n)(2n+1)}{2n+2} \\ \frac{(n+1)n}{-(4n+6)} & \frac{(n-1)(n+1)}{4n-2} & \frac{(2+n)n}{4n+6} & \frac{(n+1)n}{2-4n} & \frac{2n+1}{2} \end{pmatrix},$$

$$(2n+1)\mathbf{m}_{n+2} = \begin{pmatrix} \frac{(2+n)(n+1)}{2n+3} & \frac{(2+n)(n+1)}{2n-1} & \frac{(2+n)(n+1)}{2n+3} & \frac{(2+n)(n+1)}{2n-1} & 0 \\ \frac{n^2}{2n+3} & \frac{n^2}{2n-1} & \frac{n^2}{2n+3} & \frac{n^2}{2n-1} & 0 \\ \frac{(n+1)^2}{2n+3} & \frac{(n+1)^2}{2n-1} & \frac{(n+1)^2}{2n+3} & \frac{(n+1)^2}{2n-1} & 0 \\ \frac{(n-1)n}{2n+3} & \frac{(n-1)n}{2n-1} & \frac{(n-1)n}{2n+3} & \frac{(n-1)n}{2n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(2n+1)\mathbf{m}_{n-1} = \begin{pmatrix} \frac{2+n^2+n}{2n+2} & -\frac{2+n^2+n}{2n} & \frac{(1-n)n}{2n+2} & \frac{(2+n)(n+1)}{2n} \\ -\frac{2+n^2+n}{2n+2} & \frac{2+n^2+n}{2n} & \frac{(n-1)n}{2n+2} & -\frac{(2+n)(n+1)}{2n} \\ \frac{(1-n)(2+n)}{2n+2} & \frac{(n-1)(2+n)}{2n} & \frac{(2+n)(3n+1)}{2n+2} & \frac{(2+n)(n+1)}{2n} \\ \frac{(n-1)(2+n)}{2n+2} & \frac{(1-n)(2+n)}{2n} & \frac{(n-1)n}{2n+2} & \frac{(n-1)(3n+2)}{2n} \end{pmatrix},$$

$$(2n+1)\mathbf{m}_{n+1} = \begin{pmatrix} \frac{(3n+5)n}{2n+2} & \frac{2+n^2+n}{2n} & \frac{(n-1)n}{2n+2} & -\frac{(2+n)(n+1)}{2n} \\ \frac{2+n^2+n}{2n+2} & \frac{(n+1)(3n-2)}{2n} & \frac{(1-n)n}{2n+2} & \frac{(2+n)(n+1)}{2n} \\ \frac{(n-1)(2+n)}{2n+2} & \frac{(1-n)(2+n)}{2n} & \frac{(n-1)n}{2n+2} & -\frac{(2+n)(n+1)}{2n} \\ \frac{(1-n)(2+n)}{2n+2} & \frac{(n-1)(2+n)}{2n} & \frac{(1-n)n}{2n+2} & \frac{(2+n)(n+1)}{2n} \end{pmatrix}.$$

6.11 Counterparts to the Legendre Polynomials

Our considerations about orthogonal expansions in terms of tensor spherical harmonics motivate the following definition.

Definition 6.33. The kernel ${}^t\mathbf{p}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3, i, k \in \{1, 2, 3\}$ given by

$$ {}^t\mathbf{p}_n^{(i,k)}(\xi, \eta) = (\mu_n^{(i,k)})^{-1/2} \mathbf{o}_\xi^{(i,k)} P_n(\xi \cdot \eta) \tag{6.324}$$

is called the *(tensorial) Legendre rank-2 tensor kernel of degree n and type (i, k) with respect to the dual system of operators $\mathbf{o}^{(i,k)}, O^{(i,k)}$, $i, k \in \{1, 2, 3\}$.* The kernel

$$ {}^t\mathbf{p}_n = \sum_{i=1}^{3} \sum_{k=1}^{3} {}^t\mathbf{p}_n^{(i,k)} \tag{6.325}$$

is called *(tensorial) Legendre rank-2 tensor kernel of degree n with respect to the dual system of operators $o^{(i,k)}, O^{(i,k)}$, $i, k = 1, 2, 3$.*

Obviously, the Legendre tensors fulfill an addition theorem.

Theorem 6.34. *Let $\{Y_{n,m}\}_{m=1,\ldots,2n+1}$ be an $\mathrm{L}^2(\Omega)$–orthonormal basis of the space* Harm_n *and let $\{\mathbf{y}_{n,m}^{(i,k)}\}_{m=1,\ldots,2n+1}$ with*

$$\mathbf{y}_{n,m}^{(i,k)} = (\mu_n^{(i,k)})^{-1/2} \mathbf{o}^{(i,k)} Y_{n,m} \tag{6.326}$$

be an $\mathbf{l}^2(\Omega)$–orthonormal basis of $\mathbf{harm}_n^{(i,k)}$. Then

$$\frac{2n+1}{4\pi} {}^t\mathbf{p}_n^{(i,k)}(\xi, \eta) = \sum_{m=1}^{2n+1} \mathbf{y}_{n,m}^{(i,k)}(\xi) Y_{n,m}(\eta), \tag{6.327}$$

$i, k \in \{1, 2, 3\}$.

The relation between the Legendre polynomial of degree n and the Legendre tensors is given by the following lemma.

Lemma 6.35. *Let P_n be the one-dimensional Legendre polynomial of degree n and ${}^t\mathbf{p}_n^{(i,k)}$, $\mathbf{P}_n^{(i,k,l,m)}$ the Legendre tensors as defined above. Then, for $\xi, \eta \in \Omega$,*

$$P_n(\xi \cdot \eta) = \frac{1}{\left(\mu_n^{(i,k)}\right)^{1/2}} \frac{1}{\left(\mu_n^{(l,m)}\right)^{1/2}} O_\xi^{(l,m)} O_\eta^{(i,k)} \mathbf{P}_n^{(i,k,l,m)}(\xi, \eta) \tag{6.328}$$

and

$$P_n(\xi \cdot \eta) = \frac{1}{\left(\mu_n^{(i,k)}\right)^{1/2}} O_\xi^{(i,k)} {}^t\mathbf{p}_n^{(i,k)}(\xi, \eta). \tag{6.329}$$

By use of the addition theorem, we are able to express any rank–2 tensor field on the sphere in terms of the Legendre tensors in the following way:

$$\mathbf{f} = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \int_{\Omega} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,i,k)}(\cdot,\eta)\mathbf{f}^{(i,k)}(\eta)\, d\omega(\eta), \tag{6.330}$$

(in $\mathbf{l}^2(\Omega)$–sense), where the integral is taken componentwise and $\mathbf{f}^{(i,k)} \in \mathbf{l}^2_{(i,k)}(\Omega)$.

It should be noted that $\mathbf{K}_{\mathbf{harm}_n^{(i,k)}} = \frac{2n+1}{4\pi}\mathbf{P}_n^{(i,k,i,k)}$ is the reproducing kernel of the space $\mathbf{harm}_n^{(i,k)}$ in the sense that

(i) for all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{K}_{\mathbf{harm}_n^{(i,k)}}(\cdot,\xi) \in \mathbf{harm}_n^{(i,k)}, \tag{6.331}$$

(ii) for every $\mathbf{f} \in \mathbf{harm}_n^{(i,k)}$ and all $\xi \in \Omega$

$$O_\xi^{(i,k)} \mathbf{f}(\xi) = \left(O_\xi^{(i,k)} \mathbf{K}_{\mathbf{harm}_n^{(i,k)}}(\cdot,\xi), \mathbf{f} \right)_{\mathbf{l}^2(\Omega)}, \tag{6.332}$$

where, for (sufficiently smooth) tensor fields, $\mathbf{F} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ of the form,

$$\mathbf{F}(\xi) = \sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} F_{p,q,r,s}(\xi)\varepsilon^p \otimes \varepsilon^q \otimes \varepsilon^r \otimes \varepsilon^s \tag{6.333}$$

the operators $O^{(i,k)}$ are defined by

$$O^{(i,k)}\mathbf{F}(\xi) = \sum_{r=1}^{3}\sum_{s=1}^{3} O^{(i,k)} \left(\sum_{p=1}^{3}\sum_{q=1}^{3} F_{p,q,r,s}(\xi)\varepsilon^p \otimes \varepsilon^q \right) \varepsilon^r \otimes \varepsilon^s. \tag{6.334}$$

In the vectorial case, two approaches (based on the Legendre tensors and the Legendre vectors) were presented. The analogue to the Legendre vector approach in vectorial theory is an approach involving Legendre tensors of rank two as follows:

$$\begin{aligned} \mathbf{f}(\xi) &= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \left(\mathbf{f}^{(i,k)}, \mathbf{y}_{n,m}^{(i,k)} \right)_{\mathbf{l}^2(\Omega)} \mathbf{y}_{n,m}^{(i,k)}(\xi) \\ &= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \int_{\Omega} \mathbf{f}^{(i,k)}(\eta) \frac{\mathbf{o}_\eta^{(i,k)} Y_{n,m}(\eta)}{(\mu_n^{(i,k)})^{1/2}}\, d\omega(\eta) \frac{\mathbf{o}_\xi^{(i,k)} Y_{n,m}(\xi)}{(\mu_n^{(i,k)})^{1/2}}. \end{aligned} \tag{6.335}$$

6.12 Tensor Spherical Harmonics Related to Tensor Homogeneous Harmonic Polynomials

Up to now, we dealt with the tensor spherical harmonic system $\{\mathbf{y}_{n,m}^{(i,k)}\}_{i,k=1,2,3,\ n=0_{ik},\ldots,\ m=1,\ldots,2n+1}$ with respect to the dual system of operators $\mathbf{o}^{(i,k)}, O^{(i,k)}, i,k \in \{1,2,3\}$, concentrating on the fact that the decomposition into normal and tangential tensor fields is realized. We are now going to introduce an $\mathbf{l}^2(\Omega)$–orthonormal set of tensor spherical harmonics $\{\tilde{\mathbf{y}}_n^{(i,k)}\}_{i,k=1,2,3,\ n=\tilde{0}_{ik},\ldots,\ m=1,\ldots,2n+1}$ such that the functions $\tilde{\mathbf{y}}_{n,m}^{(i,k)}$ are eigenfunctions of the (scalar) Beltrami operator and, therefore, are useful in the (theory) of harmonic functions. More explicitly, it turns out that the new system of tensor spherical harmonics will enable us to introduce so-called inner/outer (solid spherical) tensor harmonics in such a way that they fulfill the Laplace equation in the inner/outer space of a sphere (see Chapter 10).

In the sequel, we understand $\tilde{\mathbf{o}}_n^{(i,k)} Y_n$ to be defined by

$$\tilde{\mathbf{o}}_n^{(i,k)} Y_n(\xi) = \tilde{\mathbf{o}}_n^{(i,k)} H_n(x)|_{r=1}, \tag{6.336}$$

with $H_n(x) = r^n Y_n(\xi)$, $x = r\xi$. In more detail,

$$\begin{pmatrix} \tilde{\mathbf{o}}_n^{(1,1)} Y_n \\ \tilde{\mathbf{o}}_n^{(1,2)} Y_n \\ \tilde{\mathbf{o}}_n^{(2,1)} Y_n \\ \tilde{\mathbf{o}}_n^{(2,2)} Y_n \\ \tilde{\mathbf{o}}_n^{(3,3)} Y_n \end{pmatrix} = \mathbf{a}_n \begin{pmatrix} \mathbf{o}^{(1,1)} Y_n \\ \mathbf{o}^{(1,2)} Y_n \\ \mathbf{o}^{(2,1)} Y_n \\ \mathbf{o}^{(2,2)} Y_n \\ \mathbf{o}^{(2,3)} Y_n \end{pmatrix} \tag{6.337}$$

and

$$\begin{pmatrix} \tilde{\mathbf{o}}_n^{(1,3)} Y_n \\ \tilde{\mathbf{o}}_n^{(2,3)} Y_n \\ \tilde{\mathbf{o}}_n^{(3,1)} Y_n \\ \tilde{\mathbf{o}}_n^{(3,2)} Y_n \end{pmatrix} = \mathbf{b}_n \begin{pmatrix} \mathbf{o}^{(1,3)} Y_n \\ \mathbf{o}^{(3,1)} Y_n \\ \mathbf{o}^{(3,2)} Y_n \\ \mathbf{o}^{(3,3)} Y_n \end{pmatrix}, \tag{6.338}$$

with matrixes $\mathbf{a}_n$ and $\mathbf{b}_n$ as defined in (6.246) and (6.247), respectively. The adjoint operators $\tilde{O}_n^{(i,k)}$ satisfying

$$\left(\tilde{\mathbf{o}}_n^{(i,k)} G, \mathbf{f}\right)_{\mathbf{l}^2(\Omega)} = \left(G, \tilde{O}_n^{(i,k)} \mathbf{f}\right)_{\mathrm{L}^2(\Omega)}, \tag{6.339}$$

$\mathbf{f} \in \mathbf{harm}_n$, $G \in \mathrm{Harm}_n$, are given by

$$\begin{pmatrix} \tilde{O}_n^{(1,1)} \mathbf{f} \\ \tilde{O}_n^{(1,2)} \mathbf{f} \\ \tilde{O}_n^{(2,1)} \mathbf{f} \\ \tilde{O}_n^{(2,2)} \mathbf{f} \\ \tilde{O}_n^{(3,3)} \mathbf{f} \end{pmatrix} = \mathbf{a}_n \begin{pmatrix} O^{(1,1)} G \\ O^{(1,2)} G \\ O^{(2,1)} G \\ O^{(2,2)} G \\ O^{(2,3)} G \end{pmatrix} \tag{6.340}$$

and

$$\begin{pmatrix} \tilde{O}_n^{(1,3)}\mathbf{f} \\ \tilde{O}_n^{(2,3)}\mathbf{f} \\ \tilde{O}_n^{(3,1)}\mathbf{f} \\ \tilde{O}_n^{(3,2)}\mathbf{f} \end{pmatrix} = \mathbf{b}_n \begin{pmatrix} O^{(1,3)}G \\ O^{(3,1)}G \\ O^{(3,2)}G \\ O^{(3,3)}G \end{pmatrix}. \tag{6.341}$$

Further on, by use of the constants $\tilde{\mu}_n^{(i,k)}$

$$\tilde{\mu}_n^{(i,k)} = \|\tilde{O}_n^{(i,k)}\tilde{\mathbf{o}}_n^{(i,k)}Y_n\|_{\mathrm{L}^2(\Omega)} \tag{6.342}$$

we obtain

$$\tilde{\mu}_n^{(1,1)} = (n+2)(n+1)(2n-3)(2n-1), \tag{6.343}$$
$$\tilde{\mu}_n^{(1,2)} = 3n^4, \tag{6.344}$$
$$\tilde{\mu}_n^{(2,1)} = (n+1)^2(2n-3)(2n-1), \tag{6.345}$$
$$\tilde{\mu}_n^{(2,2)} = n(n-1)(2n+1)(2n-1), \tag{6.346}$$
$$\tilde{\mu}_n^{(3,3)} = n^2(n-1)(2n+1), \tag{6.347}$$
$$\tilde{\mu}_n^{(1,3)} = n(n+1)^2(2n+1), \tag{6.348}$$
$$\tilde{\mu}_n^{(2,3)} = n^2(n+2)(n+1), \tag{6.349}$$
$$\tilde{\mu}_n^{(3,1)} = n^2(n+1)(2n+1), \tag{6.350}$$
$$\tilde{\mu}_n^{(3,2)} = n(n+1)^2(2n+1). \tag{6.351}$$

The operators $\tilde{\mathbf{o}}_n^{(i,k)} : \mathrm{Harm}_n \to \widetilde{\mathbf{harm}}_n$, $i,k \in \{1,2,3\}$, admit extensions

$$\tilde{\mathbf{o}}^{(i,k)} : \mathrm{C}^{(\infty)}(\Omega) \to \mathbf{c}^{(\infty)}(\Omega),\ i,k \in \{1,2,3\}, \tag{6.352}$$

by letting

$$\begin{pmatrix} \tilde{\mathbf{o}}^{(1,1)}Y_n \\ \tilde{\mathbf{o}}^{(1,2)}Y_n \\ \tilde{\mathbf{o}}^{(2,1)}Y_n \\ \tilde{\mathbf{o}}^{(2,2)}Y_n \\ \tilde{\mathbf{o}}^{(3,3)}Y_n \end{pmatrix} = \mathbf{a}_D \begin{pmatrix} Y_n \\ Y_n \\ Y_n \\ Y_n \\ Y_n \end{pmatrix} \tag{6.353}$$

and

$$\begin{pmatrix} \tilde{\mathbf{o}}^{(1,3)}Y_n \\ \tilde{\mathbf{o}}^{(2,3)}Y_n \\ \tilde{\mathbf{o}}^{(3,1)}Y_n \\ \tilde{\mathbf{o}}^{(3,2)}Y_n \end{pmatrix} = \mathbf{b}_D \begin{pmatrix} Y_n \\ Y_n \\ Y_n \\ Y_n \end{pmatrix}, \tag{6.354}$$

where the matricial operators $\mathbf{a}_D$ and $\mathbf{b}_D$ are defined by

$$\mathbf{a}_D = \begin{pmatrix} \mathbf{o}^{(1,1)}(D+1)(D+2) & -\mathbf{o}^{(1,2)}(D+2) & -\mathbf{o}^{(2,1)}(D+2) & -\frac{1}{2}o^{(2,2)}(D+2)(D+1) & \frac{1}{2}\mathbf{o}^{(2,3)} \\ \mathbf{o}^{(1,1)}D^2 & \mathbf{o}^{(1,2)}D & -\mathbf{o}^{(2,1)}(D-1) & -\frac{1}{2}\mathbf{o}^{(2,2)}D(D-1) & -\frac{1}{2}\mathbf{o}^{(2,3)} \\ \mathbf{o}^{(1,1)}(D+1)^2 & -\mathbf{o}^{(1,2)}(D+1) & \mathbf{o}^{(2,1)}(D+2) & \frac{1}{2}\mathbf{o}^{(2,2)}(D+2)(D+1) & -\frac{1}{2}\mathbf{o}^{(2,3)} \\ \mathbf{o}^{(1,1)}D(D-1) & \mathbf{o}^{(1,2)}(D-1) & \mathbf{o}^{(2,1)}(D-1) & -\frac{1}{2}\mathbf{o}^{(2,2)}D(D-1) & \frac{1}{2}\mathbf{o}^{(2,3)} \\ 0 & 0 & \mathbf{o}^{(2,1)} & -\frac{1}{2}\mathbf{o}^{(2,2)}D(D+1) & -\frac{1}{2}\mathbf{o}^{(2,3)} \end{pmatrix}, \tag{6.355}$$

$$\mathbf{b_D} = \begin{pmatrix} \mathbf{o}^{(1,3)}(D+1) & \mathbf{o}^{(3,1)} & -\frac{1}{2}\mathbf{o}^{(3,2)} & -\frac{1}{2}\mathbf{o}^{(3,3)}D(D+1) \\ \mathbf{o}^{(1,3)}D & -\mathbf{o}^{(3,1)} & \frac{1}{2}\mathbf{o}^{(3,2)} & \frac{1}{2}\mathbf{o}^{(3,3)}D(D+1) \\ 0 & \mathbf{o}^{(3,1)}(D+2) & -\frac{1}{2}\mathbf{o}^{(3,2)} & \frac{1}{2}\mathbf{o}^{(3,3)}(D+2)(D+1) \\ 0 & \mathbf{o}^{(3,1)}(D-1) & \frac{1}{2}\mathbf{o}^{(3,2)} & -\frac{1}{2}\mathbf{o}^{(3,3)}D(D-1) \end{pmatrix}. \tag{6.356}$$

with D being the (pseudo)differential operator given by (5.294).

In consequence, we are led to introduce the following tensor spherical harmonics

$$\tilde{\mathbf{y}}_{n,m}^{(i,k)} = \left(\tilde{\mu}_n^{(i,k)}\right)^{-1/2} \tilde{\mathbf{o}}^{(i,k)} Y_{n,m}, \tag{6.357}$$

$n = \tilde{0}_{ik}, \ldots, m = 1, \ldots, 2n+1$, where

$$\tilde{0}_{ik} = \begin{cases} 0, & (i,k) \in \{(1,1),(2,1),(3,1)\} \\ 1, & (i,k) \in \{(1,2),(1,3),(2,3),(3,3)\} \\ 2, & (i,k) \in \{(2,2),(3,2)\} \end{cases}. \tag{6.358}$$

Obviously, the system $\{\mathbf{y}_{n,m}^{(i,k)}\}_{i,k=1,2,3,n=0_{ik},\ldots,m=1,\ldots,2n+1}$ and the system $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}_{i,k=1,2,3,n=\tilde{0}_{ik},\ldots;m=1,\ldots,2n+1}$ are related in the following way

$$\begin{pmatrix} \tilde{\mathbf{y}}_{n,m}^{(1,1)} \\ \tilde{\mathbf{y}}_{n,m}^{(1,2)} \\ \tilde{\mathbf{y}}_{n,m}^{(2,1)} \\ \tilde{\mathbf{y}}_{n,m}^{(2,2)} \\ \tilde{\mathbf{y}}_{n,m}^{(3,3)} \end{pmatrix} = \tilde{\alpha}_n^{-1} \alpha_n a_n^{-1} \begin{pmatrix} \mathbf{y}_{n,m}^{(1,1)} \\ \mathbf{y}_{n,m}^{(1,2)} \\ \mathbf{y}_{n,m}^{(2,1)} \\ \mathbf{y}_{n,m}^{(2,2)} \\ \mathbf{y}_{n,m}^{(2,3)} \end{pmatrix}, \tag{6.359}$$

and

$$\begin{pmatrix} \tilde{\mathbf{y}}_{n,m}^{(1,3)} \\ \tilde{\mathbf{y}}_{n,m}^{(2,3)} \\ \tilde{\mathbf{y}}_{n,m}^{(3,1)} \\ \tilde{\mathbf{y}}_{n,m}^{(3,2)} \end{pmatrix} = \tilde{\beta}_n^{-1} \beta_n b_n^{-1} \begin{pmatrix} \mathbf{y}_{n,m}^{(1,3)} \\ \mathbf{y}_{n,m}^{(3,1)} \\ \mathbf{y}_{n,m}^{(3,2)} \\ \mathbf{y}_{n,m}^{(3,3)} \end{pmatrix}, \tag{6.360}$$

where

$$\alpha_n = \begin{pmatrix} \sqrt{\mu_n^{(1,1)}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\mu_n^{(1,2)}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\mu_n^{(2,1)}} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\mu_n^{(2,2)}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\mu_n^{(2,3)}} \end{pmatrix}$$

and

$$\beta_n = \begin{pmatrix} \sqrt{\mu_n^{(1,3)}} & 0 & 0 & 0 \\ 0 & \sqrt{\mu_n^{(2,3)}} & 0 & 0 \\ 0 & 0 & \sqrt{\mu_n^{(3,1)}} & 0 \\ 0 & 0 & 0 & \sqrt{\mu_n^{(3,2)}} \end{pmatrix},$$

and the matrices $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ are defined analogously.

Clearly, the system $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}$ is a set of eigenfunctions of the Beltrami operator. Furthermore, the functions $\tilde{\mathbf{y}}_{n,m}^{(i,k)}$ are eigenfunctions of the tensorial Beltrami operator $\blacktriangle^*$.

Theorem 6.36. *Let* $\{Y_{n,m}\}_{n=0,1,\ldots,\ m=1,\ldots,2n+1}$ *be an* $\mathrm{L}^2(\Omega)$*-orthonormal set of scalar spherical harmonics. Then, the set*

$$\left\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\right\}_{i,k=1,2,3,\ n=\tilde{0}_{ik},\ldots,\ m=1,\ldots,2n+1}, \tag{6.361}$$

as defined by (6.357) *forms an* $\mathbf{l}^2(\Omega)$*-orthonormal set of tensor spherical harmonics which is closed in* $\mathbf{c}(\Omega)$ *with respect to* $\|\cdot\|_{\mathbf{c}(\Omega)}$ *and complete in* $\mathbf{l}^2(\Omega)$ *with respect to* $(\cdot,\cdot)_{\mathbf{l}^2(\Omega)}$*. Furthermore, we have*

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(1,1)} = -(n+2)(n+3)\tilde{\mathbf{y}}_{n,m}^{(1,1)}, \tag{6.362}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(1,2)} = -n(n+1)\tilde{\mathbf{y}}_{n,m}^{(1,2)}, \tag{6.363}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(2,1)} = -n(n+1)\tilde{\mathbf{y}}_{n,m}^{(2,1)}, \tag{6.364}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(2,2)} = -(n-1)(n-2)\tilde{\mathbf{y}}_{n,m}^{(2,2)}, \tag{6.365}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(3,3)} = -n(n+1)\tilde{\mathbf{y}}_{n,m}^{(3,3)}, \tag{6.366}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(1,3)} = -(n+1)(n+2)\tilde{\mathbf{y}}_{n,m}^{(1,3)}, \tag{6.367}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(2,3)} = -n(n-1)\tilde{\mathbf{y}}_{n,m}^{(2,3)}, \tag{6.368}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(3,1)} = -(n+1)(n+2)\tilde{\mathbf{y}}_{n,m}^{(3,1)}, \tag{6.369}$$

$$\Delta_\xi^* \tilde{\mathbf{y}}_{n,m}^{(3,2)} = -n(n-1)\tilde{\mathbf{y}}_{n,m}^{(3,2)}. \tag{6.370}$$

6.13 Alternative Systems of Tensor Spherical Harmonics

Introducing the spaces

$$\widetilde{\mathbf{harm}}_n^{(i,k)} = \mathrm{span}\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}_{m=1,\ldots,2n+1}, \quad n = \tilde{0}_{ik},\ldots, \tag{6.371}$$

we find

$$\mathbf{harm}_0^{(3,3)} = \widetilde{\mathbf{harm}_0}^{(3,1)}, \tag{6.372}$$
$$\mathbf{harm}_0^{(1,1)} \oplus \mathbf{harm}_0^{(2,2)} = \widetilde{\mathbf{harm}_0}^{(1,1)} \oplus \widetilde{\mathbf{harm}_0}^{(2,1)}, \tag{6.373}$$

$$\mathbf{harm}_1^{(1,3)} \oplus \mathbf{harm}_1^{(3,1)} \oplus \mathbf{harm}_1^{(3,3)} = \widetilde{\mathbf{harm}_1}^{(1,3)} \oplus \widetilde{\mathbf{harm}_1}^{(2,3)} \oplus \widetilde{\mathbf{harm}_1}^{(3,1)}, \tag{6.374}$$
$$\mathbf{harm}_1^{(1,1)} \oplus \mathbf{harm}_1^{(1,2)} \oplus \mathbf{harm}_1^{(2,1)} \oplus \mathbf{harm}_1^{(2,2)} = \widetilde{\mathbf{harm}_1}^{(1,1)} \oplus \widetilde{\mathbf{harm}_1}^{(1,2)} \oplus \widetilde{\mathbf{harm}_1}^{(2,1)} \oplus \widetilde{\mathbf{harm}_1}^{(3,3)}, \tag{6.375}$$

and, for $n = 2, 3, \ldots$,

$$\mathbf{harm}_n^{(1,3)} \oplus \mathbf{harm}_n^{(3,1)} \oplus \mathbf{harm}_n^{(3,2)} \oplus \mathbf{harm}_n^{(3,3)} = \widetilde{\mathbf{harm}_n}^{(1,3)} \oplus \widetilde{\mathbf{harm}_n}^{(3,1)} \oplus \widetilde{\mathbf{harm}_n}^{(3,2)} \oplus \widetilde{\mathbf{harm}_n}^{(3,3)}, \tag{6.376}$$

and

$$\mathbf{harm}_n^{(1,1)} \oplus \mathbf{harm}_n^{(1,2)} \oplus \mathbf{harm}_n^{(2,1)} \oplus \mathbf{harm}_n^{(2,2)} \oplus \mathbf{harm}_n^{(2,3)} = \widetilde{\mathbf{harm}_n}^{(1,1)} \oplus \widetilde{\mathbf{harm}_n}^{(1,2)} \oplus \widetilde{\mathbf{harm}_n}^{(2,1)} \oplus \widetilde{\mathbf{harm}_n}^{(2,2)} \oplus \widetilde{\mathbf{harm}_n}^{(3,3)}. \tag{6.377}$$

Therefore, it is clear that

$$\mathbf{harm}_0 = \widetilde{\mathbf{harm}_0}^{(1,1)} \oplus \widetilde{\mathbf{harm}_0}^{(2,1)} \oplus \widetilde{\mathbf{harm}_0}^{(3,1)}, \tag{6.378}$$
$$\mathbf{harm}_1 = \bigoplus_{\substack{i,k=1 \\ (i,k)\notin\{(2,2),(3,2)\}}}^{3} \widetilde{\mathbf{harm}_1}^{(i,k)}, \tag{6.379}$$
$$\mathbf{harm}_n = \bigoplus_{i,k=1}^{3} \widetilde{\mathbf{harm}_n}^{(i,k)}, \quad n = 2, 3, \ldots . \tag{6.380}$$

In analogy to the vectorial case, we are able to formulate the following lemma.

Lemma 6.37. *Let $\varepsilon^k \otimes \varepsilon^l H_n$ be a homogenous harmonic tensor polynomial. Then*

$$\varepsilon^k \otimes \varepsilon^l H_n|\Omega = \tilde{\mathbf{y}}_{n+2}^{(2,2)} + \tilde{\mathbf{y}}_{n+1}^{(2,3)} + \tilde{\mathbf{y}}_{n-1}^{(3,2)} + \tilde{\mathbf{y}}_{n}^{(1,2)} + \tilde{\mathbf{y}}_{n}^{(2,1)} + \tilde{\mathbf{y}}_{n}^{(3,3)} + \tilde{\mathbf{y}}_{n-1}^{(1,3)} + \tilde{\mathbf{y}}_{n+1}^{(3,1)} + \tilde{\mathbf{y}}_{n-2}^{(1,1)}, \tag{6.381}$$

where

$$\tilde{\mathbf{y}}_{n+2}^{(2,2)} = \tilde{\mathbf{o}}_{n+2}^{(2,2)} Y_{n+2}, \quad Y_{n+2} \in \mathrm{Harm}_{n+2}, \tag{6.382}$$

$$\tilde{\mathbf{y}}_{n+1}^{(2,3)} = \tilde{\mathbf{o}}_{n+1}^{(2,3)} Y_{n+1}^{(1)}, \quad Y_{n+1}^{(1)} \in \mathrm{Harm}_{n+1}, \tag{6.383}$$

$$\tilde{\mathbf{y}}_{n+1}^{(3,2)} = \tilde{\mathbf{o}}_{n+1}^{(3,2)} Y_{n+1}^{(2)}, \quad Y_{n+1}^{(2)} \in \mathrm{Harm}_{n+1}, \tag{6.384}$$

$$\tilde{\mathbf{y}}_{n}^{(1,2)} = \tilde{\mathbf{o}}_{n}^{(1,2)} Y_{n}^{(1)}, \quad Y_{n}^{(1)} \in \mathrm{Harm}_{n}, \tag{6.385}$$

$$\tilde{\mathbf{y}}_{n}^{(2,1)} = \tilde{\mathbf{o}}_{n}^{(2,1)} Y_{n}^{(2)}, \quad Y_{n}^{(2)} \in \mathrm{Harm}_{n}, \tag{6.386}$$

$$\tilde{\mathbf{y}}_{n}^{(3,3)} = \tilde{\mathbf{o}}_{n}^{(3,3)} Y_{n}^{(3)}, \quad Y_{n}^{(3)} \in \mathrm{Harm}_{n}, \tag{6.387}$$

$$\tilde{\mathbf{y}}_{n-1}^{(1,3)} = \tilde{\mathbf{o}}_{n-1}^{(1,3)} Y_{n-1}^{(1)}, \quad Y_{n-1}^{(1)} \in \mathrm{Harm}_{n-1}, \tag{6.388}$$

$$\tilde{\mathbf{y}}_{n-1}^{(3,1)} = \tilde{\mathbf{o}}_{n-1}^{(3,1)} Y_{n-1}^{(2)}, \quad Y_{n-1}^{(2)} \in \mathrm{Harm}_{n-1}, \tag{6.389}$$

$$\tilde{\mathbf{y}}_{n-2}^{(1,1)} = \tilde{\mathbf{o}}_{n-2}^{(1,1)} Y_{n-2}, \quad Y_{n-2} \in \mathrm{Harm}_{n-2}. \tag{6.390}$$

Analogously to Lemma 5.55, we evaluate the terms $\tilde{O}^{(i,k)}\tilde{\mathbf{o}}^{(j,l)}Y_n$ with Y_n being a member of Harm_n:

Lemma 6.38. *For an* $\mathrm{L}^2(\Omega)$*-orthonormal system of spherical harmonics* $\{Y_{n,m}\}$*, the following identity holds true:*

$$\tilde{O}^{(i,k)}\tilde{\mathbf{o}}^{(j,l)}Y_{n,m}(\xi) = \delta_{ij}\delta_{kl}\tilde{\mu}_n^{(i,k)}Y_{n,m}(\xi). \tag{6.391}$$

Next, we are interested in deriving an addition theorem involving Legendre rank-4 tensor kernels.

Definition 6.39. The kernel

$$\tilde{\mathbf{P}}_n^{(i,k,l,m)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3, \quad i,k,l,m \in \{1,2,3\}, \tag{6.392}$$

given by

$$\tilde{\mathbf{P}}_n^{(i,k,l,m)}(\xi,\eta) = (\tilde{\mu}_n^{(i,k)})^{-1/2}(\tilde{\mu}_n^{(l,m)})^{-1/2}\tilde{\mathbf{o}}_\xi^{(i,k)}\tilde{\mathbf{o}}_\eta^{(l,m)}P_n(\xi\cdot\eta), \tag{6.393}$$

$\xi,\eta \in \Omega$, is called the *(tensorial) Legendre rank-4 tensor kernel of degree* n *and type* (i,k,l,m) *with respect to the dual system of operators* $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$. The kernel

$$\tilde{\mathbf{P}}_n = \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \tilde{\mathbf{P}}_n^{(i,k,l,m)} \tag{6.394}$$

is called *(tensorial) Legendre rank-4 tensor kernel of degree* n *with respect to the dual system of operators* $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$*,* $i,k \in \{1,2,3\}$.

The connection between the Legendre tensor $\tilde{\mathbf{P}}_n^{(i,k,l,m)}$ and the Legendre tensor $\mathbf{P}_n^{(i,k,l,m)}$ (more precisely, ${}^t\mathbf{P}_n^{(i,k,l,m)}$ and ${}^t\mathbf{P}^{(i,k,l,m)}$) can be easily calculated from their definitions. Unfortunately, the formulas are quite lengthy so that we will not show them here.

The addition theorem for the tensor fields functions $\tilde{\mathbf{y}}_{n,m}^{(i,k)}$ as defined in (6.357) reads as follows.

Theorem 6.40. *Let* $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}_{m=1,\ldots,2n+1}$ *be an* $\mathbf{l}^2(\Omega)$*–orthonormal basis of* $\widetilde{\mathbf{harm}}_n^{(i,k)}$ *(as defined by (6.357)). Then*

$$\sum_{m=1}^{2n+1} \tilde{\mathbf{y}}_{n,m}^{(i,k)}(\xi) \otimes \tilde{\mathbf{y}}_{n,m}^{(p,q)}(\eta) = \frac{2n+1}{4\pi} \tilde{\mathbf{P}}_n^{(i,k,p,q)}(\xi,\eta), \tag{6.395}$$

$i,k,p,q \in \{1,2,3\}$.

As in the case of the Legendre tensor $\mathbf{P}_n^{(i,k,p,q)}$, we are able to give an estimate of the values $|{}^t\tilde{\mathbf{P}}_n^{(i,k,p,q)}(\xi,\eta)|$.

Lemma 6.41. *If* $i,k,l,m,p,q \in \{1,2,3\}$*, then, for all* $\xi,\eta \in \Omega$*,*

(i) $|\tilde{\mathbf{P}}_n^{(i,k,l,m)}(\xi,\eta)(\varepsilon^p \otimes \varepsilon^q)| \leq 1$,
(ii) $|\tilde{\mathbf{P}}_n^{(i,k,l,m)}(\xi,\eta)| \leq 3$.

Obviously,

$$\tilde{\mathbf{K}}_{\widetilde{\mathbf{harm}}_n^{(i,k)}} = \frac{2n+1}{4\pi} \tilde{\mathbf{P}}_n^{(i,k,i,k)}$$

is the reproducing kernel of the space $\widetilde{\mathbf{harm}}_n^{(i,k)}$ in the sense that

(i) for all $\xi \in \Omega$

$$\tilde{O}^{(i,k)} \tilde{\mathbf{K}}_{\widetilde{\mathbf{harm}}_n^{(i,k)}}(\cdot,\xi) \in \widetilde{\mathbf{harm}}_n^{(i,k)}, \tag{6.396}$$

(ii) for every $\mathbf{f} \in \widetilde{\mathbf{harm}}_n^{(i,k)}$ and all $\xi \in \Omega$

$$\tilde{O}^{(i,k)} \mathbf{f}(\xi) = \left(\tilde{O}^{(i,k)} \tilde{\mathbf{K}}_{\widetilde{\mathbf{harm}}_n^{(i,k)}}(\cdot,\xi), \mathbf{f} \right)_{\mathbf{l}^2(\Omega)}, \tag{6.397}$$

where for (sufficiently smooth) tensor fields $\mathbf{F} : \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ of the form

$$\mathbf{F}(\xi) = \sum_{p=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{3} \sum_{s=1}^{3} F_{p,q,r,s}(\xi) \varepsilon^p \otimes \varepsilon^q \otimes \varepsilon^r \otimes \varepsilon^s \tag{6.398}$$

with $\sum_{p,q=1}^{3} F_{p,q,r,s}(\xi)\varepsilon^p \otimes \varepsilon^q \in \widetilde{\mathbf{harm}}_n$ for $r,s \in \{1,2,3\}$ we define the operators $\tilde{O}^{(i,k)}$ by

$$\tilde{O}^{(i,k)}\mathbf{F}(\xi) = \sum_{r=1}^{3}\sum_{s=1}^{3} \tilde{O}^{(i,k)} \left(\sum_{p=1}^{3}\sum_{q=1}^{3} F_{p,q,r,s}(\xi)\varepsilon^p \otimes \varepsilon^q \right) \varepsilon^r \otimes \varepsilon^s. \quad (6.399)$$

If we introduce the tensor fields ${}^t\tilde{\mathbf{p}}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3,\ i,k \in \{1,2,3\}$, we finally get an addition theorem involving the tensor spherical harmonics $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}$ and the scalar spherical harmonics $\{Y_{n,m}\}$.

Definition 6.42. The kernel ${}^t\tilde{\mathbf{p}}_n^{(i,k)}(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $i,k \in \{1,2,3\}$, given by

$${}^t\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta) = (\tilde{\mu}_n^{(i,k)})^{-1/2}\tilde{\mathbf{o}}_\xi^{(i,k)} P_n(\xi \cdot \eta),\ \xi,\eta \in \Omega. \quad (6.400)$$

is called the *(tensorial) Legendre rank-2 tensor kernel of degree n and type (i,k) with respect to the dual system of operators $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$, $i,k = 1,2,3$.* The kernel

$${}^t\tilde{\mathbf{p}}_n = \sum_{i=1}^{3}\sum_{k=1}^{3} {}^t\tilde{\mathbf{p}}_n^{(i,k)} \quad (6.401)$$

is called *(tensorial) Legendre rank-2 tensor kernel of degree n with respect to the dual system of operators $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$, $i,k = 1,2,3$.*

The relation between the Legendre tensors ${}^t\tilde{\mathbf{p}}_n^{(i,k)}$ and the Legendre tensors ${}^t\mathbf{p}_n^{(i,k)}$ can directly be derived using (6.337) and (6.338).

Lemma 6.43. *Let the Legendre tensors ${}^t\tilde{\mathbf{p}}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $i,k \in \{1,2,3\}$, be defined as above. Then*

$$\begin{aligned}
{}^t\tilde{\mathbf{p}}_n^{(1,1)} &= c_n^{1,1,1,1}(n+1)(n+2)\ {}^t\mathbf{p}_n^{(1,1)} - c_n^{1,2,1,1}(n+2)\ {}^t\mathbf{p}_n^{(1,2)} \\
&\quad -c_n^{2,1,1,1}(n+2)\ {}^t\mathbf{p}_n^{(2,1)} - c_n^{2,2,1,1}\frac{1}{2}(n+1)(n+2)\ {}^t\mathbf{p}_n^{(2,2)} \\
&\quad +c_n^{2,3,1,1}\frac{1}{2}\ {}^t\mathbf{p}_n^{(2,3)}, && (6.402) \\
{}^t\tilde{\mathbf{p}}_n^{(1,2)} &= c_n^{1,1,1,2}n^2\ {}^t\mathbf{p}_n^{(1,1)} + c_n^{1,2,1,2}n\ {}^t\mathbf{p}_n^{(1,2)} - c_n^{2,1,1,2}(n-1)\ {}^t\mathbf{p}_n^{(2,1)} \\
&\quad +c_n^{2,2,1,2}\frac{1}{2}n(n-1)\ {}^t\mathbf{p}_n^{(2,2)} - c_n^{2,3,1,2}\frac{1}{2}\ {}^t\mathbf{p}_n^{(2,3)}, && (6.403) \\
{}^t\tilde{\mathbf{p}}_n^{(2,1)} &= c_n^{1,1,2,1}(n+1)^2\ {}^t\mathbf{p}_n^{(1,1)} - c_n^{1,2,2,1}(n+1)\ {}^t\mathbf{p}_n^{(1,2)} \\
&\quad +c_n^{2,1,2,1}(n+2)\ {}^t\mathbf{p}_n^{(2,1)} + c_n^{2,2,2,1}\frac{1}{2}(n+1)(n+2)\ {}^t\mathbf{p}_n^{(2,2)} \\
&\quad -c_n^{2,3,2,1}\frac{1}{2}\ {}^t\mathbf{p}_n^{(2,3)}, && (6.404)
\end{aligned}$$

$$\begin{aligned}
{}^t\tilde{\mathbf{p}}_n^{(2,2)} &= c_n^{1,1,2,2} n(n-1)\ {}^t\mathbf{p}_n^{(1,1)} + c_n^{1,2,2,2}(n-1)\ {}^t\mathbf{p}_n^{(1,2)} \\
&\quad + c_n^{2,1,2,2}(n-1)\ {}^t\mathbf{p}_n^{(2,1)} - c_n^{2,2,2,2}\frac{1}{2}n(n-1)\ {}^t\mathbf{p}_n^{(2,2)} \\
&\quad + c_n^{2,3,2,2}\frac{1}{2}\ {}^t\mathbf{p}_n^{(2,3)}, \qquad (6.405)\\
{}^t\tilde{\mathbf{p}}_n^{(3,3)} &= c_n^{2,1,3,3}\ {}^t\mathbf{p}_n^{(2,1)} - c_n^{2,2,3,3}\frac{1}{2}n(n+1)\ {}^t\mathbf{p}_n^{(2,2)} \\
&\quad - c_n^{2,3,3,3}\frac{1}{2}\ {}^t\mathbf{p}_n^{(2,3)}, \qquad (6.406)\\
{}^t\tilde{\mathbf{p}}_n^{(1,3)} &= c_n^{1,3,1,3}(n+1)\ {}^t\mathbf{p}_n^{(1,3)} + c_n^{3,1,1,3}\ {}^t\mathbf{p}_n^{(3,1)} - c_n^{3,2,1,3}\frac{1}{2}\ {}^t\mathbf{p}_n^{(3,2)} \\
&\quad - c_n^{3,3,1,3}\frac{1}{2}n(n+1)\ {}^t\mathbf{p}_n^{(3,3)}, \qquad (6.407)\\
{}^t\tilde{\mathbf{p}}_n^{(2,3)} &= c_n^{1,3,2,3} n\ {}^t\mathbf{p}_n^{(1,3)} - c_n^{3,1,2,3}\ {}^t\mathbf{p}_n^{(3,1)} + c_n^{3,2,2,3}\frac{1}{2}\ {}^t\mathbf{p}_n^{(3,2)} \\
&\quad + c_n^{3,3,2,3}\frac{1}{2}n(n+1)\ {}^t\mathbf{p}_n^{(3,3)}, \qquad (6.408)\\
{}^t\tilde{\mathbf{p}}_n^{(3,1)} &= c_n^{3,1,3,1}(n+2)\ {}^t\mathbf{p}_n^{(3,1)} - c_n^{3,2,3,1}\frac{1}{2}\ {}^t\mathbf{p}_n^{(3,2)} \\
&\quad + c_n^{3,3,3,1}\frac{1}{2}(n+2)(n+1)\ {}^t\mathbf{p}_n^{(3,3)}, \qquad (6.409)\\
{}^t\tilde{\mathbf{p}}_n^{(3,2)} &= c_n^{3,1,3,2}(n-1)\ {}^t\mathbf{p}_n^{(3,1)} + c_n^{3,2,3,2}\frac{1}{2}\ {}^t\mathbf{p}_n^{(3,2)} \\
&\quad - c_n^{3,3,3,2}\frac{1}{2}n(n-1)\ {}^t\mathbf{p}_n^{(3,3)}, \qquad (6.410)
\end{aligned}$$

where the constants $c_n^{i,k,l,m}$, $i,k,l,m \in \{1,2,3\}$, *are given by*

$$c_n^{i,k,l,m} = \left(\frac{\mu_n^{(i,k)}}{\tilde{\mu}_n^{(l,m)}} \right)^{1/2}. \qquad (6.411)$$

Finally, we mention the following addition theorem.

Theorem 6.44. *Let* $\{Y_{n,m}\}_{m=1,\ldots,2n+1}$ *be an* $\mathrm{L}^2(\Omega)$*–orthonormal basis of* Harm_n. *Assume that* $\tilde{\mathbf{y}}_{n,m}^{(i,k)}$ *is defined by* (6.357). *Then*

$$\sum_{m=1}^{2n+1} \tilde{\mathbf{y}}_{n,m}^{(i,k)}(\xi) Y_{n,m}(\eta) = \frac{2n+1}{4\pi} {}^t\tilde{\mathbf{p}}_n{}^{(i,k)}(\xi,\eta), \qquad (6.412)$$

$i,k \in \{1,2,3\}$.

6.14 Orthogonal Expansions Using Tensor Legendre Kernels

For $F \in \mathrm{L}^2(\Omega)$, we already know the orthogonal expansion

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(n,j) Y_{n,j} \tag{6.413}$$

with $F^\wedge(n,j) = (F, Y_{n,j})_{\mathrm{L}^2(\Omega)}$. Using the addition theorem, (6.413) can be reformulated as follows

$$\begin{aligned} F &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_\Omega F(\eta) Y_{n,j}(\eta) Y_{n,j}(\cdot) d\omega(\eta) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\cdot\ \eta) d\omega(\eta). \end{aligned} \tag{6.414}$$

In other words, the projection of F into Harm_n, i.e., the space of all spherical harmonics with degree n, can be written as

$$\mathrm{Proj}_{\mathrm{Harm}_n}(F) = \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\cdot\ \eta) d\omega(\eta). \tag{6.415}$$

It is the aim of the remaining part of this section to show how these expansions look like for the tensorial case. For that purpose, we follow a similar way as in the vectorial case, (cf. Section 5.15). In particular, we introduce two generalizations of the Legendre polynomial for the tensorial case, which lead to two different generalizations of (6.415) for the two system of dual operators, respectively.

Let $\mathbf{f} \in \mathbf{l}^2(\Omega)$. Letting

$$(\mathbf{f}^{(i,k)})^\wedge(n,j) = \int_\Omega \mathbf{f}(\eta) \cdot \mathbf{y}_{n,j}^{(i,k)}(\eta) d\omega(\eta) \tag{6.416}$$

we have the expansion

$$\mathbf{f} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{j=1}^{2n+1} (\mathbf{f}^{(i,k)})^\wedge(n,j) \mathbf{y}_{n,j}^{(i,k)}. \tag{6.417}$$

Using the addition theorem for tensor spherical harmonics involving Legendre rank-4 tensor kernels, the expansion (6.417) may be rewritten in the form

$$\mathbf{f} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\cdot, \eta) \mathbf{f}^{(i,k)}(\eta) d\omega(\eta),$$

where the Legendre tensor $\mathbf{P}_n^{(i,k,i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ reads as follows:

$$\begin{aligned}\sum_{j=1}^{2n+1} \mathbf{y}_{n,j}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,j}^{(i,k)}(\eta) &= \frac{2n+1}{4\pi}\mathbf{P}_n^{(i,k,i,k)}(\xi,\eta) \qquad (6.418)\\ &= (\mu_n^{(i,k)})^{-1}\frac{2n+1}{4\pi}\mathbf{o}_\xi^{(i,k)}\mathbf{o}_\eta^{(i,k)}P_n(\xi\cdot\eta),\end{aligned}$$

$(\xi,\eta) \in \Omega \times \Omega$ (P_n is the usual Legendre polynomial of degree n). Explicit expressions for $\mathbf{o}_\eta^{(i,k)}P_n(\xi\cdot\eta)$ can be calculated using Lemma 6.23, where the explicit expressions for $\mathbf{P}_n^{(i,k,i,k)}$ are given in Theorem 6.26. Furthermore, it is obvious that the projection $\mathbf{l}^2(\Omega) \to \mathbf{harm}_n^{(i,k)}$ reads as follows

$$\mathrm{Proj}_{\mathbf{harm}_n^{(i,k)}}(\mathbf{f}) = \frac{2n+1}{4\pi}\int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\cdot,\eta)\mathbf{f}^{(i,k)}(\eta)d\omega(\eta). \qquad (6.419)$$

Thus, we recognize Legendre rank-4 tensor kernel $\mathbf{P}_n^{(i,k,i,k)}$ as a canonical generalization of the Legendre polynomial to the tensor case.

As in the vectorial case, there is a second variant to generalize the Legendre polynomial. Let the tensor spherical harmonics $\mathbf{y}_{n,j}^{(i,k)}$ be constructed from an orthonormal set of scalar spherical harmonics, i.e.,

$$\mathbf{y}_{n,j}^{(i,k)} = \left(\mu_n^{(i,k)}\right)^{-1/2}\mathbf{o}^{(i,k)}Y_{n,j}, \qquad (6.420)$$

$i,k = 1,2,3,\ n = 0_{ik},\ldots,\ j = 1,\ldots,2n+1$. Assuming that $\mathbf{f} \in \mathbf{l}^2(\Omega)$ is, in addition, sufficiently smooth, we can reformulate (6.417) in the following way

$$\begin{aligned}\mathbf{f} &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega \mathbf{f}(\eta)\cdot\mathbf{y}_{n,j}^{(i,k)}(\eta)d\omega(\eta)\mathbf{y}_{n,j}^{(i,k)} \qquad (6.421)\\ &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega \mathbf{f}(\eta)\cdot\frac{1}{(\mu_n^{(i,k)})^{1/2}}\mathbf{o}_\eta^{(i,k)}Y_{n,j}(\eta)d\omega(\eta)\mathbf{y}_{n,j}^{(i,k)}\\ &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega (O_\eta^{(i,k)}\mathbf{f}(\eta))\frac{1}{(\mu_n^{(i,k)})^{1/2}}Y_{n,j}(\eta)d\omega(\eta)\mathbf{y}_{n,j}^{(i,k)}\\ &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\frac{2n+1}{4\pi}\frac{1}{(\mu_n^{(i,k)})^{1/2}}\int_\Omega O_\eta^{(i,k)}\mathbf{f}(\eta)^t\mathbf{p}_n^{(i,k)}(\cdot,\eta)\ \ d\omega(\eta),\end{aligned}$$

where the Legendre tensor ${}^t\mathbf{p}_n^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is given by

$$\sum_{j=1}^{2n+1} y_n^{(i,k)}(\xi)Y_{n,j}(\eta) = \frac{2n+1}{4\pi}{}^t\mathbf{p}_n^{(i,k)}(\xi,\eta), \qquad (\xi,\eta) \in \Omega\times\Omega, \qquad (6.422)$$

and the operators $O^{(i,k)}$ which are adjoint to $\mathbf{o}^{(i,k)}$ are given by (6.132)–(6.140). The Legendre tensors can be determined using the addition theorem 6.34. Using this second generalization ${}^t\mathbf{p}_n^{(i,k)}$ of the Legendre polynomials, the projection operator (6.419) can be rewritten as

$$\mathrm{Proj}_{\mathbf{harm}_n^{(i,k)}}(\mathbf{f}) = \frac{2n+1}{4\pi}\frac{1}{(\mu_n^{(i,k)})^{1/2}}\int_\Omega O_\eta^{(i,k)}\mathbf{f}(\eta)\,{}^t\mathbf{p}_n^{(i,k)}(\cdot,\eta)d\omega(\eta). \quad (6.423)$$

For this formula to be valid, it is necessary that f is sufficiently smooth.

In addition, it should be mentioned that not only the system of dual operators $\mathbf{o}^{(i,k)}, O^{(i,k)}, i,k \in \{1,2,3\}$, define tensor spherical harmonics, but also the system of dual operators $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$ enables us to introduce tensor spherical harmonics. In more detail, using the system $\{\tilde{y}_{n,j}^{(i,k)}\}$ with

$$\tilde{\mathbf{y}}_{n,j}^{(i,k)} = \left(\tilde{\mu}_n^{(i,k)}\right)^{-1/2}\tilde{\mathbf{o}}^{(i,k)}Y_{n,j}, \quad (6.424)$$

$i,k = 1,2,3,\ n = 0_{ik},\ldots,\ j = 1,\ldots,2n+1$, we find for $\mathbf{f} \in \mathbf{l}^2(\Omega)$

$$\begin{aligned}
\mathbf{f} &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega \mathbf{f}(\eta)\cdot\tilde{\mathbf{y}}_{n,j}^{i,k}(\eta)d\omega(\eta)\tilde{\mathbf{y}}_{n,j}^{i,k} \quad (6.425)\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega \mathbf{f}(\eta)\cdot\frac{1}{(\tilde{\mu}_n^{(i,k)})^{1/2}}\tilde{\mathbf{o}}_\eta^{(i,k)}Y_{n,j}(\eta)d\omega(\eta)\tilde{\mathbf{y}}_{n,j}^{i,k}\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=1}^{2n+1}\int_\Omega (\tilde{O}_\eta^{(i,k)}\mathbf{f}(\eta))\frac{1}{(\tilde{\mu}_n^{(i,k)})^{1/2}}Y_{n,j}(\eta)d\omega(\eta)\tilde{\mathbf{y}}_{n,j}^{i,k}\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{n=0_{ik}}^{\infty}\frac{2n+1}{4\pi}\frac{1}{(\tilde{\mu}_n^{(i,k)})^{1/2}}\int_\Omega \tilde{O}_\eta^{(i,k)}\mathbf{f}(\eta)\,{}^t\tilde{\mathbf{p}}_n^{(i,k)}(\cdot,\eta)\,d\omega(\eta),
\end{aligned}$$

where the Legendre tensor ${}^t\tilde{\mathbf{p}}_n^{(i,k)} : \Omega\times\Omega \to \mathbb{R}^3\otimes\mathbb{R}^3$ is given by

$$\sum_{j=1}^{2n+1}\tilde{\mathbf{y}}_n^{(i,k)}(\xi)Y_{n,j}(\eta) = \frac{2n+1}{4\pi}{}^t\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta), \qquad (\xi,\eta)\in\Omega\times\Omega. \quad (6.426)$$

The kernel ${}^t\tilde{\mathbf{p}}_n^{(i,k)}$ leads us to the projection operator (cf. (6.419))

$$\mathrm{Proj}_{\widetilde{\mathbf{harm}}_n^{(i,k)}}(\mathbf{f}) = \frac{2n+1}{4\pi}\frac{1}{(\tilde{\mu}_n^{(i,k)})^{1/2}}\int_\Omega\left(\tilde{O}_\eta^{(i,k)}\mathbf{f}(\eta)\right){}^t\tilde{\mathbf{p}}_n^{(i,k)}(\cdot,\eta)d\omega(\eta). \quad (6.427)$$

Again, for this formula to be valid, it is necessary that f is sufficiently smooth.

6.15 Bibliographical Notes

Tensor spherical harmonics are used in many fields of application, and therefore many different approaches can be found in the literature: see for example G.E. Backus (1966); G.E. Backus (1967), F.A. Dahlen, M.L. Smith (1975), W. Freeden et al. (1994), James R.W. (1976), M.N. Jones (1980), and F.J. Zerilli (1970). K.S. Thorne (1980) has collected many of the former studies in a review paper. Our approach is a straightforward generalization of the vector spherical harmonic theory. The tensor spherical harmonics of pure normal and mixed type are also considered by F.J. Zerilli (1970), while the tangential ones have also been considered in the work of G.E. Backus (1966); G.E. Backus (1967). The decomposition theorem has been proven by W. Freeden et al. (1994), where also a detailed view on tensor spherical splines is worked out. The proof of closure and completeness based on Bernstein summability is due to W. Freeden, M. Gutting (2008). An intrinsic approach which emphasizes the tensor spherical harmonics to be eigenfunctions of a tensorial Beltrami operator is due to M. Schreiner (1994), W. Freeden et al. (1998), and H. Nutz (2002).

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

7 Scalar Zonal Kernel Functions

Any kernel function $K : \Omega \times \Omega \to \mathbb{R}$ that is characterized by the property

$$K(\xi, \eta) = K(|\xi - \eta|), \quad \xi, \eta \in \Omega \tag{7.1}$$

is called a (spherical) *radial basis function* (at least in the theory of constructive approximation). In other words, a radial basis function is a real-valued kernel function whose values depend only on the Euclidean distance $|\xi - \eta|$ of two unit vectors ξ, η (see Fig. 7.1). A well-known fact is that the distance of two unit vectors is expressible in terms of their inner product:

$$|\xi - \eta|^2 = |\xi|^2 + |\eta|^2 - 2\xi \cdot \eta = 2(1 - \xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{7.2}$$

Consequently, any radial basis function is equivalently characterized by the property of being dependent only on the inner product $\xi \cdot \eta$ of the unit vectors $\xi, \eta \in \Omega$, i.e.,

$$K(\xi, \eta) = K(|\xi - \eta|) = \hat{K}(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{7.3}$$

In the theory of special functions of mathematical physics, however, a kernel $\hat{K} : \Omega \times \Omega \to \mathbb{R}$ satisfying $\hat{K}(\xi \cdot \eta) = \hat{K}(\mathbf{t}\xi \cdot \mathbf{t}\eta)$, $\xi, \eta \in \Omega$, for all orthogonal transformation $\mathbf{t}$ is known as a *zonal kernel function* (see (2.107)). In order to point out the reducibility of $\hat{K}$ to a function defined on the interval $[-1, 1]$, the notation $(\xi, \eta) \mapsto \hat{K}(\xi \cdot \eta), (\xi, \eta) \in \Omega \times \Omega$, is used throughout this work.

7.1 Zonal Kernel Functions in Scalar Context

In what follows, we deal with essential keystones of the scalar theory of zonal kernel functions. The classical addition theorem of spherical harmonics enables us to characterize zonal kernel functions as orthogonal (Fourier) sum expansions in terms of Legendre polynomials.

We begin our considerations by recapitulating the definition of a zonal kernel function in more mathematical rigor.

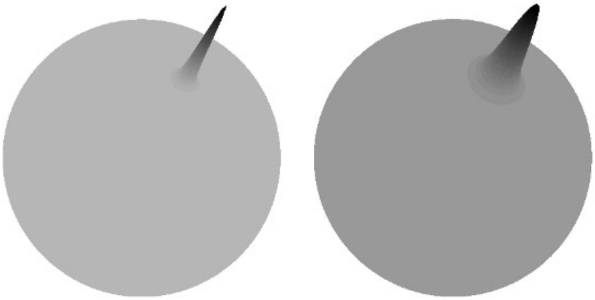


Fig. 7.1: Two examples of scalar (locally supported) zonal functions on the unit sphere Ω.

Definition 7.1. Assume that $\hat{K}$ is a real function on the interval $[-1, 1]$. A function $K_\xi : \Omega \to \mathbb{R}$, $\xi \in \Omega$ fixed, given by

$$\eta \mapsto K_\xi(\eta) = \hat{K}(\xi \cdot \eta), \quad \eta \in \Omega, \tag{7.4}$$

is called a *scalar zonal kernel function* (more accurately, ξ–*zonal kernel function* or ξ–*zonal function*).

For simplicity, we write $K(\xi\cdot)$ instead of $K(\xi \cdot \cdot)$. It is clear that ξ–zonal functions are invariant under orthogonal transformations which leave ξ fixed, such that the value $K_\xi(\eta)$ depends only on the inner product of η and ξ (isotropy). Moreover, it is customary to identify $K_\xi(\eta)$ with $K(\xi \cdot \eta)$ (instead of $\hat{K}(\xi \cdot \eta)$).

Definition 7.2. A scalar zonal function $K : [-1, 1] \to \mathbb{R}$ is called an $\mathrm{L}^2(\Omega)$–*zonal scalar kernel function*, if $K(\xi\cdot)$ is a member of the space $\mathrm{L}^2(\Omega)$ for each $\xi \in \Omega$.

From the Funk-Hecke formula, we obtain for all ξ, $\eta \in \Omega$ and $K \in \mathrm{L}^2[-1, 1]$

$$\int_\Omega K(\xi \cdot \alpha) P_n(\alpha \cdot \zeta) \, d\omega(\alpha) = K^\wedge(n) P_n(\xi \cdot \zeta) \tag{7.5}$$

with Legendre coefficients $K^\wedge(n), n \in \mathbb{N}_0$, given by

$$K^\wedge(n) = 2\pi \int_{-1}^{1} K(t) P_n(t) \, dt. \tag{7.6}$$

Using both the addition theorem (Theorem 3.26) and the Funk–Hecke formula (Theorem 3.60), we get a representation of an $\mathrm{L}^2(\Omega)$–scalar zonal kernel function K in terms of a Legendre series. Explicitly written out,

$$K(\xi\cdot) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) P_n(\xi\cdot) \tag{7.7}$$

(in $\|\cdot\|_{\mathrm{L}^2(\Omega)}$–sense), where the sequence $\{K^\wedge(n)\}_{n\in\mathbb{N}_0}$ is called the *Legendre symbol* of the zonal kernel $K(\xi\cdot)$.

From the addition theorem with $\xi = \eta$ we get

$$\sum_{m=1}^{2n+1} (Y_{n,m}(\xi))^2 = \frac{2n+1}{4\pi}, \quad \xi \in \Omega. \tag{7.8}$$

Therefore, the representation (7.7) in combination with (7.8) helps us to formulate the following theorem:

Theorem 7.3. *A scalar zonal kernel function $K : [-1,1] \to \mathbb{R}$ is an $\mathrm{L}^2(\Omega)$–scalar zonal function if and only if*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(K^\wedge(n)\right)^2 < \infty. \tag{7.9}$$

7.2 Convolutions Involving Scalar Zonal Kernel Functions

Via the Funk–Hecke formula, we are led to compositions of zonal kernels generated by convolution. An important feature is that the convolution of zonal kernel functions does not affect the property of being a zonal kernel.

Definition 7.4. Let H, K be $\mathrm{L}^2(\Omega)$-scalar zonal kernel functions. Suppose that F is of class $\mathrm{L}^2(\Omega)$. Then $K * F$ defined by

$$(K * F)(\xi) = \int_\Omega K(\xi\cdot\eta) F(\eta)\, d\omega(\eta), \tag{7.10}$$

$\xi \in \Omega$, is called the *convolution of K against F*. Furthermore, $H * K$ defined by

$$(H * K)(\xi\cdot\eta) = \int_\Omega H(\xi\cdot\zeta) K(\zeta\cdot\eta)\, d\omega(\zeta), \tag{7.11}$$

$\xi, \eta \in \Omega$, is called the *convolution of H against K*.

Note that we use the same symbol '$*$' for different specifications of convolutions. Moreover, the commutativity in (7.11) should be pointed out, such that the convolution of H against K is equal to the convolution of K against H, i.e., $H * K = K * H$.

Convolutions on the sphere have been discussed by many authors (see, for example, S. Bochner (1954), A.P. Calderon, A. Zygmund (1955)). Of particular importance in our approach are the following properties:

(i) For $G \in \mathrm{L}^2[-1,1]$, $\quad Y_n \in \mathrm{Harm}_n$,

$$(G * Y_n)(\xi) = G^\wedge(n) Y_n(\xi), \quad \xi \in \Omega, \tag{7.12}$$

(ii) For all $Y_n \in \mathrm{Harm}_n$ (cf. Corollary 3.61),

$$\int_\Omega (G * F)(\eta) Y_n(\eta)\, d\omega(\eta) = G^\wedge(n) \int_\Omega F(\eta) Y_n(\eta)\, d\omega(\eta). \tag{7.13}$$

For later use, we introduce the concept of an iterated convolution.

Definition 7.5. Assume that $K \in \mathrm{L}^2[-1,1]$ and $F \in \mathrm{L}^2(\Omega)$. For $(\xi, \zeta) \in \Omega \times \Omega$ we let

$$\begin{aligned} K^{(1)}(\xi \cdot \zeta) &= K(\xi \cdot \zeta), \\ K^{(k)}(\xi \cdot \zeta) &= \int_\Omega K^{(k-1)}(\xi \cdot \eta) K(\zeta \cdot \eta)\, d\omega(\eta), \quad k = 2, 3 \ldots . \end{aligned}$$

Then $K^{(k)} * F$ is called the k-th *iterated convolution* of K *against* F.

Obviously, the k-th iterated kernel K is a scalar zonal kernel function, and it follows immediately that

$$(K^{(k)})^\wedge(n) = (K^\wedge(n))^k, \quad n = 0, 1, \ldots, \quad k = 1, 2, \ldots . \tag{7.14}$$

Let H, K be $\mathrm{L}^2(\Omega)$-scalar zonal kernel functions. Furthermore, suppose that F is of class $\mathrm{L}^2(\Omega)$. By virtue of of the Cauchy–Schwarz inequality, it is not hard to see that, for $K \in \mathrm{L}^2(\Omega)$, $K * F$ is in $\mathrm{L}^2(\Omega)$, whereas $H * K$ is a member of class $\mathrm{C}(\Omega)$. In spectral formulation, we have

$$K * F = \sum_{n=0}^{\infty} K^\wedge(n) \sum_{m=1}^{2n+1} F^\wedge(n,m) Y_{n,m}, \tag{7.15}$$

and

$$H * K = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} H^\wedge(n) K^\wedge(n) P_n. \tag{7.16}$$

Finally, it should be mentioned that

$$F = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n * F \tag{7.17}$$

in the topology of $\| \cdot \|_{\mathrm{L}^2(\Omega)}$.

7.3 Classification of Zonal Kernel Functions

As already mentioned, spherical harmonics are an adequate and often used tool for global approximation of functions on a sphere. In fact, spherical harmonic expansions are classical means in geopotential modeling. However, spherical harmonics suffer from several drawbacks in their construction because of their global support. An essential disadvantage is the fact that they are usually not appropriate for the investigation of local structures. In this respect, it is advisable to go over to space localizing functions, e.g., zonal kernel functions that are generated by summing up certain spherical harmonic expressions. Several classes of zonal kernel functions can be distinguished, for example, bandlimited and non-bandlimited, space-limited and non-spacelimited kernel functions. But the question is what is the right zonal kernel function of local nature for local purposes of approximation? Of course, the user of a mathematical method is interested in knowing the trial system which fits 'adequately' to the problem. Actually it is necessary, in the case where several choices are possible or an optimal choice cannot be found, to choose the trial systems in close adaptation to the data width and the required smoothness of the field to be approximated. In this respect, an uncertainty principle specifying the space and frequency (in physical language 'momentum') localization is helpful to serve as a decisive criterion. The essential outcome of the uncertainty principle is a better understanding of the classification of zonal kernel functions.

We begin our mathematical explanations of an uncertainty principle on the sphere Ω with the development of suitable bounds for the quantification of space and frequency localization.

Localization in Space. Suppose that F is of class $\mathrm{L}^2(\Omega)$. Assume first that

$$\|F\|_{\mathrm{L}^2(\Omega)} = \left(\int_\Omega (F(\eta))^2 \; d\omega(\eta) \right)^{1/2} = 1. \tag{7.18}$$

We associate to F the normal (radial) field $\eta \mapsto \eta F(\eta) = o_\eta^{(1)} F(\eta)$, $\eta \in \Omega$. This function maps $\mathrm{L}^2(\Omega)$ into the associated set of normal fields on Ω. The 'center of gravity of the spherical window' is defined by the *expectation value in the space domain*

$$g_F^{o^{(1)}} = \int_\Omega \left(o_\eta^{(1)} F(\eta)\right) F(\eta)\, d\omega(\eta) = \int_\Omega \eta (F(\eta))^2\, d\omega(\eta) \in \mathbb{R}^3 \qquad (7.19)$$

thereby interpreting $(F(\eta))^2 \ d\omega(\eta)$ as surface mass distribution over the sphere Ω embedded in Euclidean space $\mathbb{R}^3$. It is clear that $g_F^{o^{(1)}}$ lies in the closed inner space $\overline{\Omega^{\rm int}}$ of Ω: $|g_F^{o^{(1)}}| \leq 1$. The *variance in the space domain* is understood in canonical sense as the variance of the operator $o^{(1)}$

$$\begin{aligned} (\sigma_F^{o^{(1)}})^2 &= \int_\Omega \left(\left(o_\eta^{(1)} - g_F^{o^{(1)}}\right) F(\eta)\right)^2 \ d\omega(\eta) \\ &= \int_\Omega \left(\eta - g_F^{o^{(1)}}\right)^2 (F(\eta))^2 \ d\omega(\eta) \in \mathbb{R}. \end{aligned} \qquad (7.20)$$

Observing the identity $(\eta - g_F^{o^{(1)}})^2 = 1 + (g_F^{o^{(1)}})^2 - 2\eta \cdot g_F^{o^{(1)}}$, $\eta \in \Omega$, it follows immediately that $(\sigma_F^{o^{(1)}})^2 = 1 - (g_F^{o^{(1)}})^2$. Obviously, $0 \leq (\sigma_F^{o^{(1)}})^2 \leq 1$.

Since we are particularly interested in bandlimited or non–bandlimited zonal (i.e., radial basis) functions on the sphere, some simplifications can be made. Let K be of class $\mathrm{L}^2[-1,1]$ and $\|K\|_{\mathrm{L}^2[-1,1]} = 1$. Then, the corresponding expectation value ('center of gravity') can be computed readily as follows ($\varepsilon^3 = (0,0,1)^T$):

$$g_{K(\cdot\varepsilon^3)}^{o^{(1)}} = \int_\Omega \eta \left(K\left(\eta\cdot\varepsilon^3\right)\right)^2 \ d\omega(\eta) = \left(2\pi \int_{-1}^1 t\,(K(t))^2 \ dt\right) \varepsilon^3.$$

Letting

$$t_K^{o^{(1)}} = \left|g_{K(\cdot\varepsilon^3)}^{o^{(1)}}\right| = 2\pi \left|\int_{-1}^1 t\,(K(t))^2 \ dt\right| \in \mathbb{R} \qquad (7.21)$$

we find for the corresponding variance

$$\begin{aligned} (\sigma_K^{o^{(1)}})^2 &= \int_\Omega \left(\eta - g_{K(\cdot\varepsilon^3)}^{o^{(1)}}\right)^2 \ \left(K\left(\eta\cdot\varepsilon^3\right)\right)^2 \ d\omega(\eta) \qquad (7.22) \\ &= 1 - \left(t_K^{o^{(1)}}\right)^2 \\ &= 1 - \left(g_{K(\cdot\varepsilon^3)}^{o^{(1)}}\right)^2 \in \mathbb{R}. \end{aligned}$$

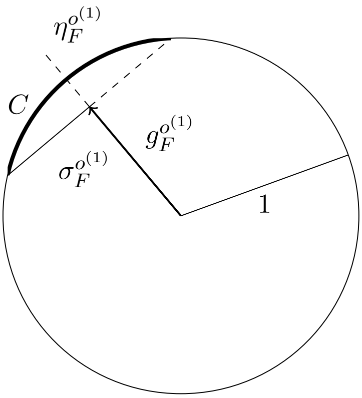


Fig. 7.2: Localization in a spherical cap.

Figure 7.2 gives a geometric interpretation of $g_F^{o^{(1)}}$ and $\sigma_F^{o^{(1)}}$. We associate to $g_F^{o^{(1)}}$, $g_F^{o^{(1)}} \neq 0$, and its projection $\eta_F^{o^{(1)}}$ onto the sphere Ω the spherical cap $C = \{\eta \in \Omega \mid 1 - \eta \cdot \eta_F^{o^{(1)}} \leq 1 - |g_F^{o^{(1)}}|\}$. Then the boundary ∂C is a circle with radius $\sigma_F^{o^{(1)}}$. As one thinks of a zonal function F to be a 'window function' on Ω, the window is determined by C, and its width is given by $\sigma_F^{o^{(1)}}$.

Localization in Frequency (once again, in physics usually called localization in momentum). Next the *expectation value in the 'frequency domain'* is introduced to be the *expectation value of the surface curl operator* $o^{(3)}$ on Ω. Then, for $F \in \mathcal{H}_{2l}(\Omega)$, $l \in \mathbb{N}$, i.e., for all $F \in \mathrm{L}^2(\Omega)$ such that there exists a function $G \in \mathrm{L}^2(\Omega)$ with $G^\wedge(n,k) = (-n(n+1))^l F^\wedge(n,k)$ for all $n = 0, 1, \ldots, k = 1, \ldots, 2n+1$, we have

$$g_F^{o^{(3)}} = \int_\Omega \left(o_\eta^{(3)} F(\eta)\right) F(\eta)\, d\omega(\eta) = 0 \in \mathbb{R}^3.$$

Correspondingly, the *variance in the 'frequency domain'* is given by

$$(\sigma_F^{o^{(3)}})^2 = \int_\Omega \left(\left(o_\eta^{(3)} - g_F^{o^{(3)}}\right) F(\eta)\right)^2 d\omega(\eta) \in \mathbb{R}.$$

The surface theorem of Stokes shows us that

$$\begin{aligned}(\sigma_F^{o^{(3)}})^2 &= \int_\Omega \left(o_\eta^{(3)} F(\eta)\right) \cdot \left(o_\eta^{(3)} F(\eta)\right) d\omega(\eta) \\ &= \int_\Omega \left(-\Delta_\eta^* F(\eta)\right) F(\eta) d\omega(\eta).\end{aligned}$$

Table 7.1: Space/frequency localization: A comparison of the operators $o^{(1)}$ and $o^{(3)}$.

	Operator	Expectation value
Space	$o^{(1)}$	$g_F^{o^{(1)}} = \int_\Omega \left(o_\eta^{(1)} F(\eta)\right) F(\eta) d\omega(\eta)$
Frequency	$o^{(3)}$	$g_F^{o^{(3)}} = \int_\Omega \left(o_\eta^{(3)} F(\eta)\right) F(\eta) d\omega(\eta)$

	Operator	Variance
Space	$o^{(1)}$	$(\sigma_F^{o^{(1)}})^2 = \int_\Omega \left(\left(o_\eta^{(1)} - g_F^{o^{(1)}}\right) F(\eta)\right)^2 d\omega(\eta)$
Frequency	$o^{(3)}$	$(\sigma_F^{o^{(3)}})^2 = \int_\Omega \left(\left(o_\eta^{(3)} - g_F^{o^{(3)}}\right) F(\eta)\right)^2 d\omega(\eta)$

Expressed in terms of spherical harmonics, we get via the Parseval identity

$$(\sigma_F^{o^{(3)}})^2 = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} n(n+1)(F^\wedge(n,k))^2.$$

Note that we require

$$\|F\|^2_{\mathrm{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(F^\wedge(n,k)\right)^2 = 1.$$

The meaning of $\sigma_F^{o^{(3)}}$ as measure for 'frequency localization' is as follows: The range of $\sigma_F^{o^{(3)}}$ is the interval $[0,\infty]$; a large value of $\sigma_F^{o^{(3)}}$ occurs if many Fourier coefficients contribute to $\sigma_F^{o^{(3)}}$. In conclusion, relating any spherical harmonic to a 'single wavelength' a large value $\sigma_F^{o^{(3)}}$ tells us that F is spread out widely in 'frequency domain'. In contrast to this statement, a small number of $\sigma_F^{o^{(3)}}$ indicates that only a few number of Fourier coefficients is significant (cf. Table 7.1).

Again we formulate our quantities in the context of zonal functions . Let $K(\cdot\varepsilon^3)$ be of class $\mathcal{H}_2(\Omega)$ satisfying $\|K(\cdot\varepsilon^3)\|_{\mathrm{L}^2(\Omega)} = 1$, then

$$\begin{aligned}(\sigma_{K(\cdot\varepsilon^3)}^{o^{(3)}})^2 &= -\int_\Omega \Delta^*_\eta K(\eta\cdot\varepsilon^3) K(\eta\cdot\varepsilon^3) d\omega(\eta) \\ &= -2\pi \int_{-1}^{1} K(t) \mathrm{L}_t K(t)\, dt\end{aligned}$$

where L_t denotes the Legendre operator as given by (3.175).

The square roots of the variances, i.e., $\sigma^{o^{(1)}}$ and $\sigma^{o^{(3)}}$, are called the *uncertainties* in $o^{(1)}$ and $o^{(3)}$, respectively. For these quantities, we get (see F.J. Narcowich, J.D. Ward (1996) and W. Freeden (1998)) the estimate $(\sigma_F^{o^{(1)}})^2(\sigma_F^{o^{(3)}})^2 \geq |g_F^{o^{(1)}}|^2$.

Summarizing our results, we are led to the following theorem.

Theorem 7.6. *Let* $F \in \mathcal{H}_2(\Omega)$ *satisfy* $\|F\|_{L^2(\Omega)} = 1$. *Then*

$$(\sigma_F^{o^{(1)}})^2(\sigma_F^{o^{(3)}})^2 \geq \left| g_F^{o^{(1)}} \right|^2 . \tag{7.23}$$

If $g_F^{o^{(1)}}$ *is non-vanishing, then*

$$\Delta_F^{o^{(1)}} \Delta_F^{o^{(3)}} \geq 1, \tag{7.24}$$

where we have used the abbreviations

$$\Delta_F^{o^{(1)}} = \frac{\sigma_F^{o^{(1)}}}{\left| g_F^{o^{(1)}} \right|} \tag{7.25}$$

and

$$\Delta_F^{o^{(3)}} = \sigma_F^{o^{(3)}} . \tag{7.26}$$

Proof. First we observe that for $F \in \mathcal{H}_2(\Omega)$ and all constant vectors $a \in \mathbb{R}^3$, $a = (a_1, a_2, a_3)^T$,

$$\begin{aligned} &\int_\Omega F(\eta) \left((\eta - a) \wedge o_\eta^{(3)} F(\eta) \right) \, d\omega(\eta) \qquad (7.27) \\ &= \int_\Omega \left(\sum_{i=1}^{3} \varepsilon^i \wedge \left(F(\eta)(\eta_i - a_i) o_\eta^{(3)} F(\eta) \right) \right) \, d\omega(\eta) \\ &= \sum_{i=1}^{3} \varepsilon^i \wedge \int_\Omega F(\eta)(\eta_i - a_i) o_\eta^{(3)} F(\eta) \, d\omega(\eta) \end{aligned}$$

(note that $o^{(3)} = L^*$). Now it is clear that for $i = 1, 2, 3$

$$F(\eta)(\eta_i - a_i) o_\eta^{(3)} F(\eta) = \sum_{k=1}^{3} \left(F(\eta)(\eta_i - a_i) \varepsilon^k \cdot o_\eta^{(3)} F(\eta) \right) \varepsilon^k . \tag{7.28}$$

This yields the identity

$$\begin{aligned}&\int_\Omega F(\eta)\left((\eta-a)\wedge o^{(3)}_\eta F(\eta)\right)\, d\omega(\eta) \\ &\quad = \sum_{i=1}^{3}\varepsilon^i\wedge\sum_{k=1}^{3}\int_\Omega F(\eta)(\eta_i-a_i)\varepsilon^k\cdot o^{(3)}_\eta F(\eta)\, d\omega(\eta)\varepsilon^k.\end{aligned} \tag{7.29}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{3}\varepsilon^i\wedge\sum_{k=1}^{3}\int_\Omega F(\eta)(\eta_i-a_i)\varepsilon^k\cdot o^{(3)}_\eta F(\eta)\, d\omega(\eta)\varepsilon^k \\ &\quad = \sum_{i=1}^{3}\varepsilon^i\wedge\sum_{k=1}^{3}(-1)\int_\Omega F(\eta)o^{(3)}_\eta\cdot\left(F(\eta)(\eta_i-a_i)\varepsilon^k\right)\, d\omega(\eta)\varepsilon^k \\ &\quad = \sum_{i=1}^{3}\varepsilon^i\wedge\sum_{k=1}^{3}(-1)\int_\Omega F(\eta)o^{(3)}_\eta\left(F(\eta)(\eta_i-a_i)\right)\cdot\varepsilon^k\, d\omega(\eta)\varepsilon^k \\ &\quad = \sum_{i=1}^{3}\varepsilon^i\wedge(-1)\int_\Omega F(\eta)o^{(3)}_\eta\left(F(\eta)(\eta_i-a_i)\right)\, d\omega(\eta).\end{aligned} \tag{7.30}$$

This leads us to the identity

$$\begin{aligned}&\int_\Omega F(\eta)\left((\eta-a)\wedge o^{(3)}_\eta F(\eta)\right)\, d\omega(\eta) \\ &\quad = \sum_{i=1}^{3}\int_\Omega F(\eta)o^{(3)}_\eta\left(F(\eta)(\eta_i-a_i)\right)\, d\omega(\eta)\wedge\varepsilon^i \\ &\quad = \int_\Omega F(\eta)o^{(3)}_\eta\wedge\left(F(\eta)(\eta-a)\right)\, d\omega(\eta) \\ &\quad = \int_\Omega F(\eta)\left(o^{(3)}_\eta\wedge\left((\eta-a)F(\eta)\right)\right)\, d\omega(\eta),\end{aligned} \tag{7.31}$$

where we used the notation

$$\mathrm{L}^*_\eta\wedge g(\eta)=\sum_{i=1}^{3}\left(\mathrm{L}^*_\eta(g(\eta)\cdot\varepsilon^i)\right)\wedge\varepsilon^i \tag{7.32}$$

in analogy to (2.132). With the help of this identity, we now verify the uncertainty principle. For that purpose, we first see by application of the Cauchy–Schwarz inequality that

$$\sigma_F^{o^{(1)}}\sigma_F^{o^{(3)}}\geq g_F, \tag{7.33}$$

where we have used the abbreviation

$$g_F=\int_\Omega\left|\left(\eta-g_F^{o^{(1)}}\right)F(\eta)\right|\left|o^{(3)}_\eta F(\eta)\right|\, d\omega(\eta). \tag{7.34}$$

The last expression can be estimated from below as follows

$$g_F \geq \left| \int_\Omega F(\eta) \left(\eta - g_F^{o^{(1)}}\right) \wedge \left(o_\eta^{(3)} F(\eta)\right) \, d\omega(\eta) \right| . \tag{7.35}$$

With our preliminary result, we then obtain

$$g_F \geq \left| \int_\Omega F(\eta) \left(o_\eta^{(3)} \wedge \left(\left(\eta - g_F^{o^{(1)}}\right) F(\eta)\right)\right) \, d\omega(\eta) \right| . \tag{7.36}$$

Furthermore, after elementary calculations, it follows that

$$\left(\eta - g_F^{o^{(1)}}\right) \wedge o_\eta^{(3)} + o_\eta^{(3)} \wedge \left(\eta - g_F^{o^{(1)}}\right) = -2\eta. \tag{7.37}$$

But this gives us

$$g_F \geq \left| \int_\Omega F(\eta)(-\eta)F(\eta) \, d\omega(\eta) \right| = \left| g_F^{o^{(1)}} \right| , \tag{7.38}$$

as required. □

In fact, the statement of Theorem 7.6 remains valid without assuming the condition $\|F\|_{\mathrm{L}^2(\Omega)} = 1$ (see S. Beth (2000)).

Corollary 7.7. *Let G be a member of class $\mathcal{H}_2(\Omega)$. Then*

$$(\sigma_G^{o^{(1)}})^2 (\sigma_G^{o^{(3)}})^2 \geq \left| g_G^{o^{(1)}} \right|^2 . \tag{7.39}$$

If $g_G^{o^{(1)}}$ is non-vanishing, then

$$\Delta_G^{o^{(1)}} \Delta_G^{o^{(3)}} \geq 1 . \tag{7.40}$$

Proof. Remember that $\sigma_G^{o^{(1)}}, \sigma_G^{o^{(3)}}$ respectively, are non-negative. Therefore, the inequality (7.39) is verified for $G = 0$ (in $\|\cdot\|_{\mathrm{L}^2(\Omega)}$–sense) by the following estimate

$$\begin{aligned} \left| g_G^{o^{(1)}} \right| &= \left| \int_\Omega \eta \, |G(\eta)|^2 \, d\omega(\eta) \right| \\ &\leq \int_\Omega |\eta| \, |G(\eta)|^2 \, d\omega(\eta) \\ &= \|G\|^2_{\mathrm{L}^2(\Omega)} \\ &= 0 . \end{aligned} \tag{7.41}$$

Without loss of generality, we suppose that $\|G\|_{\mathrm{L}^2(\Omega)} \neq 0$. Then we define $F = G / \|G\|_{\mathrm{L}^2(\Omega)}$. The application of the operator definitions leads us to

$$g_G^{o^{(1)}} = \int_\Omega \eta \, |G(\eta)|^2 \, d\omega(\eta) = \|G\|^2_{\mathrm{L}^2(\Omega)} \, g_F^{o^{(1)}}, \tag{7.42}$$

and

$$\begin{aligned}(\sigma_G^{o^{(1)}})^2 &= \int_\Omega \left|\eta - g_G^{o^{(1)}}\right|^2 |G(\eta)|^2 \, d\omega(\eta) \qquad (7.43)\\ &= \int_\Omega \left(1 - 2\eta \cdot g_G^{o^{(1)}} + \left(g_G^{o^{(1)}}\right)^2\right) |G(\eta)|^2 \, d\omega(\eta)\\ &= \|G\|^2_{\mathrm{L}^2(\Omega)} - 2\left(g_G^{o^{(1)}}\right)^2 + \|G\|^2_{\mathrm{L}^2(\Omega)}\left(g_G^{o^{(1)}}\right)^2\\ &= \|G\|^2_{\mathrm{L}^2(\Omega)}\left(1 - 2\|G\|^2_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2 + \|G\|^4_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2\right) \ .\end{aligned}$$

In order to obtain a relation between $\sigma_G^{o^{(1)}}$ and $\sigma_F^{o^{(1)}}$, we need the following estimate

$$\begin{aligned}1 - 2\|G\|^2_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2 &+ \|G\|^4_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2 - \left(1 - \left(g_F^{o^{(1)}}\right)^2\right)\\ &= \left(g_F^{o^{(1)}}\right)^2\left(1 - 2\|G\|^2_{\mathrm{L}^2(\Omega)} + \|G\|^4_{\mathrm{L}^2(\Omega)}\right)\\ &= \left(g_F^{o^{(1)}}\right)^2\left(\|G\|^2_{\mathrm{L}^2(\Omega)} - 1\right)^2\\ &\geq 0 \ . \qquad (7.44)\end{aligned}$$

Consequently,

$$1 - 2\|G\|^2_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2 + \|G\|^4_{\mathrm{L}^2(\Omega)}\left(g_F^{o^{(1)}}\right)^2 \geq 1 - \left(g_F^{o^{(1)}}\right)^2 = (\sigma_F^{o^{(1)}})^2 \ . \qquad (7.45)$$

Using (7.45) in the right hand side of (7.43), we see that $\|G\|^2_{\mathrm{L}^2(\Omega)}(\sigma_F^{o^{(1)}})^2$ is bounded by $(\sigma_G^{o^{(1)}})^2$, i.e.,

$$(\sigma_G^{o^{(1)}})^2 \geq \|G\|^2_{\mathrm{L}^2(\Omega)}(\sigma_F^{o^{(1)}})^2 \ . \qquad (7.46)$$

We already know that

$$g_G^{o^{(3)}} = \int_\Omega G(\eta)\mathrm{L}^*_\eta G(\eta) d\omega(\eta) = 0, \qquad (7.47)$$

and

$$(\sigma_G^{o^{(3)}})^2 = \int_\Omega -G(\eta)\Delta^*_\eta G(\eta) d\omega(\eta) = \|G\|^2_{\mathrm{L}^2(\Omega)}(\sigma_F^{o^{(3)}})^2 \ . \qquad (7.48)$$

From (7.46) and (7.48), respectively, we immediately find the uncertainty principle (7.39):

$$\begin{aligned}(\sigma_G^{o^{(1)}})^2(\sigma_G^{o^{(3)}})^2 &\geq \|G\|^4_{\mathrm{L}^2(\Omega)}(\sigma_F^{o^{(1)}})^2(\sigma_F^{o^{(3)}})^2 \qquad (7.49)\\ &\geq \|G\|^4_{\mathrm{L}^2(\Omega)}\left|g_F^{o^{(1)}}\right|^2 = \left|g_G^{o^{(1)}}\right|^2 \ .\end{aligned}$$

This completes the proof. □

The *uncertainty relation* measures the trade off between 'space localization' and 'frequency localization' ('spread in frequency'). It states that *sharp localization in space and 'frequency' are mutually exclusive.*

An immediate consequence of Theorem 7.6 is its reformulation for zonal functions $K(\varepsilon^3 \cdot) : \eta \mapsto K(\varepsilon^3 \cdot \eta)$, $\eta \in \Omega$.

Corollary 7.8. *Let $K(\varepsilon^3 \cdot) \in \mathcal{H}_2(\Omega)$ satisfy $\|K\|_{\mathrm{L}^2[-1,1]} = 1$. If $t_K^{o^{(1)}}$ is non-vanishing, then*

$$\Delta_K^{o^{(1)}} \Delta_K^{o^{(3)}} \geq 1,$$

where

$$\Delta_K^{o^{(1)}} = \frac{\sigma_K^{o^{(1)}}}{t_K^{o^{(1)}}}$$

and

$$\Delta_K^{o^{(3)}} = \sigma_K^{o^{(3)}}.$$

The interpretation of $(\sigma_K^{o^{(3)}})^2$ as variance in 'total angular momentum' helped us to prove Theorem 7.6. But this interpretation shows two essential drawbacks: First, the expectation value of the surface curl gradient is a vector which seems to be inadequate in 'momentum localization' in terms of scalar spherical harmonics, and secondly, the value of $g_F^{o^{(3)}}$ vanishes for all candidates F. This means that the 'center of gravitation of the spherical window' in 'momentum domain' is independent of the function F under consideration. Therefore, we are finally interested in the variance of the operator $-\Delta^*$

$$(\sigma_F^{-\Delta^*})^2 = \int_\Omega \left| \left((-\Delta_\eta^*) - g_F^{-\Delta^*} \right) F(\eta) \right|^2 \, d\omega(\eta) \tag{7.50}$$

which is a measure for the 'spread in momentum'. Now the corresponding expectation value $g_F^{-\Delta^*}$ is scalar-valued and non-vanishing. It can be easily seen that

$$(\sigma_F^{-\Delta^*})^2 = g_F^{(-\Delta^*)^2} - \left(g_F^{-\Delta^*} \right)^2. \tag{7.51}$$

In connection with Theorem 7.6, this leads to the following result.

Theorem 7.9. *Let F be of class $\mathcal{H}_4(\Omega)$ such that $\|F\|_{\mathrm{L}^2(\Omega)} = 1$. Then*

$$(\sigma_F^{o^{(1)}})^2 (\sigma_F^{-\Delta^*})^2 \geq \left| g_F^{o^{(1)}} \right| \frac{g_F^{(-\Delta^*)^2} - \left(g_F^{-\Delta^*} \right)^2}{g_F^{-\Delta^*}} \tag{7.52}$$

provided that $g_F^{-\Delta^*} \neq 0$. If the right hand side of (7.52) *is non-vanishing, then*

$$\Delta_F^{o^{(1)}} \Delta_F^{-\Delta^*} \geq 1, \tag{7.53}$$

where

$$\Delta_F^{-\Delta^*} = \left(\frac{(\sigma_F^{-\Delta^*})^2}{\frac{g_F^{(-\Delta^*)2} - \left(g_F^{-\Delta^*}\right)^2}{g_F^{-\Delta^*}}} \right)^{1/2} = \left(g_F^{-\Delta^*}\right)^{1/2} = \Delta_F^{o^{(3)}}.$$

Finally, we discuss some examples which are of particular interest for us:

Localization of Spherical Harmonics. We know that

$$\int_\Omega (Y_{n,k}(\xi))^2 \, d\omega(\xi) = 1 \quad . \tag{7.54}$$

Now it is clear that

$$g_{Y_{n,k}}^{o^{(1)}} = 0, \qquad \sigma_{Y_{n,k}}^{o^{(1)}} = 1. \tag{7.55}$$

Moreover, we have

$$g_{Y_{n,k}}^{-\Delta^*} = n(n+1), \qquad \sigma_{Y_{n,k}}^{-\Delta^*} = 0. \tag{7.56}$$

In other words, *spherical harmonics show an ideal frequency localization, but no space localization* (see Fig. 7.3 for an illustration of space and frequency localization for the Legendre polynomials).

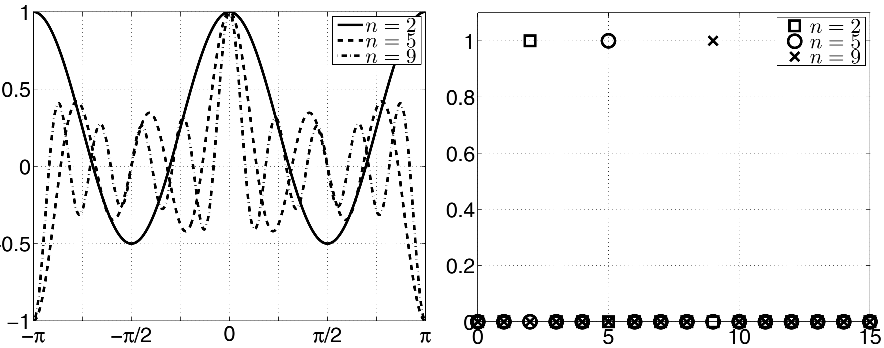


Fig. 7.3: The Legendre kernel P_n for $n = 2, 5, 9$, space representation $\vartheta \mapsto P_n(\cos(\vartheta))$ (*left*) and frequency representation $m \mapsto (P_n)^\wedge(m)$ (*right*).

Localization of the Legendre Kernel. We have with $P_n^* = \sqrt{\frac{2n+1}{2}} P_n$

$$\int_\Omega (P_n^*(\xi \cdot \zeta))^2 \, d\omega(\zeta) = 1 \tag{7.57}$$

for all $\xi \in \Omega$, such that

$$g^{o^{(1)}}_{P_n^*(\xi\cdot)} = 0, \quad \sigma^{o^{(1)}}_{P_n^*(\xi\cdot)} = 1 \tag{7.58}$$

$$g^{-\Delta^*}_{P_n^*(\xi\cdot)} = n(n+1), \quad \sigma^{-\Delta^*}_{P_n^*(\xi\cdot)} = 0. \tag{7.59}$$

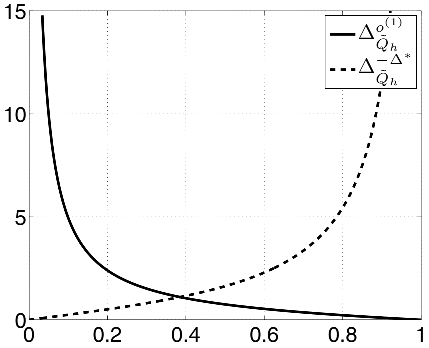


Fig. 7.4: Abel–Poisson kernel uncertainty classification: The functions $h \mapsto \Delta^{o^{(1)}}_{\tilde{Q}_h}$ and $h \mapsto \Delta^{-\Delta^*}_{\tilde{Q}_h}$.

Localization of the Abel–Poisson Kernel. Consider the function $Q_h : [-1,1] \to \mathbb{R}$, $h < 1$, given by (see Fig. 7.4)

$$Q_h(t) = \frac{1}{4\pi} \frac{1-h^2}{(1+h^2-2ht)^{3/2}} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} h^n P_n(t). \tag{7.60}$$

An easy calculation gives us

$$\|Q_h\|_{\mathrm{L}^2[-1,1]} = (Q_{h^2}(1))^{1/2} = \left(\frac{1+h^2}{4\pi}\right)^{1/2} \frac{1}{1-h^2}. \tag{7.61}$$

Furthermore, for $\tilde{Q}_h(t) = \|Q_h\|^{-1}_{\mathrm{L}^2[-1,1]} Q_h(t)$, $t \in [-1,1]$, we obtain after an elementary calculation

$$t^{o^{(1)}}_{\tilde{Q}_h} = \frac{2h}{1+h^2}, \quad (\sigma^{o^{(1)}}_{\tilde{Q}h})^2 = \left(\frac{1-h^2}{1+h^2}\right)^2, \tag{7.62}$$

$$g^{-\Delta^*}_{\tilde{Q}_h} = \frac{6h^2}{(1-h^2)^2}, \quad (\sigma^{-\Delta^*}_{\tilde{Q}_h})^2 = \frac{12h^2(h^4+5h^2+1)}{(1-h^2)^4} \tag{7.63}$$

and

$$\Delta^{o^{(1)}}_{\tilde{Q}_h} = \frac{1-h^2}{2h}, \quad \Delta^{-\Delta^*}_{\tilde{Q}_h} = \frac{\sqrt{6}h}{1-h^2}. \tag{7.64}$$

Thus, we finally obtain

$$\Delta^{o^{(1)}}_{\tilde{Q}_h} \Delta^{-\Delta^*}_{\tilde{Q}_h} = \frac{\sqrt{6}}{2} = \sqrt{\frac{3}{2}} > 1. \tag{7.65}$$

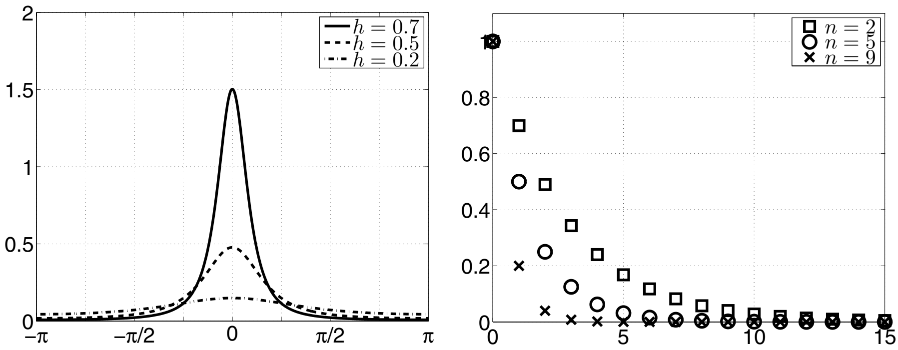


Fig. 7.5: The Abel–Poisson kernel Q_h for $h = 0.7,\ 0.5,\ 0.2$. Space representation $\vartheta \mapsto Q_h(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (Q_h)^\wedge(n)$ (*right*).

Note that in this case, the value $\Delta_{\tilde{Q}_h}^{o(1)} \Delta_{\tilde{Q}_h}^{-\Delta^*}$ is independent of h.

All intermediate cases of 'space-frequency localization' occur when discussing the Abel–Poisson kernel. In fact, it should be pointed out that the Abel–Poisson kernel does not satisfy a minimum uncertainty state.

Letting h formally tend to 1 in the results provided by the uncertainty principle for the Abel–Poisson kernel function, we are able to interpret the localization properties of the *Dirac kernel* on Ω:

$$\delta(\xi \cdot \eta) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} Y_{n,k}(\xi) Y_{n,k}(\eta), \quad \xi, \eta \in \Omega. \tag{7.66}$$

Using the addition theorem, we see that the Dirac kernel is of zonal nature satisfying $\delta^\wedge(n) = 1$ for all $n \in \mathbb{N}_0$:

$$\delta(\xi \cdot \eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{7.67}$$

where the convergence is understood in distributional sense. As a matter of fact, letting h tend to 1 shows us that the variances in the space domain take the constant value 0. On the other hand, the variances in the frequency domain converge to ∞. Hence, the *Dirac kernel shows ideal space localization, but no frequency localization* .

Bandlimited versions δ_N of the Dirac kernel, i.e., truncations of the Dirac kernel in the form

$$\delta_N(\xi \cdot \eta) = \sum_{n=0}^{N} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{7.68}$$

are called *Shannon kernel functions* of degree $N \in \mathbb{N}_0$.

The minimum uncertainty state within the uncertainty relation is provided by the bell-shaped (Gaussian) probability density function (see W. Freeden (1998), N. Laín Fernández (2003)) .

Localization of the Gaussian Function. Consider the function G_λ given by

$$G_\lambda(t) = e^{-(\lambda/2)(1-t)}, \qquad t \in [-1,1], \quad \lambda > 0. \tag{7.69}$$

An elementary calculation shows us that

$$\tilde{G}_\lambda(t) = \gamma(\lambda) e^{-(\lambda/2)(1-t)}, \tag{7.70}$$

with

$$\gamma(\lambda) = (1/\sqrt{4\pi}) \left(\frac{1}{2\lambda}(1 - e^{-2\lambda}) \right)^{-1/2}, \tag{7.71}$$

satisfies $\|\tilde{G}_\lambda\|_{\mathrm{L}^2[-1,1]} = 1$. Furthermore, it is not difficult to deduce (cf. W. Freeden, U. Windheuser (1997)) that $\Delta_{\tilde{G}_\lambda}^{o^{(1)}} \Delta_{\tilde{G}_\lambda}^{-\Delta^*} \to 1$ as $\lambda \to \infty$. This shows us that the best value of the uncertainty principle (Corollary 7.8) is 1.

Summarizing our results, we are led to the following conclusions: The uncertainty principle represents a trade off between two 'spreads', one for the position and the other for the frequency. The main statement is that *sharp localization in space and in frequency are mutually exclusive.* The reason for the validity of the uncertainty relation (Theorem 7.6) is that the operators $o^{(1)}$ and $o^{(3)}$ do not commute. Thus $o^{(1)}$ and $o^{(3)}$ cannot be sharply defined simultaneously. Extremal members in the space/momentum relation are the polynomials (i.e., spherical harmonics) and the Dirac function(al)s. An asymptotically optimal kernel is the Gaussian function.

The estimate (Corollary 7.8) allows us to give a quantitative classification in the form of a canonically defined hierarchy of the space/frequency localization properties of kernel functions of the form

$$K(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) P_n(t), \qquad t = \xi \cdot \eta, \tag{7.72}$$

$(\xi, \eta) \in \Omega \times \Omega$. In view of the amount of space/frequency localization, it is also important to distinguish bandlimited kernels (i.e., $K^\wedge(n) = 0$ for all $n \geq N$) and non-bandlimited ones. Non-bandlimited kernels show a much stronger space localization than bandlimited counterparts. It is not difficult to prove that if $K \in \mathrm{L}^2[-1,1]$ with $\|K(\xi\cdot\ )\|_{\mathrm{L}^2(\Omega)} = 1$,

$$(\sigma^{o^{(1)}}_{K(\xi\cdot\ )})^2 = 1 - \left(\sum_{n=1}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) K^\wedge(n+1)\right)^2 . \tag{7.73}$$

Thus, if $K^\wedge(n) \approx K^\wedge(n+1) \approx 1$ for many successive integers n, then the support of (7.72) in space domain is small.

The varieties of the intensity of the localization on the sphere Ω can be also illustrated by considering the kernel function (7.72). By choosing '$K^\wedge(n) = \delta_{nk}$' we obtain a Legendre kernel of degree k, i.e., we arrive at the left end of our scheme (see Table 7.2). On the other hand, if we formally take $K^\wedge(n) = 1$ for $n = 0, 1, \ldots$, we obtain the kernel which is the Dirac functional in $\mathrm{L}^2(\Omega)$. Band-limited kernels have the property $K^\wedge(n) = 0$ for all $n \geq N$, $N \in \mathbb{N}_0$. Non-bandlimited kernels satisfy $K^\wedge(n) \neq 0$ for an infinite number of integers $n \in \mathbb{N}_0$. Assuming the condition $\lim_{n\to\infty} K^\wedge(n) = 0$, it follows that the slower the sequence $\{K^\wedge(n)\}_{n=0,1,\ldots}$ converges to zero, the lower is the frequency localization, and the higher is the space localization.

Table 7.2: The uncertainty principle and its consequences.

space localization			
←			→
no space localization			ideal space localization
frequency localization			
←			→
ideal frequency localization			no frequency localization
kernel type			
←			→
Legendre kernel	bandlimited	locally supported	Dirac kernel

Altogether, Table 7.2 gives a qualitative illustration of the consequences of the uncertainty principle in the theory of zonal kernel functions on the sphere: On the left end of this scheme, we have the Legendre kernels with their ideal frequency (momentum) localization. However, they show no space localization, as they are of polynomial nature. Thus, the present standard way in applications of increasing the accuracy in spherical harmonic

(Fourier) expansions is to increase the maximum degree of the spherical harmonics expansions under consideration. On the right end of the scheme, there is the Dirac kernel which maps a function to its value at a certain point. Hence, those functionals have ideal space localization but no frequency localization. Consequently, they are used in a finite pointset approximation (see, for example, the thesis due to J. Cui (1997) and the references therein).

Zonal kernel functions exist as bandlimited and non-bandlimited functions. Every bandlimited zonal kernel function refers to a finite number of frequencies. This reduction of the frequency localization allows a finite variance of the space in the uncertainty principle, i.e., this method has both a frequency localization and a space localization. If we move from bandlimited to non-bandlimited zonal kernel functions, the frequency localization usually decreases and the space localization increases in accordance to the uncertainty principle. In consequence, if the accuracy has to be increased in zonal kernel approximation (e.g., by splines and wavelets as proposed in W. Freeden et al. (1998)), a denser point grid is required in the (local) region under investigation.

7.4 Dirac Families of Zonal Scalar Kernel Functions

As already pointed out, the spectral representation of a square-integrable function by means of spherical harmonics is essential to solving many problems in today‘s applications. In future research, however, Fourier (orthogonal) expansions in terms of spherical harmonics $\{Y_{n,j}\}$ will not be the only way of representing a square-integrable function. In order to explain this in more detail, we think of a square-integrable function as a signal in which the spectrum evolves over space in significant way. We imagine that, at each point on the sphere Ω, the function refers to a certain combination of frequencies, and that these frequencies are continuously changing. This space–evolution of the frequencies, however, is not reflected in the Fourier expansion in terms of non–space localizing spherical harmonics, at least not directly. Therefore, in theory, any member F of the space $\mathrm{L}^2(\Omega)$ can be reconstructed from its Fourier transforms, i.e., the ‘amplitude spectrum’ $\{F^\wedge(n,j)\}_{\substack{n=0,1,\ldots,\\ j=1,\ldots,2n+1}}$, but the Fourier transform contains information about the frequencies of the function over all positions instead of showing how the frequencies vary in space.

In what follows, we present a two-parameter, i.e., scale- and space-dependent method of achieving a reconstruction of a function $F \in \mathrm{L}^2(\Omega)$

involving (scalar) zonal kernel functions which we refer to as scaling (kernel) functions converging to the (zonal) Dirac kernel. Roughly speaking, a *Dirac family* as discussed here is a set of zonal kernels $\Phi_\rho : [-1,1] \to \mathbb{R}$, $\rho \in (0,\infty)$, of the form

$$\begin{aligned} \Phi_\rho(\xi \cdot \eta) &= \sum_{n=0}^{\infty} \varphi_\rho(n) \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta), \\ &= \sum_{n=0}^{\infty} \varphi_\rho(n) \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \end{aligned} \tag{7.74}$$

converging to the 'Dirac–kernel' δ as $\rho \to 0$. As shown in Section 7.3, the Dirac kernel can be formally written as zonal kernel function

$$\begin{aligned} \delta(\xi \cdot \eta) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta), \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \end{aligned} \tag{7.75}$$

Consequently, if $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ is a Dirac family of zonal kernels, its 'symbol' $\{\varphi_\rho(n)\}_{n=0,1,\ldots}$ constitutes a sequence satisfying the limit relation

$$\lim_{\substack{\rho\to 0 \\ \rho>0}} \varphi_\rho(n) = 1 \tag{7.76}$$

for each $n = 0, 1, \ldots$. Accordingly, if $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ is a Dirac family of zonal kernels, the convolution integrals

$$(\Phi_\rho * F)(\xi) = \int_\Omega \Phi_\rho(\xi \cdot \eta) F(\eta) \, d\omega(\eta), \quad \xi \in \Omega, \tag{7.77}$$

converge (in a certain topology) to the limit

$$F(\xi) = (\delta * F)(\xi) = \int_\Omega \delta(\xi \cdot \eta) F(\eta) \, d\omega(\eta), \quad \xi \in \Omega, \tag{7.78}$$

for all $\xi \in \Omega$ as ρ tends to 0. In more detail, if F is a function of class $\mathrm{L}^2(\Omega)$ and $\{\Phi_\rho\}$ is a (suitable) Dirac family (tending to the Dirac kernel), then the following limit relation holds true:

$$\lim_{\rho\to 0, \rho>0} \|F - \Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} = 0 \ . \tag{7.79}$$

It should be noted that an approximate convolution identity acts as a space and frequency localization procedure in the following way: As $\{\Phi_\rho\}$, $\rho \in (0,\infty)$, is a Dirac family of zonal scalar kernel functions tending to the

Dirac kernel, the function $\Phi_\rho(\eta\cdot)$, is highly concentrated about the point $\eta \in \Omega$ if the 'scale parameter' is a small positive value. Moreover, as ρ tends to infinity, $\Phi_\rho(\eta\cdot)$ becomes more and more localized in frequency. Correspondingly, the uncertainty principle states that the space localization of $\Phi_\rho(\eta\cdot)$ becomes more and more decreasing. In conclusion, the products $\eta \mapsto \Phi_\rho(\xi \cdot \eta)F(\eta)$, $\eta \in \Omega$, $\xi \in \Omega$, for each fixed value ρ, display information in $F \in \mathrm{L}^2(\Omega)$ at various levels of spatial resolution or frequency bands. Consequently, as ρ approaches ∞, the convolution integrals $\Phi_\rho * F = \int_\Omega \Phi_\rho(\cdot\eta)F(\eta)\, d\omega(\eta)$ display coarser, lower-frequency features. As ρ approaches 0, the integrals give sharper and sharper spatial resolution. In other words, the convolution integrals can measure the space-frequency variations of spectral components, but they have a different space-frequency resolution.

Each scale approximation $\Phi_\rho * F$ of a function $F \in \mathrm{L}^2(\Omega)$ must be made directly by computing the relevant convolution integrals. In doing so, however, it is inefficient to use no information from the approximation $\Phi_\rho * F$ within the computation of $\Phi_{\rho'} * F$ provided that $\rho' < \rho$. In fact, the efficient construction of multiscale approximation based on Dirac families begins by a *multiresolution analysis* in terms of wavelets, i.e., a recursive method which is ideal for computation (see W. Freeden et al. (1998), W. Freeden, V. Michel (2005) and the references therein). However, this aspect of constructive approximation will not be discussed here in our approach to spherical functions relevant for geoscientific purposes.

A mathematically rigorous formulation of a Dirac family is as follows.

Definition 7.10. Let $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ be a family of functions in $\mathrm{L}^2[-1,1]$ satisfying the condition

$$(\Phi_\rho)^\wedge(0) = 2\pi \int_{-1}^{1} \Phi_\rho(t)P_0(t)\, dt = 2\pi \int_{-1}^{1} \Phi_\rho(t)\, dt = 1 \tag{7.80}$$

for all $\rho \in (0,\infty)$. Then $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ is said to be a *Dirac family in* $\mathrm{L}^2(\Omega)$, if

$$\lim_{\rho\to 0} \|F - \Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} = 0 \tag{7.81}$$

for all $F \in \mathrm{L}^2(\Omega)$.

Remark 7.11. In the jargon of approximation theory, the family $\{I_\rho\}_{\rho\in(0,\infty)}$ of operators I_ρ given by $I_\rho F = \Phi_\rho * F$ is called a (spherical) *singular integral,* and $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ itself is called the *kernel of the singular integral* (or, briefly, *scaling function*). However, we want to point out the convergence of $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ to the Dirac kernel δ as $\rho \to 0$. This is the reason why we use the notation of the Dirac family in our approach.

The convergence of a Dirac family of scalar zonal kernel functions to the scalar Dirac kernel is described in more detail by the following theorem.

Theorem 7.12. *Let* $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *be a family of functions in* $\mathrm{L}^2[-1,1]$ *satisfying*

$$(\Phi_\rho)^\wedge(0) = 1 \tag{7.82}$$

and

$$2\pi \int_{-1}^{1} |\Phi_\rho(t)| \, dt \leq M \tag{7.83}$$

for all $\rho \in (0,\infty)$ *with some constant* M *independent of* ρ*. Then* $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *is a Dirac family in* $\mathrm{L}^2(\Omega)$ *if and only if*

$$\lim_{\rho\to 0} (\Phi_\rho)^\wedge(n) = 1 \tag{7.84}$$

for all $n \in \mathbb{N}_0$*.*

Proof. We have to verify the equivalence (see W. Freeden, K. Hesse (2002)).

$\Longleftarrow$: From the definition of a Dirac family in $\mathrm{L}^2(\Omega)$, we are able to deduce that

$$\lim_{\rho\to 0} \|F - \Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} = 0 \tag{7.85}$$

for all $F \in \mathrm{L}^2(\Omega)$. Particularly, this holds for all spherical harmonics Y_n of degree n. The Funk-Hecke formula implies that $\Phi_\rho * Y_n = (\Phi_\rho)^\wedge(n) Y_n$. Thus, it follows that

$$\begin{aligned} 0 &= \lim_{\rho\to 0} \|Y_n - \Phi_\rho * Y_n\|_{\mathrm{L}^2(\Omega)} \\ &= \lim_{\rho\to 0} |1 - (\Phi_\rho)^\wedge(n)| \, \|Y_n\|_{\mathrm{L}^2(\Omega)} \end{aligned}$$

and $\lim_{\rho\to 0}(\Phi_\rho)^\wedge(n) = 1$ follows because of $\|Y_n\|_{\mathrm{L}^2(\Omega)} \neq 0$ for all spherical harmonics $Y_n \neq 0$ of degree $n \in \mathbb{N}_0$.

$\Longrightarrow$: The uniform boundedness in the sense of (7.83) imposed on the functions Φ_ρ, $\rho \in (0,\infty)$, and the estimate $|P_n(t)| \leq 1$ for all $t \in [-1,1]$ and all $n \in \mathbb{N}_0$ imply that

$$(\Phi_\rho)^\wedge(n) \leq 2\pi \int_{-1}^{1} |\Phi_\rho(t)| \, |P_n(t)| \, dt \leq 2\pi \int_{-1}^{1} |\Phi_\rho(t)| \, dt \leq M,$$

$$(\Phi_\rho)^\wedge(n) \geq -2\pi \int_{-1}^{1} |\Phi_\rho(t)| \, |P_n(t)| \, dt \geq -2\pi \int_{-1}^{1} |\Phi_\rho(t)| \, dt \geq -M.$$

Hence, $(\Phi_\rho)^\wedge(n) \in [-M, M]$ for all $n \in \mathbb{N}_0$ and all $\rho \in (0,\infty)$. Therefore,

$$\begin{aligned} \|F - \Phi_\rho * F\|^2_{\mathrm{L}^2(\Omega)} &= \sum_{n=0}^{\infty} \sum_{l=1}^{2n+1} \left(1 - (\Phi_\rho)^\wedge(n)\right)^2 \left(F^\wedge(n,l)\right)^2 \\ &\leq (M+1)^2 \, \|F\|^2_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{7.86}$$

for all $\rho \in (0,\infty)$ and all $F \in \mathrm{L}^2(\Omega)$. As the upper bound $(M+1)$ of the term $|1 - (\Phi_\rho)^\wedge(n)|$ is independent of $\rho \in (0,\infty)$, the limit for $\rho \to 0$ and the infinite sum may be interchanged. Hence,

$$\begin{aligned} &\lim_{\rho\to 0} \|F - \Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} \\ &= \left(\sum_{n=0}^{\infty} \sum_{l=1}^{2n+1} \lim_{\rho\to 0} \left(1 - (\Phi_\rho)^\wedge(n)\right)^2 \left(F^\wedge(n,l)\right)^2 \right)^{1/2} = 0 \end{aligned} \tag{7.87}$$

for all $F \in \mathrm{L}^2(\Omega)$, as required. □

Restricting our attention to non-negative kernels $\{\Phi_\rho\}_{\rho\in(0,\infty)}$, (i.e., all Φ_ρ, $\rho \in (0,\infty)$, satisfy $\Phi_\rho(t) \geq 0$ for almost all $t \in [-1,1]$), more equivalent characterizations of a Dirac family are deducible. The main advantage of non-negative kernels $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ is that the property $(\Phi_\rho)^\wedge(0) = 1$ implies

$$1 = (\Phi_\rho)^\wedge(0) = 2\pi \int_{-1}^{1} \Phi_\rho(t)\, dt = 2\pi \int_{-1}^{1} |\Phi_\rho(t)|\, dt \tag{7.88}$$

i. e., the condition (7.83) is valid with the sharp bound $M = 1$.

Theorem 7.13. *Let* $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *be a family of functions in* $\mathrm{L}^2[-1,1]$*, which satisfy* $(\Phi_\rho)^\wedge(0) = 1$ *and which are non-negative. Then the following properties are equivalent:*

(i) $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *is a non-negative Dirac family in* $\mathrm{L}^2(\Omega)$*,*

(ii) $\lim_{\rho\to 0} \|F - \Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} = 0$ *for all* $F \in \mathrm{L}^2(\Omega)$*,*

(iii) $\lim_{\rho\to 0} (\Phi_\rho)^\wedge(n) = 1$ *for all* $n \in \mathbb{N}_0$*,*

(iv) $\lim_{\rho\to 0} (\Phi_\rho)^\wedge(1) = 1$*,*

(v) $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *satisfies the 'localization property'*

$$\lim_{\rho\to 0} \int_{-1}^{\delta} \Phi_\rho(t)\, dt = 0$$

for all $\delta \in (-1,1)$*.*

Proof. The statements (i) and (ii) are equivalent by definition and the equivalence of (ii) and (iii) is clear from Theorem 7.13. Obviously, (iii) implies (iv). It remains to show, that (v) follows from (iv) and that (v) implies (iii).

(iv) $\Longrightarrow$ (v): Let $\delta \in (-1,1)$ be arbitrary. Because of the non-negativity of the kernels Φ_ρ,

$$\begin{aligned} 0 \leq \int_{-1}^{\delta} \Phi_\rho(t)\,dt &\leq \frac{1}{(1-\delta)} \int_{-1}^{\delta} (1-t)\,\Phi_\rho(t)\,dt \\ &\leq \frac{1}{(1-\delta)} \int_{-1}^{1} (1-t)\,\Phi_\rho(t)\,dt \\ &= \frac{1}{2\pi}\,\frac{1}{(1-\delta)}\Big((\Phi_\rho)^\wedge(0) - (\Phi_\rho)^\wedge(1)\Big). \end{aligned} \tag{7.89}$$

Taking the limit for $\rho \to 0$ the localization property follows from (vi).

(v) $\Longrightarrow$ (iii): Property (iii) is equivalent to the following assertion: For every $n \in \mathbb{N}$ and for every $\varepsilon > 0$, there exists a value $\rho_0 = \rho_0(\varepsilon, n) \in (0,\infty)$ such that $1-\varepsilon \leq (\Phi_\rho)^\wedge(n)) \leq 1$ for all $\rho \in (0,\rho_0]$. By the non-negativity of Φ_ρ and the estimate $|P_n(t)| \leq 1$ for all $n \in \mathbb{N}_0$,

$$(\Phi_\rho)^\wedge(n) = 2\pi \int_{-1}^{1} \Phi_\rho(t)\,P_n(t)\,dt \leq 2\pi \int_{-1}^{1} \Phi_\rho(t)\,dt = (\Phi_\rho)^\wedge(0) = 1. \tag{7.90}$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. For $\delta \in (-1,1)$,

$$(\Phi_\rho)^\wedge(n) = 2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,P_n(t)\,dt + 2\pi \int_{\delta}^{1} \Phi_\rho(t)\,P_n(t)\,dt. \tag{7.91}$$

As $P_n(1) = 1$, $\delta \in (-1,1)$ can be chosen so close to 1 that $P_n(t) \geq \sqrt{1-(\varepsilon/2)}$ for all $t \in [\delta,1]$. Thus,

$$(\Phi_\rho)^\wedge(n) \geq 2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,P_n(t)\,dt + 2\pi\,\sqrt{1-(\varepsilon/2)} \int_{\delta}^{1} \Phi_\rho(t)\,dt. \tag{7.92}$$

As $|P_n(t)| \leq 1$ for all $\delta \in (-1,1)$,

$$-2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,dt \leq 2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,P_n(t)\,dt \leq 2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,dt. \tag{7.93}$$

Therefore, the localization property (v) implies that there exists ρ_1, such that the estimate $2\pi \int_{-1}^{\delta} \Phi_\rho(t)\,P_n(t)\,dt \geq -\varepsilon/2$ is valid for all $\rho \in (0,\rho_1)$. On the other hand, $(\Phi_\rho)^\wedge(0) = 1$ for all $\rho \in (0,\infty)$, and the localization property implies

$$
\begin{aligned}
\frac{1}{2\pi} &= \lim_{\rho\to 0} \int_{-1}^{1} \Phi_\rho(t)\, dt \qquad (7.94)\\
&= \lim_{\rho\to 0} \int_{-1}^{\delta} \Phi_\rho(t)\, dt + \lim_{\rho\to 0} \int_{\delta}^{1} \Phi_\rho(t)\, dt\\
&= \lim_{\rho\to 0} \int_{\delta}^{1} \Phi_\rho(t)\, dt.
\end{aligned}
$$

Hence, there exists $\rho_2 \in (0,\infty)$ such that $2\pi \int_\delta^1 \Phi_\rho(t)\, dt \geq \sqrt{1-(\varepsilon/2)}$ for all $\rho \in (0,\rho_2)$. The relation (7.92) implies $1-\varepsilon \leq (\Phi_\rho)^\wedge(n) \leq 1$ for all $\rho \in (0,\rho_0)$ with $\rho_0 = \min\{\rho_1,\rho_2\}$. □

Theorem 7.13 immediately leads us to the following notation.

Definition 7.14. A family $\{\Phi_\rho\}_{\rho\in(0,\infty)} \subset \mathrm{L}^2[-1,1]$ satisfying the conditions

(i) $(\Phi_\rho)^\wedge(0) = 1$,

(ii) Φ_ρ is non-negative on $[-1,1]$,

(iii) $\lim_{\rho\to 0} \Phi_\rho(t)\ dt = 0,\ \delta \in (-1,1)$,

is called a *Dirac family of non-negative type in* $\mathrm{L}^2(\Omega)$.

Finally, it is worth mentioning that a Dirac family of non-negative type in $\mathrm{L}^2(\Omega)$, i.e., a family $\{\Phi_\rho\}_{\rho\in(0,\infty)} \subset \mathrm{L}^2[-1,1]$, fulfilling the assumptions of Theorem 7.13 satisfies the estimate

$$
\|\Phi_\rho * F\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)} \qquad (7.95)
$$

for all $\rho \in (0,\infty)$ and for all $F \in \mathrm{L}^2(\Omega)$.

For a categorization of certain examples of Dirac families of scalar zonal kernel functions, the following definition is helpful (see, e.g., H. Berens et al. (1969)).

Definition 7.15. A family $\{I_\rho\}, I_\rho : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, $\rho \in (0,\infty)$, is called a *semigroup of contraction operators on* $\mathrm{L}^2(\Omega)$, if the following properties are satisfied:

(i) For each $\rho \in (0,\infty), I_\rho$ is a linear bounded operator mapping $\mathrm{L}^2(\Omega)$ into itself and $I_0 = I$ (identity operator).

(ii) $I_{\rho_1+\rho_2} = I_{\rho_1} I_{\rho_2}$, $0 \leq \rho_1 \leq \rho_2 < \infty$

(iii) $\lim_{\substack{\rho \to 0 \\ \rho > 0}} \|I_\rho(F) - F\|_{\mathrm{L}^2(\Omega)} = 0$, $F \in \mathrm{L}^2(\Omega)$

(iv) $\|I_\rho(F)\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)}, \rho \in (0, \infty)$, $F \in \mathrm{L}^2(\Omega)$.

Examples of semigroups of contraction operators on $\mathrm{L}^2(\Omega)$ will be discussed in Section 7.5.

We conclude this section by making some words about the spherical *wavelet transform* (for references, the reader is referred to the list in Section 7.6) The wavelet transform acts as a space and frequency localization operator in the following way: If $\{\Phi_\rho\}, \rho \in (0, \infty)$, is a Dirac family and ρ approaches infinity, the convolution integrals

$$\Phi_\rho * U = \int_\Omega \Phi_\rho(\cdot, \eta) F(\eta)\ d\omega(\eta) \tag{7.96}$$

display coarser, lower-frequency features. As ρ approaches zero, the integrals give sharper and sharper spatial resolution. In other words, the convolution integrals can measure the space-frequency variations of spectral components, but they have a different space-frequency resolution.

Each scale-space approximation $\Phi_\rho * F$ of a function $F \in \mathrm{L}^2(\Omega)$ must be made directly by computing the relevant convolution integrals. In doing so, however, it is inefficient to use no information from the approximation $\Phi_\rho * F$ within the computation of $\Phi_{\rho'} * F$ provided that $\rho' < \rho$. In fact, the efficient construction of wavelets begins by a multiresolution analysis, i.e., a completely recursive method which is therefore ideal for computation. In this context, we observe that

$$\int_R^\infty \int_\Omega \Psi_\rho(\cdot, \eta) F(\eta)\ d\omega(\eta) \frac{d\sigma}{\sigma} \to F \in \mathrm{L}^2(\Omega), \quad R \to 0, \tag{7.97}$$

i.e.,

$$\lim_{\substack{R \to 0 \\ R > 0}} \left\| F - \int_R^\infty \int_\Omega \Psi_\rho(\xi, \eta) F(\eta)\ d\omega(\eta) \frac{d\rho}{\rho} \right\|_{\mathrm{L}^2(\Omega)} = 0, \tag{7.98}$$

provided that

$$\Psi_\rho(\xi, \eta) = \sum_{n=0}^\infty \Psi_\rho^\wedge(n) \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) \quad (\xi, \eta) \in \Omega \times \Omega, \tag{7.99}$$

is given such that

$$\psi_\rho^\wedge(n) = -\rho \frac{d}{d\rho} \Phi_\rho^\wedge(n) \tag{7.100}$$

for $n = 0, 1, \ldots$ and all $\rho \in (0, \infty)$. Conventionally, the family $\{\Psi_\rho\}$, $\rho \in (0, \infty)$, is called a (scale continuous) wavelet. The (scale continuous) wavelet transform (WT): $\mathrm{L}^2(\Omega) \to \mathrm{L}^2((0, \infty) \times \Omega)$ is defined by

$$(WT)(F)(\rho; \xi) = (\Psi_\rho) * F(\xi) = \int_\Omega \Psi_\rho(\xi, \eta) F(\eta) \, d\omega(\eta). \tag{7.101}$$

In other words, the wavelet transform is defined as the $\mathrm{L}^2(\Omega)$–inner product of $F \in \mathrm{L}^2(\Omega)$ with the set of 'rotations' and 'dilations' of F. The (scale continuous) wavelet transform (WT) is invertible on $\mathrm{L}^2(\Omega)$, i.e.,

$$F = \int_\Omega \int_0^\infty (WT)(F)(\rho; \eta) \Psi_\rho(\cdot, \eta) \frac{d\rho}{\rho} \, d\omega(\eta) \tag{7.102}$$

in the sense of $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. From Parseval's identity in terms of scalar spherical harmonics, it follows that

$$\int_\Omega \int_0^\infty (\Psi_\rho * F)(\eta) \frac{d\rho}{\rho} d\omega(\eta) = \|F\|^2_{\mathrm{L}^2(\Omega)} \tag{7.103}$$

i.e., (WT) converts a function F of one variable into a function of two variables $\xi \in \Omega$ and $\rho \in (0, \infty)$ without changing its total energy.

In terms of filtering $\{\Phi_\rho\}$ and $\{\Psi_\rho\}$, $\rho \in (0, \infty)$ may be interpreted as low-pass filter and bandpass filter, respectively. Correspondingly, the convolution operators are given by

$$\Phi_\rho * F, \quad F \in \mathrm{L}^2(\Omega), \tag{7.104}$$
$$\Psi_\rho * F, \quad F \in \mathrm{L}^2(\Omega). \tag{7.105}$$

The Fourier transforms read as follows:

$$(\Phi_\rho * F)^\wedge(n, j) = F^\wedge(n, j) \Phi_\rho^\wedge(n), \tag{7.106}$$
$$(\Psi_\rho * F)^\wedge(n, j) = F^\wedge(n, j) \Psi_\rho^\wedge(n). \tag{7.107}$$

These formulas provide the transition from the wavelet transform to the Fourier transform.

Since all scales ρ are used, the reconstruction is highly redundant. Of course, the redundancy leads us to the following question which is of particular importance in data analysis:

- Given an arbitrary $H \in \mathrm{L}^2((0, \infty) \times \Omega)$, how can we know whether $H = (WT)(F)$ for some function $F \in \mathrm{L}^2(\Omega)$?

The question amounts to finding the range of the (scale continuous) wavelet transform $(WT) : \mathrm{L}^2(\Omega) \to \mathrm{L}^2((0, \infty) \times \Omega)$ (see W. Freeden et al. (1998)), i.e., the subspace

$$\mathcal{W} = (WT)(\mathrm{L}^2(\Omega)) \neq \mathrm{L}^2((0, \infty) \times \Omega). \tag{7.108}$$

Actually, it can be shown that the tendency for minimizing errors by use of the wavelet transform is again expressed in least-squares approximation:

Let H be an arbitrary element of $\mathrm{L}^2((0,\infty)\times\Omega)$. Then the unique function $F_H \in \mathrm{L}^2(\Omega)$ with the property

$$\|H-(WT)(F_H)\|_{\mathrm{L}^2((0,\infty)\times\Omega)} = \inf_{U\in\mathrm{L}^2(\Omega)} \|H-(WT)(U)\|_{\mathrm{L}^2((0,\infty)\times\Omega)} \tag{7.109}$$

is given by

$$F_H = \int_0^\infty \int_\Omega H(\rho;\eta)\Psi_\rho(\cdot,\eta)d\omega(\eta)\frac{d\rho}{\rho}. \tag{7.110}$$

$(WT)(F_H)$ is indeed the orthogonal projection of H onto $\mathcal{W}$.

Another important question in the context of the wavelet transform is:

- Given an arbitrary $H(\rho;\xi) = (WT)(F)(\rho;\xi), \rho \in (0,\infty)$, and $\xi \in \Omega$, for some $F \in \mathrm{L}^2(\Omega)$, how can we reconstruct F?

The answer is provided by the so-called least-energy representation. It states: Of all possible functions $H \in \mathrm{L}^2((0,\infty)\times\Omega)$ for $F \in \mathrm{L}^2(\Omega)$, the function $H = (WT)(F)$ is unique in that it minimizes the 'energy' $\|H\|^2_{\mathrm{L}^2((0,\infty)\times\Omega}$. More explicitly (see W. Freeden et al. (1998))

$$\|(WT)(F)\|_{\mathrm{L}^2((0,\infty)\times\Omega)} = \inf_{\substack{H\in\mathrm{L}^2((0,\infty)\times\Omega)\\ (WT)^{-1}(H)=F}} \|H\|_{\mathrm{L}^2((0,\infty)\times\Omega)}.$$

7.5 Examples of Dirac Families

Several types of Dirac families can be distinguished which are of basic interest in applications (see Table 7.3).

Space limited, i.e., locally supported kernel functions are nothing new, having been discussed in one-dimensional Euclidean space already by A. Haar (1910). In what follows, we first present the classical concept initiated by Haar in a generalization to the spherical case (see Fig. 7.6):

Example 7.16. The *Haar Dirac family* $\{H_h\}_{h\in(0,1)} \subset \mathrm{L}^2[-1,1]$, $H_h : [-1,1] \to R$, $t \mapsto H_h(t)$, $h = e^{-\rho}, \rho \in (0,\infty)$, is given by

$$H_h(t) = \begin{cases} 0 & , \quad t \in [-1,h] \\ \frac{1}{2\pi}\,\frac{1}{(1-h)} & , \quad t \in [h,1]. \end{cases} \tag{7.111}$$

Table 7.3: Three types of kernels: bandlimited, spacelimited, and non–spacelimited/non-bandlimited.

<table>
<tr><td rowspan="4">Legendre kernels

$K^\wedge(n) = \delta_{n,k}$</td><td colspan="2">Zonal kernels

general case</td><td rowspan="4">Dirac kernel

$K^\wedge(n) = 1,$
$n = 0, \ldots$</td></tr>
<tr><td>bandlimited</td><td>spacelimited</td></tr>
<tr><td>$K^\wedge(n) = 0,$
$n > N$</td><td>$K(\xi \cdot \eta) = 0,$
$1 - \xi \cdot \eta \geq \delta$</td></tr>
<tr><td>Shannon

$K^\wedge(n) = 1,$
$n \leq N$</td><td>Haar

$K(\xi \cdot \eta) = 1,$
$1 - \xi \cdot \eta \leq \delta$</td></tr>
</table>

Obviously, $H_h(t) \geq 0$ for all $t \in [-1,1]$ and $(H_h)^\wedge(0) = 2\pi \, \|H_h\|_{\mathrm{L}^2[-1,1]} = 1$ are fulfilled. Thus $\{H_h\}_{h \in (0,1)}$ generates an approximate identity in $\mathrm{L}^2(\Omega)$. Further properties of the Haar Dirac family follow in the next example by specialization to the case $k = 0$.

Example 7.17. Let k be a non-negative integer, i.e., $k \in \mathbb{N}_0$. The *smoothed Haar Dirac family* $\{L_h^{(k)}\}_{h \in (0,1)} \subset \mathrm{C}^{(k-1)}[-1,1]$, $\rho \in (0,\infty)$, is defined by $L_h^{(k)} : [-1,1] \to \mathbb{R}$, $t \mapsto L_h^{(k)}(t)$, where (cf. (3.44))

$$L_h^{(k)}(t) = ((B_h^{(k)})^\wedge(0))^{-1} B_h^{(k)}(t) \tag{7.112}$$

with

$$B_h^{(k)}(t) = \begin{cases} 0 & , \quad t \in [-1,h) \\ \dfrac{(t-h)^k}{(1-h)^k} & , \quad t \in [h,1]. \end{cases} \tag{7.113}$$

By definition, $L_h^{(k)}$ is non-negative, has the support $[h,1]$, and satisfies $(L_h^{(k)})^\wedge(0) = 1$. Hence, it is a non-negative $[h,1]$-locally supported Dirac

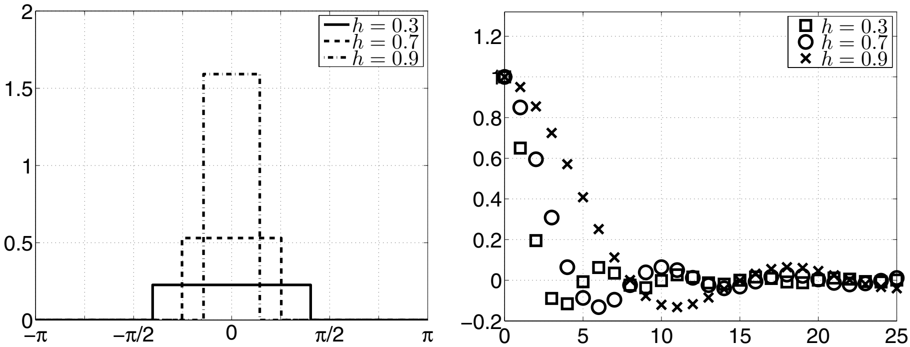


Fig. 7.6: The Haar kernel H_h for $h = 0.3,\ 0.7,\ 0.9$. Space representation $\vartheta \mapsto H_h(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (H_h)^\wedge(n)$ (*right*).

family. Obviously, the function $L_h^{(0)}$, $h \in (-1,1)$, coincides with the Haar function H_h. The Legendre coefficients of $B_h^{(k)}$ and, hence, $L_h^{(k)}$, $h \in (-1,1)$, $k \in \mathbb{N}_0$, can be calculated recursively (cf. W. Freeden et al. (1998)):

$$(B_h^{(k)})^\wedge(0) = 2\pi \left(\frac{1-h}{k+1}\right) \neq 0, \tag{7.114}$$

$$(B_h^{(k)})^\wedge(1) = 2\pi \left(\frac{1-h}{k+1}\right)\left(1 - \frac{1-h}{k+2}\right), \tag{7.115}$$

$$(B_h^{(k)})^\wedge(n+1) = \left(\frac{2n+1}{n+k+2}\right) h\,(B_h^{(k)})^\wedge(n) + \left(\frac{k+1-n}{n+k+2}\right) (B_h^{(k)})^\wedge(n-1). \tag{7.116}$$

It can be shown by use of the estimates for the Legendre polynomials that $|(L_h^{(k)})^\wedge(n)| = O\left((n(1-h))^{-(3/2)-k}\right)$ for $n \to \infty$. The functions $L_h^{(k)}$, $h \in (0,1)$, $k \in \mathbb{N}_0$, assume their maximum in $t = 1$. For $k > 2$, the Lipschitz-constant $C_h^{(k)}$ for $L_h^{(k)}$ can be chosen as the maximum of the first derivative, which is also taken in the point $t = 1$. Thus, we obtain

$$\sup_{t\in[-1,1]} |L_h^{(k)}(t)| = L_h^{(k)}(1) = \frac{1}{2\pi}\,\frac{(k+1)}{(1-h)}, \qquad k \in \mathbb{N}_0, \tag{7.117}$$

and

$$C_h^{(k)} = \sup_{t\in[-1,1]} |(L_h^{(k)})^{(1)}(t)| = (L_h^{(k)})'(1) = \frac{1}{2\pi}\,\frac{k(k+1)}{(1-h)^2}, \qquad k \geq 2. \tag{7.118}$$

The function $L_h^{(0)}$ is constant on its support. Consequently, Equation (7.118) is also valid for $k = 0$ on $\mathrm{supp}(L_h^{(0)}) = [h,1]$. For $k = 1$, the function $L_h^{(k)}$ is continuous and piecewise linear, thus the Lipschitz-constant $C_h^{(1)}$ can be chosen as the first derivative of $L_h^{(1)}$ on $\mathrm{supp}(L_h^{(1)})$. Hence, (7.118) is also true for $k = 1$.

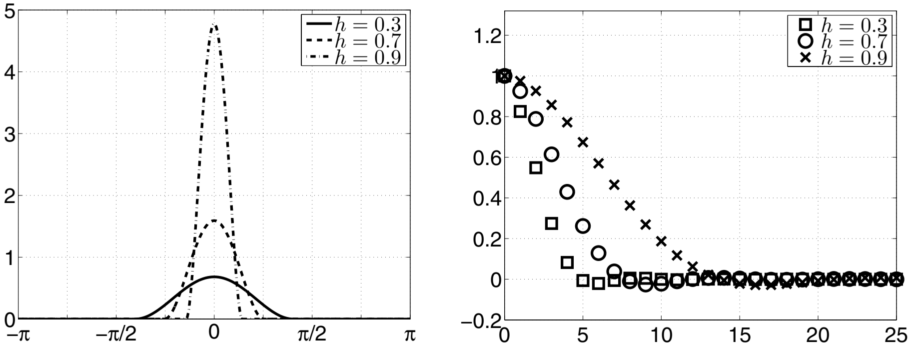


Fig. 7.7: The smoothed Haar kernel $L_h^{(k)}$ for $k = 2$ and $h = 0.3,\ 0.7,\ 0.9$. Space representation $\vartheta \mapsto L_h^{(k)}(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (L_h^{(k)})^\wedge(n)$ (*right*).

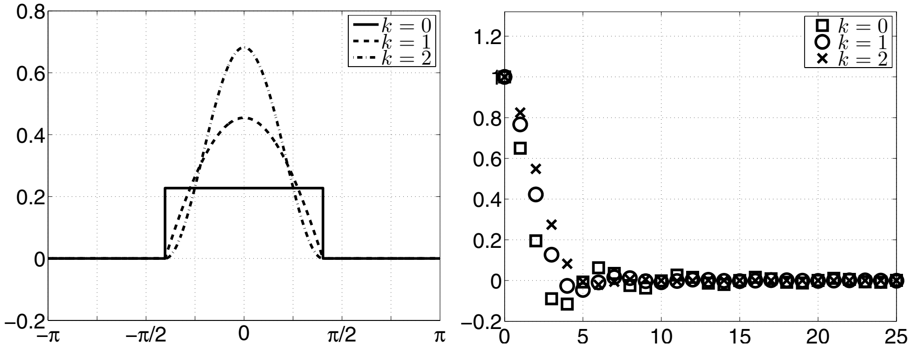


Fig. 7.8: The smoothed Haar kernel $L_h^{(k)}$ for $h = 0.3$ and $k = 0,\ 1,\ 2$. Space representation $\vartheta \mapsto L_h^{(k)}(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (L_h^{(k)})^\wedge(n)$ (*right*).

In order to discuss the Dirac property for the Haar functions in more detail, we consider the averages

$$M_h^{(k)}(F)(\xi) = \int_\Omega L_h^{(k)}(\xi \cdot \eta) F(\eta)\, d\omega(\eta), \quad \xi \in \Omega, k \geq 0, \tag{7.119}$$

where (see Figs. 7.7 and 7.8 for a graphical illustration)

$$L_h^{(k)}(\xi \cdot \eta) = \left((B_h^{(k)})^\wedge(0) \right)^{-1} B_h^{(k)}(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{7.120}$$

Clearly, the case $k = 0$ describes an equally weighted average over a spherical cap. For $\xi \in \Omega$ and $F \in \mathrm{C}(\Omega)$, we have

$$
\begin{aligned}
|M_h^{(0)}(F)(\xi) - F(\xi)| \quad & (7.121)\\
= \|\Omega(\xi;h)\|^{-1} \left| \int_{\xi\cdot\eta \geq h, |\eta|=1} (F(\eta) - F(\xi))\, d\omega(\eta) \right| & \\
\leq \sup_{h \leq \xi\cdot\eta \leq 1} |F(\eta) - F(\xi)|. \quad & (7.122)
\end{aligned}
$$

Thus, it is easy to see that

$$\|M_h^{(0)}(F) - F\|_{\mathrm{C}(\Omega)} \leq \mu(F; 1-h). \tag{7.123}$$

Moreover, for $h \in [0,1)$ and $F \in \mathrm{L}^2(\Omega)$,

$$\|M_h^{(0)}(F)\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)} \tag{7.124}$$

and

$$\lim_{\substack{h\to 1\\ h<1}} \|M_h^{(0)}(F) - F\|_{\mathrm{L}^2(\Omega)} = 0. \tag{7.125}$$

For $F, H \in \mathrm{L}^2(\Omega)$ the identity

$$-n(n+1)F^\wedge(n,j) = H^\wedge(n,j), n = 0,1,\ldots, j = 1,\ldots,2n+1, \tag{7.126}$$

is equivalent to the limit relation

$$\lim_{\substack{h\to 1\\ h<1}} \left\| \frac{M_h^{(0)}(F) - F}{h^2} - H \right\|_{\mathrm{L}^2(\Omega)} = 0. \tag{7.127}$$

In fact, as W. Freeden et al. (1998) have pointed out, we have

$$\lim_{\substack{h\to 1\\ h<1}} \frac{M_h^{(0)}(F)(\xi) - F(\xi)}{h^2} = \Delta_\xi^* F(\xi), \quad \xi \in \Omega, \tag{7.128}$$

provided that F is a member of class $\mathrm{C}^{(2)}(\Omega)$. By similar arguments (see, e.g., H. Berens et al. (1969)) we are able to show that, for $F \in \mathrm{L}^2(\Omega)$, the relation

$$\|M_h^{(0)}(F) - F\|_{\mathrm{L}^2(\Omega)} = O((1-h)^2) \tag{7.129}$$

is equivalent to the fact that there exists a function $G \in \mathrm{L}^2(\Omega)$ such that

$$G^\wedge(n,j) = -n(n+1)F^\wedge(n,j), \quad n = 0,1,\ldots,\ j = 1,\ldots,2n+1. \tag{7.130}$$

Next, we discuss higher order averages, i.e., $k \geq 1$. For $k = 1,2,\ldots$, $h \in (0,1)$, and $F \in (\Omega)$ it can be readily seen that

$$\left\| M_h^{(k)}(F) \right\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)} \tag{7.131}$$

and

$$\lim_{\substack{h\to 1\\ h<1}} \|M_h^{(k)}(F) - F\|_{L^2(\Omega)} = 0. \tag{7.132}$$

Using the modulus of continuity, we obtain for $F \in C(\Omega)$

$$\| M_h^{(k)}(F) - F \|_{C(\Omega)} \le \mu(F; 1-h). \tag{7.133}$$

Moreover, it follows that

$$\int_\Omega P_n(\xi\cdot\zeta) M_h^{(k)}(F)(\zeta)\, d\omega(\zeta) = \frac{(B_h^{(k)})^\wedge(n)}{(B_h^{(k)})^\wedge(0)} \int_\Omega P_n(\xi\cdot\zeta)F(\zeta)\, d\omega(\zeta). \tag{7.134}$$

Furthermore, we find

$$L_h^{(k)} = \sum_{n=0}^\infty \frac{2n+1}{4\pi}(L_h^{(k)})^\wedge(n) P_n \tag{7.135}$$

in the sense of $\|\cdot\|_{L^2[-1,1]}$. For $t\in[-1,1]$ and $k = 1,2,\ldots$ we have

$$\left| L_h^{(k)}(t) \right| \le \frac{1}{2\pi}\frac{k+1}{1-h}. \tag{7.136}$$

Moreover, for all $t, t' \in [-1,1]$ and $k = 1,2,\ldots$

$$\left| L_h^{(k)}(t) - L_h^{(k)}(t') \right| \le \frac{1}{2\pi}\frac{2^{k-1}k(k+1)}{(1-h)^{k+1}} |t-t'|. \tag{7.137}$$

The values $(B_h^{(k)})^\wedge(n)$ are the eigenvalues of the operator $M_h^{(k)} : C(\Omega) \to C(\Omega)$, i.e., $M_h^{(k)} Y_n = (B_h^{(k)})^\wedge(n) Y_n$ for all $Y_n \in \mathrm{Harm}_n$.

Finally, we mention that the second iterations $(L_h^{(k)})^{(2)}$ are given by

$$(L_h^{(k)})^{(2)}(\xi\cdot\eta) = ((B_h^{(k)})^\wedge(0))^{-2} \int_\Omega B_h^{(k)}(\xi\cdot\zeta) B_h^{(k)}(\zeta\cdot\eta)\, d\omega(\zeta). \tag{7.138}$$

It is obvious from (7.136) and (7.137) that for all $\xi, \xi', \eta \in \Omega$, and $k = 1,2,\ldots$

$$\left| (L_h^{(k)})^{(2)}(\xi\cdot\eta) \right| \le 2\frac{(k+1)^2}{(1-h)^2}, \tag{7.139}$$

and

$$\left| (L_h^{(k)})^{(2)}(\xi\cdot\eta) - (L_h^{(k)})^{(2)}(\xi'\cdot\eta) \right| \le \frac{1}{2\pi}\frac{2^k k^2 (k+1)^2}{(1-h)^{k+2}} |\xi - \xi'|. \tag{7.140}$$

The operators $(M_h^{(k)})^{(2)} : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ are defined by

$$\begin{aligned} (M_h^{(k)})^{(2)}(F)(\xi) &= M_h^{(k)}(M_h^{(k)}(F))(\xi) \\ &= \int_\Omega (L_h^{(k)})^{(2)}(\xi\cdot\eta)F(\eta)\,d\omega(\eta), \quad \xi\in\Omega. \end{aligned} \tag{7.141}$$

For all $\xi \in \Omega$, we find that

$$(M_h^{(k)})^{(2)}(F)(\xi) - F(\xi) = \int\limits_{-1+2h^2\le\xi\cdot\eta\le 1} (L_h^{(k)})^{(2)}(\xi\cdot\eta)(F(\eta)-F(\xi))\,d\omega(\eta). \tag{7.142}$$

The function $F \in \mathrm{C}(\Omega)$ is uniformly continuous on Ω. Hence,

$$\begin{aligned} &|(M_h^{(k)})^{(2)}(F)(\xi) - F(\xi)| \\ &\le \max_{-1+2h^2\le\xi\cdot\eta\le 1} |F(\xi)-F(\eta)| \int\limits_{-1+2h^2\le\xi\cdot\eta\le 1} (L_h^{(k)})^{(2)}(\xi\cdot\eta)\,d\omega(\eta). \end{aligned} \tag{7.143}$$

But this gives us

$$|(M_h^{(k)})^{(2)}(F)(\xi) - F(\xi)| \le \max_{-1+2h^2\le\xi\cdot\eta\le 1} |F(\xi)-F(\eta)|. \tag{7.144}$$

For $h \in (-1,1)$ and $F \in \mathrm{C}(\Omega)$, we have

$$\|(M_h^{(k)})^{(2)}(F)\|_{\mathrm{C}(\Omega)} \le \|F\|_{\mathrm{C}(\Omega)} \tag{7.145}$$

and

$$\|(M_h^{(k)})^{(2)}(F) - F\|_{\mathrm{C}(\Omega)} \le \mu(F; 2(1-h^2)). \tag{7.146}$$

The function $B_h^{(k)}(\cdot\eta) : \xi \mapsto B_h^{(k)}(\xi\cdot\eta), \xi \in \Omega$, admits an expansion in terms of Legendre polynomials as follows:

$$B_h^{(k)}(\xi\cdot\eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (B_h^{(k)})^\wedge(n) P_n(\xi\cdot\eta). \tag{7.147}$$

Thus it is easy to see that

$$\begin{aligned} &(L_h^{(k)})^{(2)}(\xi,\eta) = (L_h^{(k)})^{(2)}(\xi\cdot\eta) \\ &= \left((B_h^{(k)})^\wedge(0)\right)^{-2} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left((B_h^{(k)})^\wedge(n)\right)^2 P_n(\xi\cdot\eta). \end{aligned} \tag{7.148}$$

Scaling functions with local support turn out to be of particular efficiency and economy, for example, in numerical integration of convolution integrals, since the numerical effort must be done only on the local support. We know already that

$$\operatorname{supp}(L_h^{(k)} * L_{h'}^{(k)}) = [h \uplus h', 1], \quad 0 < h, h' < 1 \tag{7.149}$$

where the operation $\uplus : [-1,1]^2 \to [-1,1]$ is defined by

$$h \uplus h' = \cos(\min\{\pi, \arccos(h) + \arccos(h')\}) \tag{7.150}$$

However, the statement

$$L_h^{(k)} * L_{h'}^{(k)} = L_{h \uplus h'}^{(k)}, \quad 0 < h', h < 1, \tag{7.151}$$

does *not* hold true in general. Our Dirac families with local support, therefore, do not generate a semigroup of contraction operators (contrary to the Abel–Poisson or Gauß–Weierstraß singular integrals that will be discussed later on). Nevertheless, we are able to find a way to achieve an *approximate contraction procedure* as follows (see W. Freeden, U. Windheuser (1996)): For $h_1 \in (0,1)$ and $k \geq 0$, let $L_1^{[k]} \in \mathrm{L}^2[-1,1]$, be given by

$$L_1^{[k]} = L_{h_1}^{(k)}. \tag{7.152}$$

Consequently, we have with $(L_1^{[k]})^\wedge(n) = (L_{h_1}^{(k)})^\wedge(n), n = 0, 1, \ldots,$

$$L_1^{[k]} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (L_1^{[k]})^\wedge(n) P_n \tag{7.153}$$

and

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\left(L_1^{[k]} \right)^\wedge (n) \right)^2 < \infty. \tag{7.154}$$

Moreover,

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\left(L_1^{[k]} \right)^\wedge (n) \right)^{2j} < \infty \tag{7.155}$$

holds for all integers $j = 1, 2, \ldots$. This makes it possible to introduce kernels $\{L_j^{[k]}\} \subset \mathrm{L}^2[-1,1], j \in \mathbb{N}$ of the following representation:

$$L_j^{[k]}(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\left(L_1^{[k]} \right)^\wedge (n) \right)^j P_n(t), \quad t \in [-1,1]. \tag{7.156}$$

But this yields the property

$$L_j^{[k]} * L_{j'}^{[k]} = L_{j+j'}^{[k]} \tag{7.157}$$

for $j, j' \in \mathbb{N}$. According to our construction $L_j^{[k]}$ is the jth iteration of $L_1^{[k]}$ which in turn means that

$$\mathrm{supp} L_j^{[k]} = [h_j, 1] \tag{7.158}$$

with

$$h_j = \underbrace{h_1 \uplus \ldots \uplus h_1}_{j\text{times}}. \tag{7.159}$$

Hence, our procedure results in a locally spacelimited supported kernel $\{L_j^{[k]}\}$ showing the property (7.157). Consequently, the results obtained for the singular integral

$$M_j^{[k]}(F)(\xi) = (L_j^{[k]} * F)(\xi), \quad \xi \in \Omega, F \in \mathrm{L}^2(\Omega) \tag{7.160}$$

can be summarized as follows.

Theorem 7.18. $M_j^{[k]}$ *defines for each* $j \in \mathbb{N}, k \geq 1$, *a linear bounded operator from* $\mathcal{X}(\Omega)$ *to* $\mathrm{C}^{(k-1)}(\Omega)$. *For all* $j \in \mathbb{N}$, *and* $F \in \mathrm{L}^2(\Omega)$

$$\|M_j^{[k]}(F)\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)},$$

and, for $j, j' \in \mathbb{N}$

$$M_{j+j'}^{[k]} = M_j^{[k]} M_{j'}^{[k]}.$$

Seen in comparison with a semigroup of contraction operators, we are confronted with a simulated procedure that stops after finite steps for $j = 1$ that is for a fixed 'window size' h_1. Therefore, approximation cannot be performed with arbitrary accuracy. Nevertheless, our approach is of practical importance. In applications, only discrete data material is available, so an arbitrarily close approximation doesn't make sense. Instead, we are confronted with the problem of reaching in a finite number of steps a numerically relevant approximation (based on a parameter h_1 chosen in close adaptation to the data situation). But, of course, it remains to find the right choice h_1 in practical applications.

Example 7.19. The *spherical up function* (see W. Freeden et al. (1998), M. Schreiner (2003), W. Freeden, M. Schreiner (2006)) is introduced by an infinite convolution of locally supported functions, where the support of each of the building blocks is carefully chosen to ensure that the resulting convolution is also locally supported. Furthermore, the infinite convolution is infinitely often differentiable, since the symbol $\{(Up_h^{(\lambda)})^\wedge(n)\}_{n\in\mathbb{N}_0}$ decays for $n \to \infty$ faster than any rational function.

Our version of the up function is defined as follows (see M. Schreiner (2003)): Let $h \in (-1,1)$, and $\lambda > -1$. We let $\varphi_0 = \arccos h$ and introduce

$$\varphi_i = 2^{-i}\varphi_0, \ , \ h_i = \cos\frac{\varphi_i}{2}, \quad i = 1, 2, \ldots \tag{7.161}$$

With these preliminaries, we define $Up_h^{(\lambda)}$ as follows

$$Up_h^{(\lambda)} = (L_{h_1}^{(\lambda)})^{(2)} * (L_{h_2}^{(\lambda)})^{(2)} * \ldots = \mathop{\ast}_{i=1}^{\infty} (L_{h_i}^{(\lambda)})^{(2)}, \tag{7.162}$$

where

$$L_h^{(\lambda)}(t) = \frac{\lambda+1}{2\pi(1-h)^{\lambda+1}} B_h^{(\lambda)}(t) \tag{7.163}$$

with

$$B_h^{(\lambda)}(t) = \begin{cases} 0 & , \quad -1 \leq t \leq h \\ (t-h)^\lambda & , \quad h < t \leq 1. \end{cases} \tag{7.164}$$

Each $\vartheta \mapsto L_{h_i}^{(\lambda)}(\cos\vartheta)$ has the support $[0, \varphi_i/2]$ so that $\vartheta \mapsto (L_{h_i}^{(\lambda)})^{(2)}(\cos\vartheta)$ has the support $[0, \varphi_i]$. Thus, the function $\vartheta \mapsto Up_h^{(\lambda)}(\cos\vartheta)$ has the support $[0, \sum_{i=1}^{\infty} \varphi_i] = [0, \varphi_0]$, such that $\operatorname{supp} Up_h^{(\lambda)}(t) = [h, 1]$.

We know that, for each i, we have

$$0 \leq ((L_{h_i}^{(\lambda)})^{(2)})^\wedge(n) \leq ((L_{h_i}^{(\lambda)})^{(2)})^\wedge(0) = 1, \quad n = 1, 2, \ldots. \tag{7.165}$$

In other words, (7.162) is well-defined, and we have

$$(Up_h^{(\lambda)})^\wedge(n) = \prod_{i=1}^{\infty} \left(((L_{h_i}^{(\lambda)})^{(2)})^\wedge(n) \right)^2. \tag{7.166}$$

In particular,

$$0 \leq (Up_h^{(\lambda)})^\wedge(n) \leq (Up_h^{(\lambda)})^\wedge(0) = 1, \quad n = 1, 2, \ldots \tag{7.167}$$

We summarize the properties of the spherical up function (see Fig. 7.9) in the following theorem (for the proof see W. Freeden, M. Schreiner (2006)):

Theorem 7.20. *Let, for $h \in (-1,1)$ and $\lambda > -1$, the function $Up_h^{(\lambda)}$: $[-1,1] \to \mathbb{R}$ be defined by* (7.162)*, then the following statements are valid:*

(i) *$Up_h^{(\lambda)}$ is locally supported with* $\operatorname{supp} Up_h^{(\lambda)} = [h, 1]$.

(ii) *For all* $\eta \in \Omega$, $Up_h^{(\lambda)}(\eta \cdot\ )$ *is of class* $\mathrm{C}^{(\infty)}(\Omega)$,

(iii) $Up_h^{(\lambda)} : [-1,1] \to \mathbb{R}$ *can be expressed with the uniformly convergent series*

$$Up_h^{(\lambda)} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (Up_h^{(\lambda)})^\wedge(n) P_n, \tag{7.168}$$

where $(Up_h^{(\lambda)})^\wedge(0) = 1$ *and*

$$0 \leq (Up_h^{(\lambda)})^\wedge(n) = \prod_{i=1}^{\infty} \left((L_{h_i}^{(\lambda)})^\wedge(n) \right)^2 \leq 1, \ n = 0, 1, 2, \ldots, \tag{7.169}$$

(iv) *For all* $n = 1, 2, \ldots$

$$\lim_{h \to 1} (Up_h^{(\lambda)})^\wedge(n) = 1, \tag{7.170}$$

(v) *For all* $t \in [-1,1]$

$$0 \leq Up_h^{(\lambda)}(t) \leq Up_h^{(\lambda)}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (Up_h^{(\lambda)})^\wedge(n), \tag{7.171}$$

(vi) *For any* $k \in \mathbb{N}$

$$(Up_h^{(\lambda)})^\wedge(n) = O(n^{-k}), \quad n \to \infty. \tag{7.172}$$

From now on, we assume that the values $h \in (-1,1)$ and $\lambda > -1$ are fixed. Under this choice of h, the numbers h_i, $i = 1, 2, \ldots$, are defined as in (7.161). Using again the kernels

$$(Up_h^{(\lambda)})^{j,\ldots,\infty} = \mathop{\Huge *}_{i=j}^{\infty} (L_{h_i}^{(\lambda)})^{(2)} \tag{7.173}$$

we are able to define a Dirac family $\Phi_j : [-1,1] \to \mathbb{R}$ by (see Fig. 7.9)

$$\Phi_j = (Up_h^{(\lambda)})^{j,\ldots,\infty}, \quad \text{j = 1,2,}\ldots. \tag{7.174}$$

By construction, $\mathrm{supp}\Phi_j = [h_{j-1}, 1]$, and we have the refinement equation

$$\Phi_{j+1} * (L_{h_j}^{(\lambda)})^{(2)} = \Phi_j, \quad j \geq 1. \tag{7.175}$$

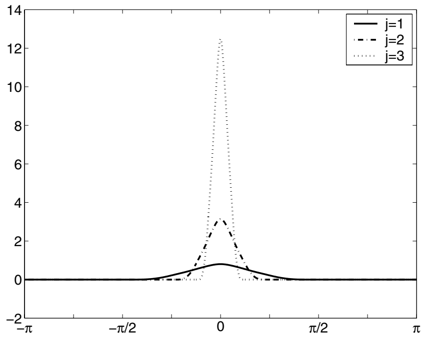


Fig. 7.9: The Dirac family based on the Up function Φ_j for $j = 1, 2, 3$, $\lambda = -0.9$ and $h = -1$.

Using the statements (iv) and (v) of the last theorem, we obtain for every $F \in \mathrm{L}^2(\Omega)$

$$\lim_{j\to\infty} \|\Phi_j * F - F\|_{\mathrm{L}^2(\Omega)} = 0 \tag{7.176}$$

and

$$\|\Phi_j * F\|_{\mathrm{L}^2(\Omega)} \leq \|\Phi_{j+1} * F\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)}. \tag{7.177}$$

Next, we list *non-spacelimited, non-bandlimited Dirac families in* $\mathrm{L}^2(\Omega)$. We confine ourselves to two types of Dirac families which are of basic interest in applications. The first kernel is the Abel–Poisson kernel whose analogue in Euclidean spaces $\mathbb{R}^n$ is well known (see for example E.M. Stein, G. Weiss (1971)). A somewhat similar kernel is the Gauß–Weierstraß kernel. We shall show, in particular, that Abel–Poisson and Gauß–Weierstraß means converge in uniform sense. Both Abel–Poisson and Gauß–Weierstraß integral means define a semigroup of contraction operators on $\mathrm{L}^2(\Omega)$.

Example 7.21. The family $\{Q_h\}_{h\in(0,1)}$, $h = e^{-\rho}$, given by

$$Q_h(t) = \frac{1}{4\pi}\frac{1-h^2}{(1+h^2-2ht)^{3/2}} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} h^n P_n(t), \tag{7.178}$$

$t \in [-1, 1]$, is called the *Abel-Poisson scaling function* (see Fig. 7.5).

The function $A_h(F) : \xi \mapsto A_h(F)(\xi)$, $\xi \in \Omega$, $F \in \mathrm{L}^2(\Omega)$, defined as convolution integral by

$$A_h(F)(\xi) = \int_\Omega Q_h(\xi \cdot \eta) F(\eta)\, d\omega(\eta), \quad \xi \in \Omega, \tag{7.179}$$

is called the '*Abel–Poisson mean*'. The integral (7.179) may be rewritten as follows

$$A_h(F)(\xi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} h^n \int_\Omega F(\eta) P_n(\xi \cdot \eta)\, d\omega(\eta) \tag{7.180}$$

so that

$$\int_\Omega A_h(F)(\xi) Y_n(\xi)\, d\omega(\xi) = h^n \int_\Omega F(\xi) Y_n(\xi)\, d\omega(\xi) \tag{7.181}$$

for all $Y_n \in \mathrm{Harm}_n$, i.e., $(Q_h)^\wedge(n) = h^n, n = 0, 1, \ldots$.

For all $F \in \mathrm{C}(\Omega)$, we have

$$\| A_h(F) \|_{\mathrm{C}(\Omega)} \leq \| F \|_{\mathrm{C}(\Omega)} \,. \tag{7.182}$$

More generally,

$$\|A_h(F)\|_{\mathrm{L}^2(\Omega)} \leq \|F\|_{\mathrm{L}^2(\Omega)} \tag{7.183}$$

for all $F \in \mathrm{L}^2(\Omega)$. If F is Lipschitz-continuous with Lipschitz constant C_F (i.e., $F \in \mathrm{Lip}(\Omega)$), then

$$\|A_h(F) - F\|_{\mathrm{C}(\Omega)} \leq \sqrt{2}(C_F + {h_0}^{-1}\|F\|_{\mathrm{C}(\Omega)})\sqrt{1-h} \tag{7.184}$$

for all $h \in (h_0, 1)$, $h_0 \in (0,1)$ fixed. If $F(\xi) > 0$ for all $\xi \in \Omega$, then $A_h(F)(\xi) > 0$ for all $\xi \in \Omega$. Moreover, because of the limit relation,

$$\lim_{h \to 1, h<1} (1-h)^{-1}(h^n - 1) = -n, \tag{7.185}$$

it is not difficult to show that the equations

$$-nF^\wedge(n,j) = G^\wedge(n,j), \quad n = 0, 1, \ldots, \quad j = 1, \ldots, 2n+1, \tag{7.186}$$

are equivalent to

$$\lim_{h \to 1, h<1} \left\| \frac{A_h(F) - F}{1-h} - G \right\|_{\mathrm{C}(\Omega)} = 0 \tag{7.187}$$

provided that $F, G \in \mathrm{C}(\Omega)$. A similar result holds in $(\mathrm{L}^2(\Omega), \| \cdot \|_{\mathrm{L}^2(\Omega)})$. Furthermore, H. Berens et al. (1969) have shown that, for $F \in \mathrm{L}^2(\Omega)$, the relation

$$\|A_h(F) - F\|_{\mathrm{L}^2(\Omega)} = O(1-h) \tag{7.188}$$

is equivalent to the fact there exists a function $G \in \mathrm{L}^2(\Omega)$ such that $G^\wedge(n,j) = -nF^\wedge(n,j), n = 0, 1, \ldots,\ j = 1, \ldots, 2n+1$. In other words, the 'saturation class' of the Abel–Poisson operators $\{A_{e-\rho}\}$, $\rho \in (0,\infty)$, is given by

$$\mathcal{H}(\mathrm{L}^2(\Omega); -n) = \{F \in \mathrm{L}^2(\Omega) | \exists G \in \mathrm{L}^2(\Omega) : G^\wedge(n,j) = -nF^\wedge(n,j)\}, \tag{7.189}$$

whereas the 'saturation order' of $\{A_{\mathrm{e}^{-\rho}}\}, \rho \in (0,\infty)$ is $O(1-\mathrm{e}^{-\rho}), \rho \to 0$.

A classical problem involving the Abel–Poisson mean is the Dirichlet problem of potential theory corresponding to a spherical boundary. More explicitly, for given $F \in \mathrm{C}(\Omega)$, the function $V : \overline{\Omega_{\text{int}}} \to \mathbb{R}$ given by $V(x) = A_r(F)(\xi)$, $x = r\xi$, $\xi \in \Omega$, is the only solution of the interior Dirichlet problem (i) $V \in \mathrm{C}^{(2)}(\Omega_{\text{int}}) \cap \mathrm{C}(\overline{\Omega_{\text{int}}})$ (ii) $\Delta V = 0$ in Ω_{int}, (iii) $V|\Omega = F$. This is the reason why this kernel is particularly useful in the approximation of harmonic functions. By virtue of the maximum/minimum principle of potential theory, we get for all $\xi \in \Omega$

$$\min_{\eta\in\Omega} F(\eta) \le A_h(F)(\xi) \le \max_{\eta\in\Omega} F(\eta), \quad F \in \mathrm{C}(\Omega). \tag{7.190}$$

Altogether, we are able to conclude that $\{A_{\mathrm{e}^{-\rho}}\}, \rho \in (0,\infty)$, forms a *semi-group of contraction operators on* $\mathrm{L}^2(\Omega)$.

Example 7.22. Next, we deal with the so-called *'Gauß-Weierstraß' scaling function* $\{W_\rho\}_{\rho\in(0,\infty)}$, given by (see Fig. 7.10)

$$W_\rho(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} e^{-n(n+1)\rho} P_n(t), \quad t \in [-1,1]. \tag{7.191}$$

The integrals

$$G_\rho(F)(\xi) = \int_\Omega W_\rho(\xi\cdot\eta) F(\eta)\, d\omega(\eta), \quad \xi \in \Omega, \quad F \in \mathrm{L}^2(\Omega), \tag{7.192}$$

are called the *'Gauß-Weierstraß' means.* $\{G_\rho\}_{\rho\in(0,\infty)}$ satisfies the relation

$$\|G_\rho(F)\|_{\mathrm{L}^2(\Omega)} \le \|F\|_{\mathrm{L}^2(\Omega)} \tag{7.193}$$

for all $F \in \mathrm{L}^2(\Omega)$. Thus $G_\rho : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ defines a bounded linear operator for every $\rho \in (0,\infty)$ such that $G_{\rho+\rho'} = G_\rho G_{\rho'}$. The set of all operators $G_\rho, 0 < \rho < \infty$, forms a semigroup of contraction operators on $\mathrm{L}^2(\Omega)$.

Since the series is absolutely and uniformly convergent, $G_\rho(F)$ can be rewritten in the form

$$G_\rho(F)(\xi) = \sum_{n=0}^{\infty} \mathrm{e}^{-n(n+1)\rho} \frac{2n+1}{4\pi} \int_\Omega F(\eta) P_n(\xi\cdot\eta)\, d\omega(\eta). \tag{7.194}$$

For sufficiently small $\rho > 0$, the series

$$\frac{1}{\rho}\sum_{n=0}^{\infty}(\mathrm{e}^{-n(n+1)\rho}-1)\frac{2n+1}{4\pi}\int_{\Omega}F(\eta)P_n(\xi\cdot\eta)\,d\omega(\eta) \tag{7.195}$$

represents an approximation for the Beltrami derivative of F at $\xi \in \Omega$.

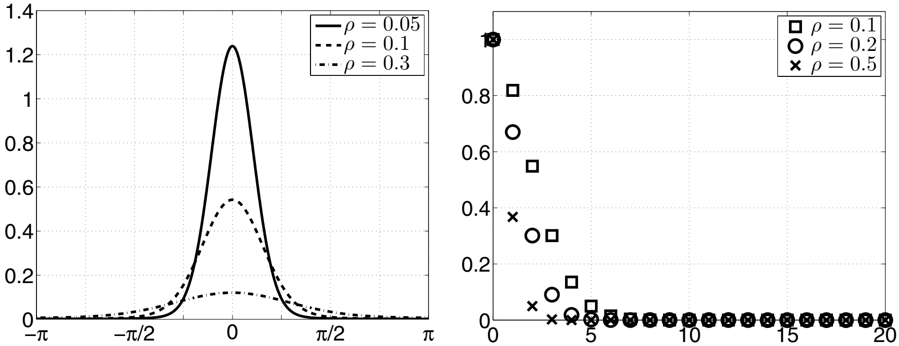


Fig. 7.10: The Gauß-Weierstraß scaling function. Space representation (*left*) and frequency representation (*right*). Both for various parameters ρ.

Theorem 7.23. *Suppose that F is twice continuously differentiable on Ω. Then*

$$\lim_{\rho\to 0,\rho>0}\left\|\frac{G_\rho(F)-F}{\rho}-\Delta^*F\right\|_{\mathrm{C}(\Omega)}=0.$$

Proof. The integral

$$\int_0^\rho G_\tau(\Delta^*F)(\xi)\,d\tau \tag{7.196}$$

exists for all $\rho > 0$, $\xi \in \Omega$ and for all $F \in \mathrm{C}^{(2)}(\Omega)$. Moreover, it is not difficult to see that

$$\begin{aligned}&\int_\Omega\int_0^\rho G_\tau(\Delta^*F)(\xi)\,d\tau Y_{n,j}(\xi)\,d\omega(\xi) \\ &=\int_0^\rho\int_\Omega G_\tau(\Delta^*F)(\xi)Y_{n,j}(\xi)\,d\omega(\xi)\,d\tau \\ &=-\frac{\mathrm{e}^{-n(n+1)\rho}-1}{n(n+1)}\int_\Omega \Delta^*_\xi F(\xi)Y_{n,j}(\xi)\,d\omega(\xi).\end{aligned} \tag{7.197}$$

Since $(\Delta^*)^\wedge(n) = -n(n+1)$ are the eigenvalues of the Beltrami operator Δ^*, we find for $n = 0, 1, ...$

$$\int_\Omega\int_0^\rho G_\tau(\Delta^*F)(\xi)\,d\tau Y_{n,j}(\xi)\,d\omega(\xi)=(\mathrm{e}^{-n(n+1)\rho}-1)\int_\Omega F(\xi)Y_{n,j}(\xi)\,d\omega(\xi). \tag{7.198}$$

On the other hand,

$$\int_{\Omega}(G_\rho(F)(\xi) - F(\xi))Y_{n,j}(\xi)\,d\omega(\xi) = (\mathrm{e}^{-n(n+1)\rho} - 1)\int_{\Omega} F(\xi)Y_{n,j}(\xi)\,d\omega(\xi). \tag{7.199}$$

By comparison of (7.198) and (7.199), we obtain

$$G_\rho(F)(\xi) - F(\xi) = \int_0^\rho G_\tau(\Delta^* F)(\xi)\,d\tau \tag{7.200}$$

for all $\rho > 0$ and $\xi \in \Omega$. Therefore, it follows that

$$\begin{aligned} \left\| \frac{G_\rho(F) - F}{\rho} - \Delta^* F \right\|_{\mathrm{C}(\Omega)} &= \sup_{\xi\in\Omega} \left| \frac{1}{\rho}\int_0^\rho (G_\tau(\Delta^* F)(\xi) - (\Delta^* F)(\xi))\,d\tau \right| \\ &\leq \frac{1}{\rho}\int_0^\rho \sup_{\xi\in\Omega} |G_\tau(\Delta^* F)(\xi) - \Delta^*_\xi F(\xi)|\,d\tau \\ &\leq \sup_{0\leq\tau\leq\rho} ||G_\tau(\Delta^* F) - \Delta^* F||_{\mathrm{C}(\Omega)}. \end{aligned} \tag{7.201}$$

Letting ρ tend to 0, we obtain the desired result. □

In the same way, we obtain the following corollary:

Corollary 7.24. *For $F, H \in \mathrm{L}^2(\Omega)$ the following statements are equivalent:*

$$-n(n+1)F^\wedge(n,j) = H^\wedge(n,j), \quad n = 0,1,\ldots, \quad j = 1,\ldots,2n+1,$$

and

$$\lim_{\substack{\rho\to 0\\ \rho>0}} \left\| \frac{G_\rho(F) - F}{\rho} - H \right\|_{\mathrm{L}^2(\Omega)} = 0$$

If $H = 0$ in $\mathrm{L}^2(\Omega)$, then $F =$ const.

Moreover, H. Berens et al. (1969) have shown that the 'saturation class' of the Gauß–Weierstraß singular integral operators $\{G_\rho\}, \rho \in (0,\infty)$, is given by

$$\begin{aligned} &\mathcal{H}(\mathrm{L}^2(\Omega); -n(n+1)) \\ &= \{F \in \mathrm{L}^2(\Omega) | \exists G \in \mathrm{L}^2(\Omega) : G^\wedge(n,j) = -n(n+1)F^\wedge(n,j)\}, \end{aligned} \tag{7.202}$$

and the 'saturation order' of $\{G_\rho\}, \rho \in (0,\infty)$, is $O(\rho), \rho \to 0$.

A problem involving the Gauß–Weierstraß kernel (cf. W. Freeden, M. Schreiner (1995)) is the initial-value problem (heat equation)

$$\frac{\partial}{\partial t} U(t;\xi) = \Delta^*_\xi U(t;\xi), \quad t \geq 0, \quad \xi \in \Omega, \tag{7.203}$$

$$U(0;\xi) = F(\xi), \quad \xi \in \Omega. \tag{7.204}$$

The solution is given by convolution against the Gauß–Weierstraß kernel

$$U(t;\xi) = G_t(F)(\xi) = \int_\Omega W_t(\xi\cdot\eta)F(\eta)\,d\omega(\eta). \tag{7.205}$$

Formula (7.205) is of fundamental importance in multiscale descriptions of spherical images.

All kernels that will be discussed now are chosen in such a way that the support of their spectral generators, i.e., the Legendre symbol is compact. In other words, our interest now is to list *bandlimited Dirac families.*

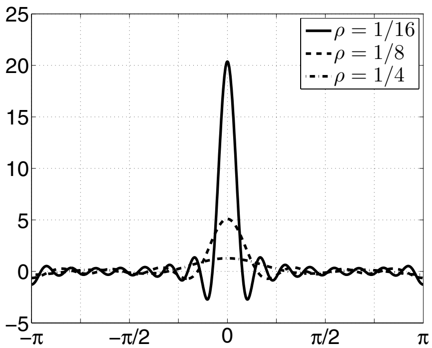
Example 7.25. The generator of the *Shannon Dirac family*, Φ_ρ, $\rho \in (0,\infty)$, is given by

$$(\Phi_\rho)^\wedge(n) = \begin{cases} 1 & , \quad n \in [0,\rho^{-1}) \\ 0 & , \quad n \in [\rho^{-1},\infty). \end{cases} \tag{7.206}$$

Its associated kernel (see Fig. 7.11 for a graphical impression) reads

$$\Phi_\rho(\xi\cdot\eta) = \sum_{n\le\rho^{-1}} \frac{2n+1}{4\pi} P_n(\xi\cdot\eta), \quad \xi,\eta\in\Omega. \tag{7.207}$$

As already known, the kernel Φ_ρ may be interpreted as truncated Dirac kernel. It is not surprising that the Shannon kernel as 'finite polynomial kernel' shows strong oscillations in space. This is the price to be paid for the sharp separation in frequency space. To suppress the oscillations, we are led to 'smoothed versions' of the Shannon kernel (dependent on an additional parameter $\alpha \in (0,1)$).



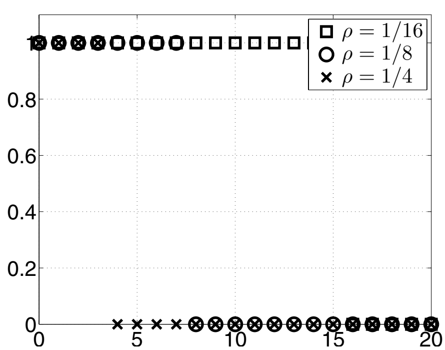


Fig. 7.11: The Shannon scaling function Φ_ρ for $\rho = 1/16,\ 1/8,\ 1/4$. Space representation $\vartheta \mapsto \Phi_\rho(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (\Phi_\rho)^\wedge(n)$ (*right*).

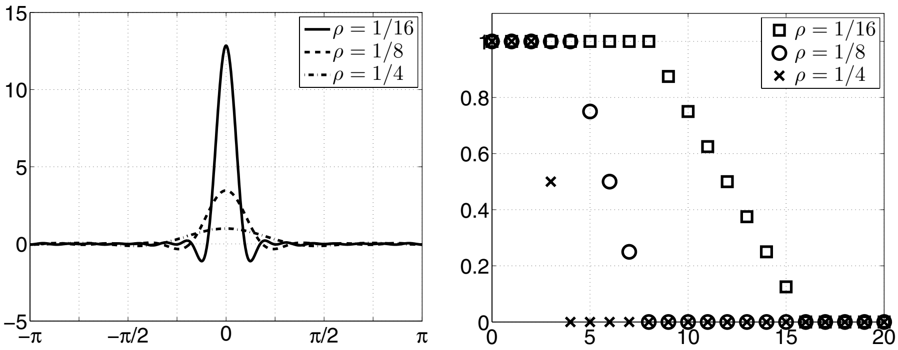


Fig. 7.12: The smoothed Shannon scaling function Φ_ρ for $\rho = 1/16,\ 1/8,\ 1/4$, and $\alpha = 0.5$. Space representation $\vartheta \mapsto \Phi_\rho(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (\Phi_\rho)^\wedge(n)$ (*right*).

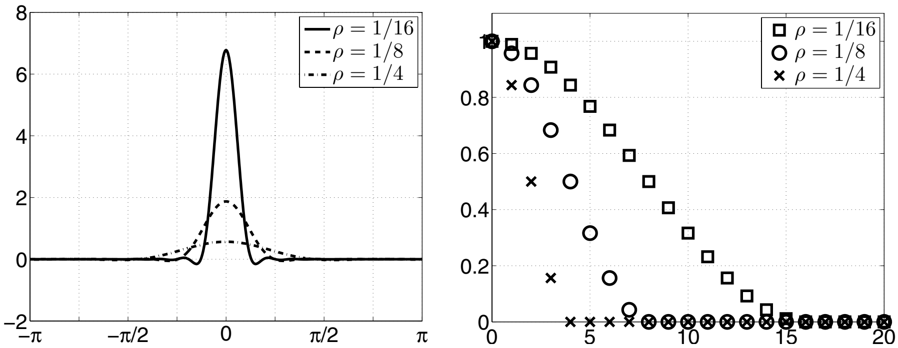


Fig. 7.13: The CUP–Dirac family Φ_ρ for $\rho = 1/16,\ 1/8,\ 1/4$. Space representation $\vartheta \mapsto \Phi_\rho(\cos(\vartheta))$ (*left*) and frequency representation $n \mapsto (\Phi_\rho)^\wedge(n)$ (*right*).

Example 7.26. The generator of the *smoothed Shannon Dirac family* (see Figs. 7.12 and 7.13) reads as follows

$$\Phi_\rho^\wedge(n) = \begin{cases} 1 & ,\quad n \in [0, \rho^{-1}\alpha) \\ \frac{1-\alpha n}{1-\alpha} & ,\quad n \in [\rho^{-1}\alpha, \rho^{-1}] \\ 0 & ,\quad n \in [\rho^{-1}, \infty) \end{cases} \tag{7.208}$$

Compared with the Shannon case, there is a linear transition from the value 1 at $[0, \rho^{-1}\alpha]$ to the value 0 at $[\rho^{-1}, \infty)$.

Example 7.27. Of course, many other suitable choices for $\Phi_\rho^\wedge(n)$ can be found for practical purposes. We only mention the CUP-Dirac family (see M. Schreiner (1996)).

$$\Phi_\rho^\wedge(n) = \begin{cases} (1-\rho n)^2(1+2\rho n) & ,\quad n \in [0, \rho^{-1}) \\ 0 & ,\quad [\rho^{-1}, \infty). \end{cases} \tag{7.209}$$

The illustrations show that the phenomena of oscillation that are still existent for the smoothed Shannon Dirac family can be suppressed by this choice.

Next, we are interested in the kernels $B_h^{(k)}$ in view of the uncertainty relation. Using

$$\begin{aligned} \|B_h^{(k)}\|^2 &= 2\pi \int_{-1}^{1} [B_h^{(k)}(t)]^2 \, dt \\ &= 2\pi \frac{1-h}{2k+1}, \end{aligned} \tag{7.210}$$

we define the kernel

$$\tilde{B}_h^{(k)} = \sqrt{\frac{2k+1}{2\pi(1-h)}} B_h^{(k)}, \tag{7.211}$$

since the uncertainty properties are normally defined for kernels with norm one. We find

$$g^{o^{(1)}}_{\tilde{B}_h^{(k)}(\,\cdot\varepsilon^3)} = 2\pi \int_{-1}^{1} t \left(\tilde{B}_h^{(k)}(t)\right)^2 \, dt \, \varepsilon^3 = \frac{1+h+2k}{2+2k} \, \varepsilon^3. \tag{7.212}$$

Consequently,

$$(\sigma^{o^{(1)}}_{\tilde{B}_h^{(k)}})^2 = 1 - \left(\frac{1+h+2k}{2+2k}\right)^2 = \frac{(1-h)(h+4k+3)}{(2k+2)^2}. \tag{7.213}$$

Using (7.25), we finally arrive at

$$\Delta^{o^{(1)}}_{\tilde{B}_h^{(k)}} = \frac{1}{1+h+2k} \sqrt{(1-h)(h+4k+3)}. \tag{7.214}$$

For the localization in frequency, we assume $k \geq 2$. We have

$$\begin{aligned} (\sigma^{o^{(3)}}_{\tilde{B}_h^{(k)}(\,\cdot\varepsilon^3)})^2 &= -2\pi \int_{-1}^{1} \tilde{B}_h^{(k)}(t) \mathrm{L}_t \tilde{B}_h^{(k)}(t) \, dt \\ &= \frac{2k+1}{2\pi(1-h)} \frac{-2\pi}{(1-h)^{2k}} \int_h^1 (t-h)^k \mathrm{L}_t (t-h)^k \, dt \\ &= \frac{k(h+2k)}{(1-h)(2k-1)}, \end{aligned} \tag{7.215}$$

so that

$$\Delta^{o^{(3)}}_{\tilde{B}_h^{(k)}} = \sqrt{\frac{k(h+2k)}{(1-h)(2k-1)}}. \tag{7.216}$$

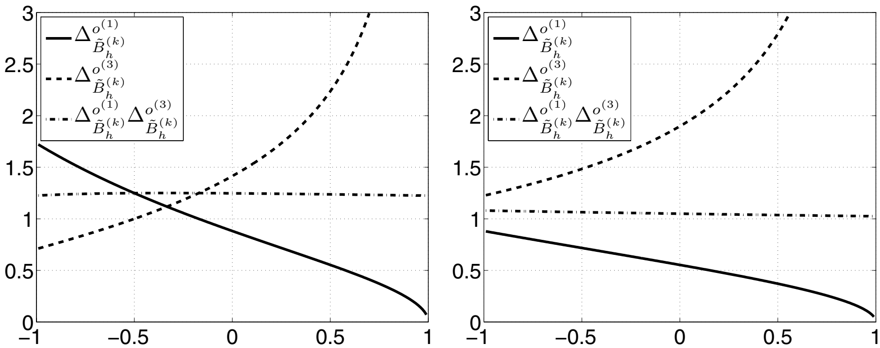


Fig. 7.14: Uncertainty classification of the normalized smoothed Haar scaling function $\tilde{B}_h^{(k)}$ ($k = 1$, left; $k = 3$ right). $\Delta^{o^{(1)}}_{\tilde{B}_h^{(k)}}$, $\Delta^{o^{(3)}}_{\tilde{B}_h^{(k)}}$ and the product $\Delta^{o^{(1)}}_{\tilde{B}_h^{(k)}} \Delta^{o^{(3)}}_{\tilde{B}_h^{(k)}}$ are shown as functions of h.

The application of L_t requires that the kernel is twice differentiable. However, using integration by parts, the results immediately carry over to the case $k = 1$. Figure 7.14 gives a graphical impression of these results for the special cases $k = 1$ and $k = 3$.

For the investigation of the uncertainty properties of the Shannon kernels, we start from

$$\|\Phi_\rho\|^2 = \sum_{n=0}^{\lfloor \rho^{-1} \rfloor} \frac{2n+1}{4\pi} = \frac{1}{4\pi} \left((\lfloor \rho^{-1} \rfloor + 1) + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} + 1 \rfloor \right) \tag{7.217}$$

where, as usual, $\lfloor \rho^{-1} \rfloor$ is the largest integer which is less or equal ρ^{-1}. Observing this result, we define the normalized Shannon kernel by

$$\tilde{\Phi}_\rho = \frac{1}{\|\Phi_\rho\|} \Phi_\rho. \tag{7.218}$$

$$\begin{aligned} (\sigma^{o^{(1)}}_{\tilde{\Phi}_\rho})^2 &= 1 - \frac{1}{\|\Phi_\rho\|^2} \left(\sum_{n=1}^{\lfloor \rho^{-1} - 1 \rfloor} \frac{2n+2}{4\pi} \right)^2 \\ &= 1 - \left(\frac{2\lfloor \rho^{-1} - 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} - 1 \rfloor}{\lfloor \rho^{-1} + 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} + 1 \rfloor} \right)^2, \end{aligned} \tag{7.219}$$

so that

$$\Delta^{o^{(1)}}_{\tilde{\Phi}_\rho} = \sqrt{\frac{1 - \left(\frac{2\lfloor \rho^{-1} - 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} - 1 \rfloor}{\lfloor \rho^{-1} + 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} + 1 \rfloor} \right)^2}{\frac{2\lfloor \rho^{-1} - 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} - 1 \rfloor}{\lfloor \rho^{-1} + 1 \rfloor + \lfloor \rho^{-1} \rfloor \lfloor \rho^{-1} + 1 \rfloor}}}. \tag{7.220}$$

Moreover, we find

$$
\begin{aligned}
(\sigma^{o(3)}_{\tilde{\Phi}_\rho})^2 &= \frac{4\pi}{\lfloor\rho^{-1}\rfloor + 1 + \lfloor\rho^{-1}\rfloor\lfloor\rho^{-1}+1\rfloor} \sum_{n=0}^{\lfloor\rho^{-1}\rfloor} \frac{2n+1}{4\pi} n(n+1) \\
&= \frac{1}{2} \frac{\lfloor\rho^{-1}\rfloor(1+\lfloor\rho^{-1}\rfloor)^2(2+\lfloor\rho^{-1}\rfloor)}{\lfloor\rho^{-1}\rfloor + 1 + \lfloor\rho^{-1}\rfloor\lfloor\rho^{-1}+1\rfloor} \qquad (7.221)
\end{aligned}
$$

such that

$$
\Delta^{o(3)}_{\tilde{\Phi}_\rho} = \sqrt{\frac{1}{2} \frac{\lfloor\rho^{-1}\rfloor(1+\lfloor\rho^{-1}\rfloor)^2(2+\lfloor\rho^{-1}\rfloor)}{\lfloor\rho^{-1}\rfloor + 1 + \lfloor\rho^{-1}\rfloor\lfloor\rho^{-1}+1\rfloor}}. \qquad (7.222)
$$

The results are illustrated in Fig. 7.15.

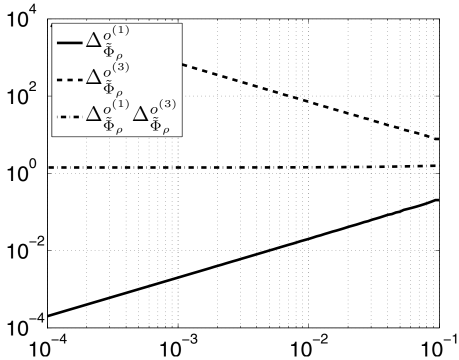


Fig. 7.15: Uncertainty classification of the normalized Shannon Dirac family $\tilde{\Phi}_\rho$. Presented are $\Delta^{o(1)}_{\tilde{\Phi}_\rho}$, $\Delta^{o(3)}_{\tilde{\Phi}_\rho}$, and the product $\Delta^{o(1)}_{\tilde{\Phi}_\rho}\Delta^{o(3)}_{\tilde{\Phi}_\rho}$ as functions of ρ in a double logarithmic setting.

7.6 Bibliographical Notes

There is a long history of zonal kernel functions (also called radial basis functions in the language of approximation theory). First essential results are due to H. Funk (1916), E. Hecke (1918). The investigations involving spherical convolutions lead back to S. Bochner (1954), W. Rudin (1950), A.P. Calderon, A. Zygmund (1955) and many others. H. Berens et al. (1969) presents an overview about the activities in the first half of the last century. F.J. Narcowich, J.D. Ward (1996) introduced an uncertainty principle for the unit sphere $\Omega \subset \mathbb{R}^3$. Another approach which involves the usage of a differential operator of second order has been made by W. Freeden (1998). The articles by N. Laín Fernández (2003), N. Laín Fernández, J., Prestin (2003) are further significant contributions to the topic of space/frequency localization. The correspondence of the Dirac delta kernel to the so-called Dirac families is well-known in the analysis of symmetries in Euclidean space. Zonal Dirac families of Abel-Poisson, Gauß-Weierstraß type were studied e.g., by H. Berens et al. (1969), C. Müller (1998), W. Freeden et al. (1998).

Scalar zonal kernel functions are basic tools for constructing spherical splines (compare, e.g., W. Freeden (1981a), G. Wahba (1981), G.E. Backus (1986), W. Freeden (1990), W. Freeden et al. (1996), W. Freeden et al. (1997), W. Freeden, F. Schneider (1999), W. Freeden (1999). In the last years, wavelets on the sphere have been the focus of several research groups which led to different wavelet approaches. Common to all these proposals is a multiresolution analysis which enables a balanced amount of both frequency (more accurately, angular momentum) and space localization (see e.g., S. Dahlke et al. (1995) and I. Weinreich (2001), D. Potts, M. Tasche (1995), T. Lyche, L. Schumaker (2000), P. Schröder, W. Sweldens (1995)). A group theoretical approach to a continuous wavelet transform on the sphere is followed by J.-P. Antoine, P. Vandergheynst (1999), J.-P. Antoine et al. (2002), and M. Holschneider (1996). The parameter choice of their continuous wavelet transform is the product of $SO(3)$ (for the motion on the sphere) and $\mathbb{R}^+$ (for the dilations). A continuous wavelet transform approach for analyzing functions on the sphere is presented by Dahlke and Maass (S. Dahlke, P. Maass (1996)).

The constructions of the Geomathematics Group in Kaiserslautern on spherical wavelets (W. Freeden, M. Schreiner (1995), W. Freeden, U. Windheuser (1996), W. Freeden, U. Windheuser (1997), W. Freeden et al. (1998), W. Freeden, K. Hesse (2002), W. Freeden, C. Mayer (2003), W. Freeden et al. (2003), W. Freeden, M. Schreiner (2007)) are intrinsically based on the specific properties concerning the theory of spherical harmonics. W. Freeden, M. Schreiner (2007) are interested in a compromise connecting zonal function expressions and structured grids on the sphere to obtain fast algorithms.

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

8 Vector Zonal Kernel Functions

In vector theory, the points of the departure to zonal kernel fields are the addition theorems relating vector spherical harmonics to Legendre vector rank-2 tensor fields and the counterparts of the Funk-Hecke formula in the vectorial context. The corresponding kernel functions obtained by summing up the vectorial Legendre kernel functions to certain (bandlimited or non-bandlimited) orthogonal series expansions are called zonal vector kernel functions due to their intimate similarities in definition and structure to scalar zonal functions and their relevance to (geo-)physically motivated applications.

Of particular significance in the theory of vector fields is the coordinate-free representation by vector zonal kernel functions. As is well known, coordinate representations of vector spherical harmonics are not calculable without singularities at the poles. Zonal vector functions, i.e., vector Legendre kernel expansions, however, avoid this problem completely, as they are constructed by application of the surface gradient and the surface curl gradient to a scalar zonal kernel function. In fact, zonal vector functions consist of a 'directional term' linked to a scalar zonal kernel function (for the normal part) or a one-dimensional derivative (for the tangential parts). Moreover, differential operators of vectorial nature like the surface gradient or the surface curl gradient can completely be treated within an isotropic (vector) framework. It should be pointed out that isotropic vector operators, i.e., operators mapping a scalar function to a vector field (or vice versa) – thereby maintaining their form when subjected to orthogonal transformations – can be expressed by means of convolutions against a vector zonal kernel function. In that sense, vector zonal functions form the canonical bridge between scalar functions and vector fields. In addition, the inherent orthogonal invariance reduces the structural complexity and dimension. It should be mentioned that, in vectorial case, two different techniques can be formulated for representing isotropic operators by convolution. Zonal kernel functions to be used in the vector context can either be formulated as vector fields generated by applying the operator $o^{(i)}$ once on scalar zonal functions, or as rank-2 tensor fields by a double application of the operators $o^{(i)}$ (see, e.g., M. Bayer et al. (1998), S. Beth (2000), H. Nutz (2002), C. Mayer (2003)). In the first case, we are led to vector zonal kernel functions, whereas the second case leads to vectorial zonal rank-2 tensor kernel functions (to be used within the vector context (see Table 8.1).

Table 8.1: Overview on (vectorial) zonal rank-2 tensor/vector kernel functions in relation to Legendre kernel functions.

Zonal kernel (generating system)	**Linear approach**		**Bilinear approach**	
Scalar field ($\{Y_{n,j}\}$–system)	scalar zonal kernel function ($\{P_n\}$–system)	K		
Vector field ($\{y_{n,j}^{(i)}\}$-system)	(vectorial) zonal rank-2 tensor kernel function ($\{{}^v\mathbf{p}_n^{(i,i)}\}$–system)	${}^v\mathbf{k}^{(i,i)}$	(vectorial) zonal vector kernel function($\{p_n^{(i)}\}$–system)	$k^{(i)}$
Vector field ($\{\tilde{y}_{n,j}^{(i)}\}$-system)	(vectorial) zonal rank-2 tensor kernel function ($\{{}^v\tilde{\mathbf{p}}_n^{(i,i)}\}$–system)	${}^v\tilde{\mathbf{k}}^{(i,i)}$	(vectorial) zonal vector kernel function($\{\tilde{p}_n^{(i)}\}$–system)	$\tilde{k}^{(i)}$

8.1 Preparatory Material

As already mentioned, two approaches are evident based on the addition theorems (Theorem 5.31 and Theorem 5.46): Zonal rank-2 tensor kernel functions (within the vectorial context) – in this approach called (vectorial) zonal rank-2 tensor kernel function – are defined by a double application of the differential dual operators $o^{(i)}$ on scalar zonal kernel functions, whereas the zonal vectorial kernel functions are derived by a single application of these operators to scalar zonal kernel functions. In doing so, we take advantage of the features from the operators $o^{(i)}$, $O^{(i)}$, $i = 1, 2, 3$. Indeed, the operators $o^{(i)}$ can be easily applied to scalar zonal kernel functions. For example, for a (sufficiently smooth) scalar zonal kernel function K, the following identities are well-known:

$$o_\xi^{(1)} K(\xi \cdot \eta) = K(\xi \cdot \eta)\eta, \tag{8.1}$$

$$o_\xi^{(2)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi), \tag{8.2}$$

$$o_\xi^{(3)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)(\xi \wedge \eta). \tag{8.3}$$

Furthermore, we mention

$$o_\xi^{(1)} o_\eta^{(1)} K(\xi \cdot \eta) = K(\xi \cdot \eta)\xi \otimes \eta, \tag{8.4}$$

$$\begin{aligned} o_\xi^{(2)} o_\eta^{(2)} K(\xi \cdot \eta) &= \nabla_\xi^* \otimes (K'(\xi \cdot \eta)(\xi - (\xi \cdot \eta)\eta)) \\ &= (\nabla_\xi^* K'(\xi \cdot \eta)) \otimes (\xi - (\xi \cdot \eta)\eta) \\ &\quad + K'(\xi \cdot \eta)\nabla_\eta^* \otimes (\xi - (\xi \cdot \eta)\eta) \\ &= K''(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi) \otimes (\xi - (\xi \cdot \eta)\eta) \\ &\quad + K'(\xi \cdot \eta)(\mathbf{i}_{\tan}(\xi) - (\eta - (\xi \cdot \eta)\xi) \otimes \eta), \end{aligned} \tag{8.5}$$

$$\begin{aligned} o_\xi^{(3)} o_\eta^{(3)} K(\xi \cdot \eta) &= K'(\xi \cdot \eta)\xi \wedge \eta \otimes \eta \wedge \xi \\ &\quad + K'(\xi \cdot \eta)((\xi \cdot \eta)\mathbf{i}_{\tan}(\xi) - (\eta - (\xi \cdot \eta)\xi) \otimes \xi), \end{aligned} \tag{8.6}$$

provided that $K : [-1, 1] \to \mathbb{R}$ is sufficiently often differentiable.

These formulas show two advantages: First, we only have to calculate the one-dimensional derivatives of a scalar zonal kernel function, which reduces the operational effort enormously (note that, in the case of double application of the operators $o^{(i)}$, the two-dimensional derivatives of a scalar zonal kernel function have to be evaluated). Second, no singularities occur when the operators $o^{(2)}$ and $o^{(3)}$ are applied to scalar zonal kernel functions. Moreover, the basic principles that are governed by the uncertainty relation canonically extend from the scalar to the vector context of zonal kernel functions (even for tangential vector kernel fields).

8.2 Tensor Zonal Kernel Functions of Rank Two in Vectorial Context

We start with the characterization of zonal rank-2 tensor kernel functions (within the vectorial context). This approach arises directly from the scalar theory (see W. Freeden et al. (1998)). To be more concrete, the zonal tensor kernel functions are defined in terms of scalar zonal kernel functions by a double application of the operators $o^{(i)}$.

Definition 8.1. Assume that $K^{(1)} \in \mathrm{C}[-1,1]$ and $K^{(i)} \in \mathrm{C}^{(2)}[-1,1]$, $i \in \{2,3\}$, are scalar zonal kernel functions. A function ${}^v\mathbf{k}^{(i,i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$,

$$ {}^v\mathbf{k}^{(i,i)}(\xi,\eta) = o_\xi^{(i)} o_\eta^{(i)} K^{(i)}(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{8.7} $$

is called a *(vectorial) zonal rank-2 tensor kernel function of type* (i,i) (with respect to $\{{}^v\mathbf{p}_n^{(i,i)}\}$). Moreover,

$$ {}^v\mathbf{k} = \sum_{i=1}^{3} {}^v\mathbf{k}^{(i,i)} \tag{8.8} $$

is called a *(vectorial) zonal rank-2 tensor kernel function* (with respect to $\{{}^v\mathbf{p}_n\}$).

Definition 8.2. A (vectorial) zonal rank-2 tensor kernel function of type (i,i), ${}^v\mathbf{k}^{(i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, $i \in \{1,2,3\}$, is called an $\mathrm{l}^2_{(i,i)}(\Omega)$*-zonal rank-2 tensor kernel function*, if ${}^v\mathbf{k}^{(i,i)}(\xi,\cdot)$ is square-integrable on Ω for each $\xi \in \Omega$. Furthermore, ${}^v\mathbf{k} = \sum_{i=1}^{3} {}^v\mathbf{k}^{(i,i)}$ is called an $\mathrm{l}^2(\Omega)$*-zonal rank-2 tensor kernel function.*

For the explicit representation of the (vectorial) zonal rank-2 tensor kernel functions in terms of the Legendre polynomials, we take advantage of the already known vectorial variants of the Funk–Hecke formula, which should be recapitulated for the convenience of the reader.

Theorem 8.3. *(Funk–Hecke Formula in Vectorial Context) Let $\eta \in \Omega$ be fixed. Assume that $g(\cdot,\eta) \in \mathrm{c}^{(1)}(\Omega)$ satisfies $\mathbf{t}g(\xi,\eta) = g(\mathbf{t}\xi,\eta)$ for all orthogonal transformations $\mathbf{t} \in SO_\eta(3)$ and all $\xi \in \Omega$. Then, for all $\zeta \in \Omega$ and $i \in \{1,2,3\}$, we have*

$$ \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\zeta,\xi) g(\xi,\eta)\, d\omega(\xi) = \left(\mu_n^{(i)}\right)^{-1} (O^{(i)}g)^\wedge(n) o_\zeta^{(i)} P_n(\zeta \cdot \eta), \tag{8.9} $$

where

$$ (O^{(i)}g)^\wedge(n) = 2\pi \int_{-1}^{1} G_i(t) P_n(t)\, dt, \tag{8.10} $$

and

$$ G_i(\xi \cdot \eta) = O_\xi^{(i)} g(\xi,\eta). \tag{8.11} $$

Within the concept of (vectorial) zonal kernel functions, this theorem leads us to the following statement.

Theorem 8.4. *Any $\mathrm{l}^2_{(i,i)}(\Omega)$-zonal rank-2 tensor kernel function ${}^v\mathbf{k}^{(i,i)}$ can be represented as a Legendre series of the form*

$$ {}^v\mathbf{k}^{(i,i)}(\xi,\cdot) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (\mathbf{k}^{(i,i)})^\wedge(n)\, {}^v\mathbf{p}_n^{(i,i)}(\xi,\cdot), \tag{8.12} $$

where

$$ ({}^v\mathbf{k}^{(i,i)})^\wedge(n) = 2\pi\mu_n^{(i)} \int_{-1}^{1} K^{(i)}(t) P_n(t)\, dt = \mu_n^{(i)}\, (K^{(i)})^\wedge(n). \tag{8.13} $$

Proof. We deal with the case $i=1$. As ${}^v\mathbf{k}^{(1,1)}(\xi,\cdot)$ is a member of the space $\mathbf{l}^2(\Omega)$, we have

$$ \int_\Omega o_\xi^{(1)} o_\eta^{(1)} K^{(1)}(\xi\cdot\eta) \cdot o_\xi^{(1)} o_\eta^{(1)} K^{(1)}(\xi\cdot\eta)\, d\omega(\eta) < \infty. \tag{8.14} $$

Furthermore, for $\mathbf{f} \in \mathbf{l}^2(\Omega)$ and $g \in \mathrm{l}^2(\Omega)$ we get

$$ \int_\Omega \mathbf{f}(\eta) \cdot o_\eta^{(1)} g(\eta)\, d\omega(\eta) = \int_\Omega O_\eta^{(1)} \mathbf{f}(\eta) \cdot g(\eta)\, d\omega(\eta). \tag{8.15} $$

Hence, it follows that

$$ \begin{aligned} &\int_\Omega o_\eta^{(1)} K^{(1)}(\xi\cdot\eta) \cdot o_\eta^{(1)} K^{(1)}(\xi\cdot\eta)\, dw(\eta) \qquad (8.16) \\ &= \int_\Omega O_\xi^{(1)} o_\xi^{(1)} o_\eta^{(1)} K^{(1)}(\xi\cdot\eta) \cdot o_\eta^{(1)} K^{(1)}(\xi\cdot\eta)\, d\omega(\eta) \\ &= \int_\Omega o_\xi^{(1)} o_\eta^{(1)} K^{(1)}(\xi\cdot\eta) \cdot o_\xi^{(1)} o_\eta^{(1)} K^{(1)}(\xi\cdot\eta)\, d\omega(\eta). \end{aligned} $$

Defining $g_\xi(\eta) = o_\eta^{(i)} K^{(1)}(\xi\cdot\eta)$, $\eta\in\Omega$, we can express g_ξ as a Fourier (orthogonal) series. More explicitly, using the addition theorem and the vectorial Funk–Hecke formula, we readily find

$$ \begin{aligned} o_\eta^{(i)} K^{(1)}(\xi\cdot\eta) &= g_\xi(\eta) \qquad (8.17) \\ &= \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \left((g_\xi)^{(i)}\right)^\wedge (n,m) y_{n,m}^{(i)}(\eta) \\ &= \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \int_\Omega g_\xi(\zeta) \cdot y_{n,m}^{(i)}(\zeta)\, d\omega(\zeta)\; y_{n,m}^{(i)}(\eta) \\ &= \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \int_\Omega {}^v\mathbf{p}_n^{(i,i)}(\eta,\zeta) g_\xi(\zeta)\, d\omega(\zeta) \\ &= \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \left(\mu_n^{(i)}\right)^{-1} \left(O^{(i)} g_\xi\right)^\wedge (n)\; o_\eta^{(i)} P_n(\eta\cdot\xi), \end{aligned} $$

where

$$(O^{(i)}g_\xi)^\wedge(n) = 2\pi \int_{-1}^{1} G_i(t)P_n(t)\ dt, \tag{8.18}$$

with

$$G_i(\xi\cdot\zeta) = O^{(i)}_\zeta g_\xi(\zeta). \tag{8.19}$$

Therefore, for $i=1$, we have

$$(O^{(1)}g_\xi)^\wedge(n) = 2\pi \int_{-1}^{1} \hat{K}(t)P_n(t)\,dt = 2\pi \int_{-1}^{1} K^{(1)}(t)P_n(t)\,dt, \tag{8.20}$$

where

$$\hat{K}(\xi\cdot\eta) = O^{(1)}_\xi o^{(1)}_\eta K^{(1)}(\xi\cdot\eta) = (\xi\cdot\xi)\ K^{(1)}(\xi\cdot\eta) = K^{(1)}(\xi\cdot\eta), \tag{8.21}$$

thereby identifying, as usual, $\hat{K}:\Omega\times\Omega\to\mathbb{R}^3$ with $\hat{K}:[-1,1]\to\mathbb{R}^3$, (i.e., $\hat{K}(t)=\hat{K}(\xi\cdot\eta)$).

Summarizing our results for $i=1$, we therefore obtain

$$\begin{aligned}
{}^v\mathbf{k}^{(1,1)}(\xi\cdot\eta) &= o^{(1)}_\xi o^{(1)}_\eta K^{(1)}(\xi\cdot\eta)\\
&= o^{(1)}_\xi g_\xi(\eta)\\
&= o^{(1)}_\xi \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}(\mu^{(1)}_n)^{-1}(O^{(1)}g_\xi)^\wedge(n) o^{(1)}_\eta P_n(\xi\cdot\eta)\\
&= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}(O^{(1)}g_\xi)^\wedge(n)\,{}^v\mathbf{p}^{(1,1)}_n(\xi,\eta).
\end{aligned} \tag{8.22}$$

Observing the fact that $\mu^{(1)}_n = 1$, we finally get the wanted assertion.

Next, we come to the cases $i=2,3$. Now, $K^{(i)}(\xi\cdot)$ is differentiable and, thus, in $\mathrm{L}^2(\Omega)$. The kernel $K^{(i)}$ admits the Legendre series expansion

$$K^{(i)} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}(K^{(i)})^\wedge(n)P_n, \tag{8.23}$$

where

$$(K^{(i)})^\wedge(n) = 2\pi \int_{-1}^{1} K^{(i)}(t)P_n(t)\,dt. \tag{8.24}$$

This leads us to the desired identity

$$\begin{aligned}
\mathbf{k}^{(i)}(\xi,\eta) &= o^{(i)}_\xi o^{(i)}_\eta K^{(i)}(\xi\cdot\eta) \qquad (8.25)\\
&= \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi}(K^{(i)})^\wedge(n)\mu^{(i)}_n\ {}^v\mathbf{p}^{(i,i)}_n(\xi,\eta).
\end{aligned}$$

Altogether, Theorem 8.4 is verified. □

A key property of an $\mathrm{l}^2(\Omega)$-tensor zonal kernel function ${}^v\mathbf{k}$ is its invariance under orthogonal transformations $\mathbf{t}$, i.e., ${}^v\mathbf{k}(\mathbf{t}\xi, \mathbf{t}\eta) = \mathbf{t}\,{}^v\mathbf{k}(\xi,\eta)\mathbf{t}^T, \xi, \eta \in \Omega$. In addition, it is not difficult to show the following result.

Theorem 8.5. *A (vectorial) zonal rank-2 tensor kernel function of type i* ${}^v\mathbf{k}^{(i,i)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ *is an* $\mathrm{l}^2_{(i,i)}(\Omega)$*-zonal rank–2 tensor kernel function, if and only if*

$$\sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \left(({}^v\mathbf{k}^{(i,i)})^\wedge(n) \right)^2 < \infty, \tag{8.26}$$

where

$$({}^v\mathbf{k}^{(i,i)})^\wedge(n) = \mu_n^{(i)} (K^{(i)})^\wedge(n). \tag{8.27}$$

Of course, this theorem follows directly from the addition theorem

$$\sum_{m=1}^{2n+1} (y_{n,m}^{(i)}(\xi))^2 = \frac{2n+1}{4\pi}, \quad \xi \in \Omega. \tag{8.28}$$

The introduction of convolutions involving (vectorial) zonal rank-2 tensor kernel functions is quite similar to the scalar case.

Definition 8.6. Let ${}^v\mathbf{k}$, ${}^v\mathbf{h}$ be $\mathrm{l}^2(\Omega)$-zonal rank-2 tensor kernel functions. Suppose that f is a vector field of class $\mathrm{l}^2(\Omega)$. Then, ${}^v\mathbf{k} * f$ defined by

$$({}^v\mathbf{k} * f)(\xi) = \int_\Omega {}^v\mathbf{k}(\xi,\eta) f(\eta)\, d\omega(\eta), \quad \xi \in \Omega, \tag{8.29}$$

is called the *convolution of* ${}^v\mathbf{k}$ *against* f. Furthermore, ${}^v\mathbf{h} * {}^v\mathbf{k}$ defined by

$$({}^v\mathbf{h} * {}^v\mathbf{k})(\xi,\eta) = \int_\Omega {}^v\mathbf{h}(\xi,\zeta)\,{}^v\mathbf{k}(\zeta,\eta)\, d\omega(\zeta), \quad \xi,\eta \in \Omega, \tag{8.30}$$

is said to be the *convolution of* ${}^v\mathbf{h}$ *against* ${}^v\mathbf{k}$.

Note that the symbol '$*$' is again used simultaneously for different types of convolutions.

Obviously, ${}^v\mathbf{k} * f$ is a member of class $\mathrm{l}^2(\Omega)$. In spectral formulation, we have

$${}^v\mathbf{k}^{(i,i)} * f = \sum_{n=0_i}^{\infty} ({}^v\mathbf{k}^{(i,i)})^\wedge(n) \sum_{m=1}^{2n+1} (f^{(i)})^\wedge(n,m) y_{n,m}^{(i)}, \tag{8.31}$$

and

$${}^v\mathbf{h}^{(i)} * {}^v\mathbf{k} = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} ({}^v\mathbf{h}^{(i,i)})^\wedge(n) ({}^v\mathbf{k}^{(i,i)})^\wedge(n)\, {}^v\mathbf{p}_n^{(i,i)}, \tag{8.32}$$

$i = 1, 2, 3$.

By virtue of the orthogonal expansion in terms of Legendre tensors (8.32), it is not hard to verify that, for every point $\xi \in \Omega$, $({}^v\mathbf{h} * {}^v\mathbf{k})(\xi, \cdot)$ is continuous on the sphere Ω.

Lemma 8.7. *Let* ${}^v\mathbf{k} = \sum_{i=1}^{3} {}^v\mathbf{k}^{(i,i)}$ *be an* $\mathrm{l}^2(\Omega)$*-zonal rank-2 tensor kernel function. Then*

$$ {}^v\mathbf{k}^{(i,i)}(\xi, \eta) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} {}^v\mathbf{p}_n^{(i,i)}(\xi, \cdot) * {}^v\mathbf{k}(\cdot, \eta), \tag{8.33} $$

$\xi, \eta \in \Omega$.

Finally, we mention the representation of an $\mathrm{l}^2(\Omega)$-vector field f in terms of Legendre tensors ${}^v\mathbf{p}_n^{(i,i)}$

$$ f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} {}^v\mathbf{p}_n^{(i,i)} * f, \tag{8.34} $$

where the equality in (8.34) is understood in $\|\cdot\|_{\mathrm{l}^2(\Omega)}$–sense.

Remark 8.8. As we have shown, the Legendre tensor fields ${}^v\tilde{\mathbf{p}}_n^{(i,i)}$ can be expressed in terms of the tensor fields ${}^v\mathbf{p}_n^{(i,i)}$. This is the reason why it also makes sense to introduce (vectorial) zonal rank-2 tensor kernel functions with respect to the $\{{}^v\tilde{\mathbf{p}}_n^{(i,i)}\}$-system by letting

$$ {}^v\tilde{\mathbf{k}}^{(i,i)}(\xi, \eta) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \left({}^v\tilde{\mathbf{k}}^{(i,i)}\right)^{\wedge}(n) {}^v\tilde{\mathbf{p}}_n^{(i,i)}(\xi, \eta), \tag{8.35} $$

$(\xi, \eta) \in \Omega \times \Omega$, where

$$ ({}^v\tilde{\mathbf{k}}^{(i,i)})^{\wedge}(n) {}^v\mathbf{p}_n^{(i,i)}(\xi, \eta) = \int_{\Omega} {}^v\tilde{\mathbf{k}}^{(i,i)}(\xi, \zeta) \frac{2n+1}{4\pi} {}^v\tilde{\mathbf{p}}_n^{(i,i)}(\zeta, \eta)\, d\omega(\eta). \tag{8.36} $$

Clearly, all results being valid for the $\{{}^v\tilde{\mathbf{p}}_n^{(i,i)}\}$-system can be formulated in parallel.

8.3 Vector Zonal Kernel Functions in Vectorial Context

Remembering the second vectorial variant of the addition theorem (Theorem 5.46) in vector theory, we now turn to the definition of vector zonal kernel functions. As already announced, they are created by a single application of the operators $o^{(i)}$ to scalar zonal kernel functions.

Definition 8.9. Assume that $K^{(i)} \in \mathrm{C}^{(0_i)}[-1,1]$, $i \in \{1,2,3\}$, are scalar zonal kernel functions, respectively. A function $k^{(i)} : \Omega \times \Omega \to \mathbb{R}^3$ (more precisely, ${}^v k^{(i)}$), given by

$$k^{(i)}(\xi,\eta) = o_\xi^{(i)} K^{(i)}(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{8.37}$$

is called a *(vectorial) zonal vector kernel function of type* i (with respect to $\{p_n^{(i)}\}$), and

$$k = \sum_{i=1}^{3} k^{(i)} \tag{8.38}$$

is called a *(vectorial) zonal vector kernel function (with respect to* $\{p_n\}$*).*

Definition 8.10. A zonal vector kernel function $k^{(i)} : \Omega \times \Omega \to \mathbb{R}^3$ of type i is called an $\mathrm{l}^2_{(i)}(\Omega)$*–zonal vector kernel function*, if $k^{(i)}(\xi,\cdot)$ is in $\mathrm{l}^2(\Omega)$ for each $\xi \in \Omega$. Furthermore, $k = \sum_{i=1}^3 k^{(i)}$ is called an $\mathrm{l}^2(\Omega)$*–zonal vector kernel function.*

The following result can be directly derived from the identities (8.4), (8.5), and (8.6).

Theorem 8.11. *An* $\mathrm{l}^2_{(i)}(\Omega)$*–zonal vector kernel function* $k^{(i)} : \Omega \times \Omega \to \mathbb{R}^3$ *of type* i *can be expressed as a Legendre series in the form*

$$k^{(i)}(\xi,\cdot) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (k^{(i)})^\wedge(n) {p_n}^{(i)}(\xi,\cdot), \tag{8.39}$$

where

$$(k^{(i)})^\wedge(n) = \left(\mu_n^{(i)}\right)^{1/2} (K^{(i)})^\wedge(n). \tag{8.40}$$

Proof. Again, we first deal with the case $i=1$. As $k^{(1)}(\xi,\cdot)$ is in $\mathrm{l}^2(\Omega)$, it is easy to see that

$$\begin{aligned}
\int_\Omega K^{(1)}(\xi\cdot\eta) K^{(1)}(\xi\cdot\eta)\, d\omega(\eta) &= \int_\Omega K^{(1)}(\xi\cdot\eta) O_\xi^{(1)} o_\xi^{(1)} K^{(1)}(\xi\cdot\eta)\, d\omega(\eta) \\
&= \int_\Omega o_\xi^{(1)} K^{(1)}(\xi\cdot\eta) \cdot o_\xi^{(1)} K^{(1)}(\xi\cdot)\, d\omega(\eta) \\
&= \int_\Omega k^{(1)}(\xi\cdot\eta) \cdot k^{(1)}(\xi\cdot\eta)\, d\omega(\eta) < \infty.
\end{aligned}$$

Thus we have $K^{(1)}(\xi\cdot) \in \mathrm{L}^2(\Omega)$, such that $K^{(1)}(\xi\cdot)$ can be written as a Legendre series. This leads us to the identities

$$\begin{aligned} k^{(1)}(\xi,\cdot) &= o^{(1)} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (K^{(1)})^\wedge(n) P_n(\xi,\cdot) \\ &= \sum_{n=0}^{\infty} \left(\mu_n^{(1)}\right)^{1/2} \frac{2n+1}{4\pi} (K^{(1)})^\wedge(n) p_n^{(1)}(\xi,\cdot) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (k^{(1)})^\wedge(n) p^{(1)}(\xi,\cdot), \end{aligned} \tag{8.41}$$

where

$$(k^{(1)})^\wedge(n) = \left(\mu_n^{(1)}\right)^{1/2} (K^{(1)})^\wedge(n). \tag{8.42}$$

This is the required result for $i = 1$. For the cases $i = 2, 3$, we observe that $K^{(i)}(\xi\cdot)$ is assumed to be differentiable and, therefore, in $\mathrm{L}^2(\Omega)$. The assertion of our theorem follows by the same arguments as shown for the case $i = 1$. □

Theorem 8.12. *A vector zonal kernel function $k^{(i)} : \Omega \times \Omega \to \mathbb{R}^3$ of type i is an $\mathrm{l}^2_{(i)}(\Omega)$–vector zonal kernel function, if and only if*

$$\sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \left((k^{(i)})^\wedge(n)\right)^2 < \infty, \tag{8.43}$$

where

$$(k^{(i)})^\wedge(n) = \left(\mu_n^{(i)}\right)^{1/2} (K^{(i)})^\wedge(n). \tag{8.44}$$

Proof. Observing that $x \cdot y = \operatorname{trace} x \otimes y$, $x, y \in \mathbb{R}^3$, we get

$$\begin{aligned} &(k(\xi,\cdot), k(\xi,\cdot))_{\mathrm{l}^2(\Omega)} \\ &= \int_\Omega \left(\sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (k^{(i)})^\wedge(n) p_n^{(i)}(\xi,\eta) \right) \\ &\qquad \cdot \left(\sum_{j=1}^{3} \sum_{l=0_j}^{\infty} \frac{2l+1}{4\pi} (k^{(j)})^\wedge(l) p_l^{(j)}(\xi,\eta) \right) d\omega(\eta) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{n=\max(0_i,0_j)}^{\infty} \sum_{m=1}^{2n+1} (k^{(i)})^\wedge(n) (k^{(j)})^\wedge(n) y_{n,m}^{(i)}(\xi) \cdot y_{n,m}^{(j)}(\xi) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{n=\max(0_i,0_j)}^{\infty} (k^{(i)})^\wedge(n) (k^{(j)})^\wedge(n) \frac{2n+1}{4\pi} \operatorname{trace}\left({}^v\mathbf{p}_n^{(i,j)}(\xi,\xi)\right). \end{aligned} \tag{8.45}$$

The desired assertion follows by taking into account that

$$\mathrm{trace}({}^v\mathbf{p}_n^{(i,j)}(\xi,\xi)) = \delta_{ij}. \tag{8.46}$$

This yields the proof of Theorem 8.12. □

Remark 8.13. From Lemma 5.63, we know that the vector Legendre kernels $\tilde{p}_n^{(i)}$ can be written in terms of $p_n^{(i)}$. Therefore, it also makes sense to introduce, in parallel, (vectorial) zonal vector kernel functions with respect to $\{\tilde{p}_n^{(i)}\}$ by letting

$$\tilde{k}^{(i)}(\xi,\eta) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} (\tilde{k}^{(i)})^\wedge(n) \tilde{p}_n^{(i)}(\xi,\eta), \tag{8.47}$$

$(\xi,\eta) \in \Omega \times \Omega$, where

$$(\tilde{k}^{(i)})^\wedge(n) \tilde{p}_n^{(i)}(\xi,\eta) = \int_\Omega \tilde{k}^{(i)}(\xi,\zeta) \frac{2n+1}{4\pi} \tilde{p}_n^{(i)}(\eta,\zeta)\ d\omega. \tag{8.48}$$

8.4 Convolutions Involving Vector Zonal Kernel Functions

Next, we introduce convolutions in the vectorial context.

Definition 8.14. Let k be an $\mathrm{l}^2(\Omega)$-zonal vector kernel function, $f \in \mathrm{l}^2(\Omega)$, $F \in \mathrm{L}^2(\Omega)$. Then $k * f$ defined by

$$k * f(\xi) = \int_\Omega k(\eta,\xi) \cdot f(\eta)\, d\omega(\eta) \tag{8.49}$$

is called the *convolution of* k *against* f. Moreover, $k^{(i)} \star F$, $i = 1,2,3$, given by

$$k^{(i)} \star F(\xi) = \int_\Omega k^{(i)}(\xi,\eta) F(\eta)\, d\omega(\eta) \tag{8.50}$$

is called the *convolution of* $k^{(i)}$ *against* F.

Note that we use different symbols for the convolutions to point out their different nature.

For $k, \hat{k}$ being $\mathrm{l}^2(\Omega)$-zonal vector kernel functions we let

$$\hat{k} \star (k * f) = \sum_{i=1}^{3} \hat{k}^{(i)} \star \left(k^{(i)} * f\right). \tag{8.51}$$

Note that

$$\hat{k}^{(i)} \star \left(k^{(i)} * f\right) = \int_\Omega \int_\Omega \hat{k}^{(i)}(\cdot,\xi) \otimes k^{(i)}(\eta,\xi) f(\eta)\, d\omega(\eta) d\omega(\xi). \tag{8.52}$$

This motivates the following rank-2 tensorial setting.

Definition 8.15. Let $\hat{k}^{(i)}$, $k^{(i)}$ be two $\mathrm{l}^2_{(i)}(\Omega)$-zonal vector kernel functions. Then, we define the convolution of $\hat{k}^{(i)}$ against $k^{(i)}$ by

$$\left(\hat{k}^{(i)} \star k^{(i)}\right)(\xi,\eta) = \int_\Omega \hat{k}^{(i)}(\xi,\zeta) \otimes k^{(i)}(\eta,\zeta)\, d\omega(\zeta). \tag{8.53}$$

Furthermore, $\hat{k} \star k$ is understood to be

$$\hat{k} \star k = \sum_{i=1}^{3} \hat{k}^{(i)} \star k^{(i)}. \tag{8.54}$$

The following theorem can be verified easily by standard arguments.

Theorem 8.16. *Let $\hat{k}^{(i)}$, $k^{(i)}$ be two $\mathrm{l}^2_{(i)}(\Omega)$-zonal vector kernel functions. Then the convolution $\hat{k}^{(i)} \star k^{(i)}$ is an $\mathrm{l}^2_{(i)}(\Omega)$-zonal tensor kernel function, such that*

$$\hat{k}^{(i)} \star k^{(i)}(\xi,\cdot) = \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} (\hat{k}^{(i)})^\wedge(n)(k^{(i)})^\wedge(n) \frac{2n+1}{4\pi} {}^v\mathbf{p}_n^{(i,i)}(\xi,\cdot). \tag{8.55}$$

From the expansion (8.55) in terms of Legendre rank-2 tensors, it can be derived that $\hat{k}^{(i)} \star k^{(i)}(\xi,\cdot)$ is continuous on the sphere Ω for every $\xi \in \Omega$.

The Parseval identity for vector spherical harmonics enables us to verify the following theorem (see M. Bayer et al. (1998), S. Beth (2000)).

Theorem 8.17. *Let f be of class $\mathrm{l}^2(\Omega)$. Assume that $\hat{k}$, k are $\mathrm{l}^2(\Omega)$-zonal vector kernel functions, whereas $\hat{\mathbf{k}}$, $\mathbf{k}$ are $\mathrm{l}^2(\Omega)$-zonal rank-2 tensor kernel functions with*

$$({}^v\hat{\mathbf{k}}^{(i,i)})^\wedge(n) = (\hat{k}^{(i)})^\wedge(n), \tag{8.56}$$

and

$$({}^v\mathbf{k}^{(i,i)})^\wedge(n) = (k^{(i)})^\wedge(n), \tag{8.57}$$

for all $i = 1,2,3$. Then

$${}^v\hat{\mathbf{k}} * {}^v\mathbf{k} * f = \hat{k} \star (k * f). \tag{8.58}$$

In other words, the different ways of forming convolutions (8.58) against vector fields either by tensor zonal kernel function or vector zonal kernels are equivalent. This is, in fact, a remarkable result. In consequence, due to Theorem 8.17, we can substitute rank-2 tensor zonal kernel functions by vector zonal kernel basis functions which is of importance not only for numerical purposes: once again, vector zonal functions require first order derivatives of the Legendre polynomials, whereas tensor zonal kernel functions make it necessary to compute the second order derivatives. Furthermore, the operational effort is reduced as we do not have to calculate tensor products when we turn over to vector zonal kernel functions. The structural price that must be paid in comparison to the tensor approach, however, is a bilinear framework for the vectorial kernels involved in the convolutions.

8.5 Dirac Families of Zonal Vector Kernel Functions

Starting from a Dirac family $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ of scalar zonal kernel functions, we are able to construct a Dirac family of zonal rank-2 tensor kernel functions $\{\boldsymbol{\varphi}_\rho\}_{\rho\in(0,\infty)}$ as follows (note that we restrict ourselves to the system of dual operators $o^{(i)}, O^{(i)}$, $i \in \{1,2,3\}$):

$$\boldsymbol{\varphi}_\rho(\xi,\eta) = \sum_{i=1}^{3} {}^{v}\boldsymbol{\varphi}_\rho^{(i,i)}(\xi,\eta) \tag{8.59}$$

with

$${}^{v}\boldsymbol{\varphi}_\rho^{(i,i)}(\xi,\eta) = \sum_{n=O_i}^{\infty} (\Phi_\rho)^\wedge(n) \sum_{j=1}^{2n+1} y_{n,j}^{(i)}(\xi) \otimes y_{n,j}^{(i)}(\eta), \tag{8.60}$$

$\xi,\eta \in \Omega$. Correspondingly, a *Dirac family* $\{\varphi_\rho\}_{\rho\in(0,\infty)}$ of vector zonal kernel functions φ_ρ reads as follows:

$$\varphi_\rho(\xi,\eta) = \sum_{i=1}^{3} \varphi_\rho^{(i)}(\xi,\eta) \tag{8.61}$$

with

$$\varphi_\rho^{(i)}(\xi,\eta) = \sum_{n=O_i}^{\infty} (\Phi_\rho)^\wedge(n) Y_{n,j}(\xi) y_{n,j}^{(i)}(\eta), \tag{8.62}$$

$\xi,\eta \in \Omega$.

As an immediate consequence, we obtain the following linear and bilinear approach for rank-2 tensor Dirac families and the bilinear approach for vector Dirac families.

Theorem 8.18. *Let* $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ *be a scalar Dirac family. Then*

$$\lim_{\rho\to 0} \|f - \boldsymbol{\varphi}_\rho * f\|_{\mathrm{l}^2(\Omega)} = 0, \tag{8.63}$$

$$\lim_{\rho\to 0} \|f - \boldsymbol{\varphi}_\rho * \boldsymbol{\varphi}_\rho * f\|_{\mathrm{l}^2(\Omega)} = 0 \tag{8.64}$$

and

$$\lim_{\rho\to 0} \|f - \varphi_\rho \star \varphi_\rho * f\|_{\mathrm{l}^2(\Omega)} = 0 \tag{8.65}$$

for all $f \in \mathrm{l}^2(\Omega)$.

This means that we have extended the notion of an approximate identity in a canonical way to spherical vector field thereby using two different, but (in bilinear sense) equivalent approaches to Dirac families.

Seen from the point of spherical functions on the sphere, we should have a closer look to the Dirac families involved in the approximation. It is clear that

$$^{v}\boldsymbol{\varphi}_\rho^{(1,1)}(\xi,\eta) = o_\xi^{(1)} o_\eta^{(1)} \Phi_\rho(\xi\cdot\eta) \tag{8.66}$$

and

$$^{v}\boldsymbol{\varphi}_\rho^{(i,i)}(\xi,\eta) = -o_\xi^{(i)} o_\eta^{(i)} \int_\Omega G(\Delta^*;\xi\cdot\zeta)\Phi_\rho(\zeta\cdot\eta)\, d\omega(\xi), \tag{8.67}$$

$\xi,\eta\in\Omega$, $i=2,3$. Equivalently, we have

$$\boldsymbol{\varphi}_\rho^{(i,i)}(\xi,\eta) = o_\xi^{(i)} o_\eta^{(i)} \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{n(n+1)} (\Phi_\rho)^\wedge(n) P_n(\xi\cdot\eta), \tag{8.68}$$

$\xi,\eta\in\Omega, i=2,3$. In an analogous way, we find

$$\varphi_\rho^{(i)}(\xi,\eta) = o_\eta^{(i)} \sum_{n=1}^{\infty} \frac{2n+1}{4\pi} \frac{1}{\sqrt{n(n+1)}} (\Phi_\rho)^\wedge(n) P_n(\xi\cdot\eta), \tag{8.69}$$

$i=2,3$.

Remark 8.19. Our approach has shown that an isotropic operator mapping a vector field onto a vector field refuses the representation in terms of a vector zonal kernel function. In that context, in fact, zonal tensor functions have to be taken into account. Zonal tensor functions, indeed, fall back upon an addition theorem involving the tensor product of vector spherical harmonics. Although they do not allow us to describe isotropic vector fields, zonal rank-2 tensor fields are of advantage for the approximation of vector fields in form of splines (W. Freeden, T. Gervens (1991)), W. Freeden et al.

(1994) and wavelets (W. Freeden et al. (1998)) (by matrix-vector multiplications with constant vectors). The numerical disadvantage of a representation of vector fields based on tensor zonal kernel fields is easily understood. We have to deal with matrix-vector multiplications. In comparison with vector zonal functions, a further drawback comes up. While vector zonal functions require only the first derivative of a scalar radial zonal function, zonal tensor functions even need second derivatives. This makes them more difficult to handle in vector field modeling, particularly when the scalar zonal function is not known elementary in a closed representation, but only as series expansion in terms of Legendre polynomials. Nevertheless, tensor zonal functions are an important tool in the characterization of vector fields (comparable to the scalar case). Of course, tensor zonal functions are natural structures to observe rotational symmetry within a tensor framework, and in this case, second derivatives for the occurring Legendre polynomials are canonical.

8.6 Bibliographical Notes

Zonal kernel functions in the vector context have been introduced in a double sense in twofold way (i) as vector fields generated by applying the operators $o^{(i)}, \tilde{o}^{(i)}$, respectively, on scalar zonal kernel functions (see M. Bayer et al. (1998), S. Beth (2000), H. Nutz (2002), C. Mayer (2003)) (ii) as tensor fields generated by a double application of the operators $o^{(i)}, \tilde{o}^{(i)}$, respectively, on scalar zonal functions (see W. Freeden et al. (1998), S. Beth (2000), H. Nutz (2002), M.K. Abeyratne (2003)). In the first case, we are led to zonal vector kernel functions, whereas the second case leads to zonal tensor kernel functions (to be used within the vector context). Both types of isotropic functions are basic tools for approximation techniques like spherical splines and wavelets (see, e.g., G. Wahba (1982), W. Freeden, T. Gervens (1991), W. Freeden, U. Windheuser (1996), U. Windheuser (1995), W. Freeden et al. (1998), W. Freeden, M. Schreiner (1997), H. Nutz (2002), M.J. Fengler (2005), W. Freeden, M. Schreiner (2006), S. Gramsch (2006), T. Fehlinger et al. (2007)).

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

9 Tensorial Zonal Kernel Functions

Next, we come to zonal kernel functions in the tensor context. In analogy to the vectorial case, we are able to derive two variants based on the known addition theorems (Theorem 6.21 and Theorem 6.34). In more detail, we obtain zonal kernel functions in the tensor context by applying the operators $\mathbf{o}^{(i,k)}$, $i, k \in \{1, 2, 3\}$, once and twice to scalar zonal kernel functions (see Table 9.1).

Table 9.1: Overview on (tensorial) zonal rank-4/rank-2 tensor kernel functions in relation to Legendre kernel functions.

Zonal kernel (generating system)	**Linear approach**		**Bilinear approach**	
Scalar field ($\{Y_{n,j}\}$-system)	scalar zonal kernel function ($\{P_n\}$–system)	K		
Tensor field ($\{\mathbf{y}_{n,j}^{(i,k)}\}$-system)	(tensorial) zonal rank-4 tensor kernel function ($\{\mathbf{P}_n^{(i,k,i,k)}\}$-system)	$\mathbf{K}^{(i,k)}$	(tensorial) zonal rank-2 tensor kernel function ($\{{}^t\mathbf{p}_n^{(i,k)}\}$-system)	${}^t\mathbf{k}^{(i,k)}$
Tensor field ($\{\tilde{\mathbf{y}}_{n,j}^{(i,k)}\}$-system)	(tensorial) zonal rank-4 tensor kernel function ($\{\tilde{\mathbf{P}}_n^{(i,k,i,k)}\}$-system)	$\tilde{\mathbf{K}}^{(i,k)}$	(tensorial) zonal rank-2 tensor kernel function ($\{{}^t\tilde{\mathbf{p}}_n^{(i,k)}\}$-system)	${}^t\tilde{\mathbf{k}}^{(i,k)}$

9.1 Preparatory Material

Our work is based on the following already known lemma, which demonstrates that the operators $\mathbf{o}^{(i,k)}$ are easily applicable to scalar zonal kernel functions.

Lemma 9.1. *Let K be of class $\mathrm{C}^{(0_{ik})}[-1,1]$. Suppose that $\eta \in \Omega$ is fixed. Then, for all $\xi \in \Omega$,*

$$\mathbf{o}_\xi^{(1,1)} K(\xi \cdot \eta) = K(\xi \cdot \eta)\xi \otimes \eta, \tag{9.1}$$

$$\mathbf{o}_\xi^{(1,2)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)\xi \otimes (\eta - (\xi \cdot \eta)\xi), \tag{9.2}$$

$$\mathbf{o}_\xi^{(1,3)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)\xi \otimes (\xi \wedge \eta), \tag{9.3}$$

$$\mathbf{o}_\xi^{(2,1)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi) \otimes \xi, \tag{9.4}$$

$$\mathbf{o}_\xi^{(2,2)} K(\xi \cdot \eta) = K(\xi \cdot \eta)\mathbf{i}_{\mathrm{tan}}(\xi), \tag{9.5}$$

$$\begin{aligned}\mathbf{o}_\xi^{(2,3)} K(\xi \cdot \eta) = {} & K''(\xi \cdot \eta)\left((\eta - (\xi \cdot \eta)\xi) \otimes (\eta - (\xi \cdot \eta)\xi)\right. \\ & \left. - (\xi \wedge \eta) \otimes (\xi \wedge \eta)\right),\end{aligned} \tag{9.6}$$

$$\mathbf{o}_\xi^{(3,1)} K(\xi \cdot \eta) = K'(\xi \cdot \eta)(\xi \wedge \eta) \otimes \xi, \tag{9.7}$$

$$\begin{aligned}\mathbf{o}_\xi^{(3,2)} K(\xi \cdot \eta) = {} & K''(\xi \cdot \eta)\left((\eta - (\xi \cdot \eta)\xi) \otimes (\xi \wedge \eta)\right. \\ & \left. + (\xi \wedge \eta) \otimes (\eta - (\xi \cdot \eta)\xi)\right),\end{aligned} \tag{9.8}$$

$$\mathbf{o}_\xi^{(3,3)} K(\xi \cdot \eta) = K(\xi \cdot \eta)\mathbf{j}_{\mathrm{tan}}(\xi). \tag{9.9}$$

For simplicity, we omit the explicit representations of a double application of the operators $o^{(i,k)}$ to the scalar kernel K.

9.2 Tensor Zonal Kernel Functions of Rank Four in Tensorial Context

First, we are interested in defining (tensorial) rank-4 tensor zonal kernel functions by a double application of operators $\mathbf{o}^{(i,k)}$ on (sufficiently often differentiable) scalar zonal kernel functions.

Definition 9.2. Assume that $K^{(i,k)} \in \mathrm{C}^{(2 \cdot 0_{ik})}[-1,1]$, $i,k \in \{1,2,3\}$, are scalar zonal kernel functions. A function $\mathbf{K}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$, (more precisely, ${}^t\mathbf{K}^{(i,k)}$) given by

$$\mathbf{K}^{(i,k)}(\xi,\eta) = \mathbf{o}_\xi^{(i,k)} \mathbf{o}_\eta^{(i,k)} K^{(i,k)}(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{9.10}$$

is called a *(tensorial) zonal rank-4 tensor kernel function of type* (i,k) (with respect to $\{\mathbf{P}_n^{(i,k,i,k)}\}$). Furthermore, we let

$$\mathbf{K} = \sum_{i=1}^{3} \sum_{k=1}^{3} \mathbf{K}^{(i,k)} \tag{9.11}$$

$\mathbf{K}$ is called *a (tensorial) zonal rank-4 tensor kernel function* (with respect to $\{\mathbf{P}_n\}$).

In close analogy to the vector case, we introduce the following setting.

Definition 9.3. A (tensorial) zonal rank-4 tensor kernel function of type (i,k), $\mathbf{K}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$, $i,k \in \{1,2,3\}$, is called an $\mathbf{l}^2_{(i,k)}(\Omega)$*–(tensorial) zonal rank-4 tensor kernel function*, if $\mathbf{K}^{(i,k)}(\xi,\cdot)$ is square-integrable on Ω for every $\xi \in \Omega$. Furthermore,

$$\mathbf{K} = \sum_{i=1}^{3} \sum_{k=1}^{3} \mathbf{K}^{(i,k)} \tag{9.12}$$

is called an $\mathbf{l}^2(\Omega)$-*tensorial zonal rank-4 tensor kernel function*, if $\mathbf{K}^{(i,k)}$ are $\mathbf{l}^2_{(i,k)}(\Omega)$-tensorial zonal rank-4 tensor kernel functions.

Clearly, (tensorial) zonal rank-4 tensor kernel functions can be expanded in terms of the Legendre functions. For that purpose, we need the corresponding tensorial variant of the Funk–Hecke formula. As in the vector case, for the convenience of the reader, it will be recapitulated briefly.

Theorem 9.4. *(Funke-Hecke Formula in Tensorial Context) Let* $\eta \in \Omega$ *be fixed. Assume that* $\mathbf{h}(\cdot,\eta) \in \mathbf{c}^{(2)}(\Omega)$ *satisfies* $\mathbf{h}(\mathbf{t}\xi,\eta) = \mathbf{t}\mathbf{h}(\xi,\eta)\mathbf{t}^T$ *for all orthogonal transformations* $\mathbf{t} \in SO_\eta(3)$ *and all* $\xi \in \Omega$*. Then, for all* $\zeta \in \Omega$ *and for all* $i,k \in \{1,2,3\}$,

$$\int_\Omega \mathbf{P}_n^{(i,k,i,k)}(\zeta,\xi)\mathbf{h}(\zeta,\eta)\, d\omega(\xi) = \left(\mu_n^{(i,k)}\right)^{-1} \left(O^{(i,k)}\mathbf{h}\right)^\wedge(n)\mathbf{o}_\zeta^{(i,k)} P_n(\zeta \cdot \eta), \tag{9.13}$$

where

$$\left(O^{(i,k)}\mathbf{h}\right)^\wedge(n) = 2\pi \int_{-1}^{1} H_{i,k}(t) P_n(t)\, dt, \tag{9.14}$$

and

$$H_{i,k}(\xi \cdot \eta) = O_\xi^{(i,k)} \mathbf{h}(\xi, \eta). \tag{9.15}$$

In parallel to the vector case, we formulate the following result.

Theorem 9.5. *An $\mathbf{l}^2_{(i,k)}(\Omega)$-tensorial zonal rank-4 tensor kernel function $\mathbf{K}^{(i,k)}$ can be represented as a Legendre series in the form*

$$\mathbf{K}^{(i,k)}(\xi, \cdot) = \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} (\mathbf{K}^{(i,k)})^\wedge(n) \mathbf{P}_n^{(i,k,i,k)}(\xi, \cdot), \tag{9.16}$$

where

$$(\mathbf{K}^{(i,k)})^\wedge(n) = \mu_n^{(i,k)} (K^{(i,k)})^\wedge(n). \tag{9.17}$$

The proof of Theorem 9.5 follows in close analogy to its vectorial counterpart (Theorem 8.4). Thus, it is omitted here.

Theorem 9.6. *A (tensorial) zonal rank-4 tensor kernel function of type (i,k), $\mathbf{K}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ is an $\mathbf{l}^2_{(i,k)}(\Omega)$ (tensorial) zonal rank–4 tensor kernel function, if and only if*

$$\sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \left((\mathbf{K}^{(i,k)})^\wedge(n) \right)^2 < \infty, \tag{9.18}$$

where

$$(\mathbf{K}^{(i,k)})^\wedge(n) = \mu_n^{(i,k)} (K^{(i,k)})^\wedge(n). \tag{9.19}$$

Theorem 9.6 follows directly from the identity

$$\sum_{m=1}^{2n+1} |\mathbf{y}_{n,m}^{(i,k)}(\xi)|^2 = \frac{2n+1}{4\pi}. \tag{9.20}$$

9.3 Convolutions Involving Zonal Tensor Kernel Functions

We are now going to introduce convolutions in the tensor context.

Definition 9.7. Let $\mathbf{H}$, $\mathbf{K}$ be $\mathbf{l}^2(\Omega)$-tensorial zonal rank-4 tensor kernel functions. Suppose that $\mathbf{f}$ is of class $\mathbf{l}^2(\Omega)$. Then $\mathbf{K} * \mathbf{f}$ defined by

$$(\mathbf{K} * \mathbf{f})(\xi) = \int_\Omega \mathbf{K}(\xi, \eta) \mathbf{f}(\eta) \, d\omega(\eta), \tag{9.21}$$

$\xi \in \Omega$, is called the *convolution of* $\mathbf{K}$ *against* $\mathbf{f}$. Furthermore, $\mathbf{H} * \mathbf{K}$ defined by

$$(\mathbf{H} * \mathbf{K})(\xi, \eta) = \int_\Omega \mathbf{H}(\xi, \zeta)\mathbf{K}(\zeta, \eta)\, d\omega(\zeta), \tag{9.22}$$

$\xi, \eta \in \Omega$, is called the *convolution of* $\mathbf{H}$ *against* $\mathbf{K}$.

Different variants of convolutions are definable.

From the addition theorem, the orthogonality of the tensor spherical harmonics, together with the identity $\mathbf{f}(\mathbf{g} \cdot \mathbf{h}) = (\mathbf{f} \otimes \mathbf{g})\mathbf{h}$, we are able to show that

$$\begin{aligned} \mathbf{K}^{(i,k)} * \mathbf{f} & \\ = & \int_\Omega \sum_{n=0_{ik}}^{\infty} (\mathbf{K}^{(i,k)})^\wedge(n) \sum_{m=1}^{2n+1} \mathbf{y}_{n,m}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,m}^{(i,k)}(\eta) \\ & \qquad \times \sum_{p=0_{ik}}^{\infty} \sum_{q=1}^{2p+1} (\mathbf{f}^{(i,k)})^\wedge(p,q)\mathbf{y}_{p,q}^{(i,k)}(\eta)\, d\omega(\eta) \\ = & \sum_{n=0_{ik}}^{\infty} (\mathbf{K}^{(i,k)})^\wedge(n) \sum_{m=1}^{2n+1} (\mathbf{f}^{(i,k)})^\wedge(n,m)\, \mathbf{y}_{n,m}^{(i,k)}. \end{aligned} \tag{9.23}$$

Further on, because of $(\mathbf{F} \otimes \mathbf{G})(\mathbf{H} \otimes \mathbf{I}) = (\mathbf{G} \cdot \mathbf{H})\mathbf{F} \otimes \mathbf{I}$, we find

$$\mathbf{H}^{(i,k)} * \mathbf{K}^{(i,k)} = \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} (\mathbf{H}^{(i,k)})^\wedge(n)(\mathbf{K}^{(i,k)})^\wedge(n)\mathbf{P}_n^{(i,k,i,k)}. \tag{9.24}$$

Using the Legendre series expansion (9.24), we easily see that the convolution $\mathbf{H}^{(i,k)} * \mathbf{K}(\xi, \cdot)$ is continuous on Ω for each point $\xi \in \Omega$.

Lemma 9.8. *Let* $\mathbf{K} = \sum_{i,k=1}^{3} \mathbf{K}^{(i,k)}$ *be an* $\mathbf{l}^2(\Omega)$*-tensorial zonal rank-4 tensor kernel function. Then* $\mathbf{K}^{(i,k)}$ *is expressible in the form*

$$\mathbf{K}^{(i,k)}(\xi, \eta) = \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,i,k)}(\xi, \cdot) * \mathbf{K}(\cdot, \eta), \tag{9.25}$$

$\xi, \eta \in \Omega$.

Finally, we are able to deduce from (6.330) that every $\mathbf{f} \in \mathbf{l}^2(\Omega)$ can be represented in the form

$$\mathbf{f} = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,i,k)} * \mathbf{f} \tag{9.26}$$

(in $\|\cdot\|_{\mathbf{l}^2(\Omega)}$-sense).

Remark 9.9. The (tensorial) Legendre rank-4 tensor kernel functions $\tilde{\mathbf{P}}_n^{(i,k,i,k)}$ are expressible in terms of $\mathbf{P}_n^{(i,k,i,k)}$. Therefore, (tensorial) zonal rank-4 tensor kernel functions with respect to the $\{\tilde{\mathbf{P}}_n^{(i,k,i,k)}\}$-system read as follows:

$$\tilde{\mathbf{K}}^{(i,k)}(\xi,\eta) = \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \left(\tilde{\mathbf{K}}^{(i,k)}\right)^{\wedge}(n)\tilde{\mathbf{P}}_n^{(i,k,i,k)}, \tag{9.27}$$

$(\xi,\eta) \in \Omega \times \Omega$, where

$$(\tilde{\mathbf{K}}^{(i,k)})^{\wedge}(n)\tilde{\mathbf{P}}_n^{(i,k,i,k)}(\xi,\eta) = \int_{\Omega} \tilde{\mathbf{K}}^{(i,k)}(\xi,\zeta)\frac{2n+1}{4\pi}\tilde{\mathbf{P}}_n^{(i,k,i,k)}(\xi,\eta)\ d\omega(\zeta). \tag{9.28}$$

9.4 Tensor Zonal Kernel Functions of Rank Two in Tensorial Context

Tensorial rank-2 tensor zonal kernel functions are defined by a single application of the operators $\mathbf{o}^{(i,k)}$ to (sufficiently smooth) scalar zonal kernel functions. Seen from operational point of view, they are of importance for two reasons: First, the computational effort is reduced because we do not have to calculate the tensor product of a tensor of rank four and a tensor of rank two. Furthermore, according to Lemma 9.1, we only need the second order derivatives of the generating scalar zonal kernel functions, whereas in the first approach involving tensorial rank-4 tensor zonal kernel functions, we need fourth order derivatives.

Definition 9.10. Assume that $K^{(i,k)} : [-1,1] \to \mathbb{R}$ are (sufficiently often differentiable) scalar zonal kernel functions, i.e., $K^{(i,k)} \in \mathrm{C}^{(0_{ik})}[-1,1]$, $i,k \in \{1,2,3\}$. A function ${}^t\mathbf{k}_{\xi}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$, given by

$${}^t\mathbf{k}_{\xi}^{(i,k)}(\xi,\eta) = \mathbf{o}_{\xi}^{(i,k)} K^{(i,k)}(\xi \cdot \eta), \quad \xi,\eta \in \Omega, \tag{9.29}$$

is called a *(tensorial) zonal rank-2 tensor kernel function of type* (i,k) (with respect to $\{{}^t\mathbf{p}_n^{(i,k)}\}$), while

$${}^t\mathbf{k} = \sum_{i=1}^{3}\sum_{k=1}^{3} {}^t\mathbf{k}^{(i,k)} \tag{9.30}$$

is called a *(tensorial) zonal rank-2 tensor kernel function* (with respect to $\{{}^t\mathbf{p}_n^{(i,k)}\}$).

In analogy to our above considerations, we introduce the following definition.

Definition 9.11. A (tensorial) zonal rank-2 tensor kernel function of kind (i,k), $\mathbf{k}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is called an $\mathbf{l}^2_{(i,k)}(\Omega)$-*tensorial zonal rank-2 tensor kernel function*, if ${}^t\mathbf{k}^{(i,k)}(\xi,\cdot)$ is in $\mathbf{l}^2(\Omega)$ for each $\xi \in \Omega$. Furthermore, ${}^t\mathbf{k} = \sum_{i=1}^3 \sum_{k=1}^3 {}^t\mathbf{k}^{(i,k)}$ is called an $\mathbf{l}^2(\Omega)$-*(tensorial) zonal rank-2 tensor kernel function*, if $\mathbf{k}^{(i,k)}$ are $\mathbf{l}^2_{(i,k)}(\Omega)$-tensorial zonal rank-2 tensor kernel functions.

In accordance with our approach, we are immediately able to prove the following property of an $\mathbf{l}^2_{(i,k)}(\Omega)$-(tensorial) zonal rank-2 tensor kernel function.

Theorem 9.12. *An $\mathbf{l}^2_{(i,k)}(\Omega)$-zonal rank-2 tensor zonal kernel function ${}^t\mathbf{k}^{(i,k)}$ can be represented as a Legendre series in the form*

$$ {}^t\mathbf{k}^{(i,k)}(\xi,\cdot) = \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} ({}^t\mathbf{k}^{(i,k)})^\wedge(n) {}^t\mathbf{p}_n^{(i,k)}(\xi,\cdot), \tag{9.31} $$

where

$$ ({}^t\mathbf{k}^{(i,k)})^\wedge(n) = \left(\mu_n^{(i,k)}\right)^{1/2} (K^{(i,k)})^\wedge(n). \tag{9.32} $$

The proof of Theorem 9.12 parallels that one known from the vectorial case, hence, we do not formulate it.

Theorem 9.13. *A (tensorial) zonal rank-2 tensor function of type (i,k) ${}^t\mathbf{k}^{(i,k)} : \Omega \times \Omega \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is an $\mathbf{l}^2_{(i,k)}(\Omega)$-zonal rank-2 tensor kernel function, if and only if*

$$ \sum_{n=0_{ik}}^{\infty} \frac{2n+1}{4\pi} \left(({}^t\mathbf{k}^{(i,k)})^\wedge(n)\right)^2 < \infty, \tag{9.33} $$

where

$$ ({}^t\mathbf{k}^{(i,k)})^\wedge(n) = \left(\mu_n^{(i,k)}\right)^{1/2} (K^{(i,k)}) \wedge (n). \tag{9.34} $$

Remark 9.14. From Lemma 6.43, we know that the (tensorial) Legendre rank-2 tensor kernel functions ${}^t\tilde{\mathbf{p}}_n^{(i,k)}$ are expressible in terms of $\mathbf{p}_n^{(i,k)}$. Therefore, we are able to introduce zonal rank-2 tensor kernel functions with respect to the $\{{}^t\tilde{\mathbf{p}}_n^{(i,k)}\}$-system

$$ {}^t\tilde{\mathbf{k}}^{(i,k)}(\xi,\eta) = \sum_{n=0_i}^{\infty} \frac{2n+1}{4\pi} \left({}^t\tilde{\mathbf{k}}^{(i,k)}\right)^\wedge (n) {}^t\tilde{\mathbf{p}}_n^{(i,k)}(\xi,\eta), \tag{9.35} $$

$(\xi, \eta) \in \Omega \times \Omega$, where

$$({}^t\tilde{\mathbf{k}}^{(i,k)})^\wedge(n)\,{}^t\mathbf{p}_n^{(i,k)}(\xi,\eta) = \int_\Omega {}^t\tilde{\mathbf{k}}^{(i,k)}(\xi,\zeta)\frac{2n+1}{4\pi}\,{}^t\tilde{\mathbf{p}}_n^{(i,k)}(\zeta,\eta)\; d\omega(\eta). \quad (9.36)$$

Next, we want to define the convolution in a (tensorial) rank-2 tensor context .

Definition 9.15. Let ${}^t\mathbf{k}$ be a $\mathbf{l}^2(\Omega)$-tensorial zonal rank-2 tensor kernel functions. Furthermore, assume that $\mathbf{f} \in \mathbf{l}^2(\Omega)$, $F \in \mathrm{L}^2(\Omega)$. Then ${}^t\mathbf{k} * \mathbf{f}$ defined by

$${}^t\mathbf{k} * \mathbf{f}(\xi) = \int_\Omega {}^t\mathbf{k}(\eta,\xi)\cdot\mathbf{f}(\eta)\, d\omega(\eta) \quad (9.37)$$

is called the *convolution of* ${}^t\mathbf{k}$ *against* $\mathbf{f}$. Moreover, ${}^t\mathbf{k}^{(i,k)} \star F$, $i,k \in \{1,2,3\}$, given by

$${}^t\mathbf{k}^{(i,k)} \star F(\xi) = \int_\Omega {}^t\mathbf{k}^{(i,k)}(\xi,\eta)F(\eta)\, d\omega(\eta) \quad (9.38)$$

is called the *convolution of* $\mathbf{k}^{(i,k)}$ *against* F.

For brevity, we write

$${}^t\hat{\mathbf{k}} \star ({}^t\mathbf{k} * \mathbf{f}) = \sum_{i=1}^3\sum_{k=1}^3 {}^t\hat{\mathbf{k}}^{(i,k)} \star \left({}^t\mathbf{k}^{(i,k)} * \mathbf{f}\right). \quad (9.39)$$

Since it is not difficult to see that

$${}^t\mathbf{h}^{(i,k)} \star \left({}^t\mathbf{k}^{(i,k)} * \mathbf{f}\right) = \int_\Omega\int_\Omega {}^t\mathbf{h}^{(i,k)}(\cdot,\xi)\otimes {}^t\mathbf{k}^{(i,k)}(\eta,\xi)\mathbf{f}(\eta)d\omega(\eta)\, d\omega(\xi), \quad (9.40)$$

we are finally led to the following setting.

Definition 9.16. Let ${}^t\mathbf{h}^{(i,k)}$, ${}^t\mathbf{k}^{(i,k)}$ be two $\mathbf{l}^2_{(i,k)}(\Omega)$–(tensorial) zonal rank–2 tensor kernel functions. Then, we define the *convolution of* ${}^t\mathbf{h}^{(i,k)}$ *against* ${}^t\mathbf{k}^{(i,k)}$ by

$${}^t\mathbf{h}^{(i,k)} \star {}^t\mathbf{k}^{(i,k)}(\xi,\eta) = \int_\Omega {}^t\mathbf{h}^{(i,k)}(\xi,\zeta)\otimes {}^t\mathbf{k}^{(i,k)}(\eta,\zeta)\, d\omega(\zeta). \quad (9.41)$$

Moreover, ${}^t\mathbf{h} \star {}^t\mathbf{k}$ is given by

$${}^t\mathbf{h} \star {}^t\mathbf{k} = \sum_{i=1}^3\sum_{k=1}^3 {}^t\mathbf{h}^{(i,k)} \star {}^t\mathbf{k}^{(i,k)}. \quad (9.42)$$

Collecting our material on the rank-2 tensor context, we are led to formulate the following result.

Theorem 9.17. *Let* ${}^t\mathbf{h}^{(i,k)}$, ${}^t\mathbf{k}^{(i,k)}$ *be two* $\mathbf{l}^2_{(i,k)}(\Omega)$*-(tensorial) zonal rank-2 tensor kernel functions. Then the convolution* ${}^t\mathbf{h}^{(i,k)} \star {}^t\mathbf{k}^{(i,k)}$ *is an* $\mathbf{l}^2_{(i)}(\Omega)$*-tensorial zonal rank-4 tensor kernel function, and we have*

$$ {}^t\mathbf{h}^{(i,k)} \star {}^t\mathbf{k}^{(i,k)}(\xi,\cdot) = \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} ({}^t\mathbf{h}^{(i,k)})^\wedge(n)({}^t\mathbf{k}^{(i,k)})^\wedge(n) \frac{2n+1}{4\pi} \mathbf{P}_n^{(i,k,i,k)}(\xi,\cdot). \tag{9.43} $$

By observing the property (9.43) we are able to deduce that, for every $\xi \in \Omega$, $\mathbf{h}^{(i,k)} \star \mathbf{k}^{(i,k)}(\xi,\cdot)$ is continuous on Ω.

Theorem 9.18. *Let* $\mathbf{f}$ *of class* $\mathbf{l}^2(\Omega)$*. Suppose that* ${}^t\mathbf{h}$ *and* ${}^t\mathbf{k}$ *are* $\mathbf{l}^2_{(i,k)}(\Omega)$*-zonal rank-2 tensor kernel functions, whereas* $\tilde{\mathbf{K}}$, $\mathbf{K}$ *are* $\mathbf{l}^2_{(i,k)}(\Omega)$*-zonal rank-4 tensor kernel functions satisfying*

$$ (\mathbf{H}^{(i,k)})^\wedge(n) = ({}^t\mathbf{h}^{(i,k)})^\wedge(n), \tag{9.44} $$

and

$$ (\mathbf{K}^{(i,k)})^\wedge(n) = (\mathbf{k}^{(i,k)})^\wedge(n), \tag{9.45} $$

for all $i,k \in \{1,2,3\}$, $n \geq 0_{ik}$*. Then*

$$ \mathbf{H} * \mathbf{K} * \mathbf{f} = {}^t\mathbf{h} \star {}^t\mathbf{k} * \mathbf{f}. \tag{9.46} $$

Proof. Observing the Legendre series expansion of $\mathbf{f}$ and of the zonal kernel functions, the addition theorem, and the orthogonality of the spherical harmonics, we get for the left hand side

$$ \begin{aligned} &\mathbf{H} * \mathbf{K} * \mathbf{f} \\ &= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} (\mathbf{H}^{(i,k)})^\wedge(n)(\mathbf{K}^{(i,k)})^\wedge(n) \sum_{m=1}^{2n+1} (\mathbf{f}^{(i,k)})^\wedge(n,m)\, \mathbf{y}_{n,m}^{(i,k)}. \end{aligned} \tag{9.47} $$

For the right hand side, we find

$$ \begin{aligned} {}^t\mathbf{h} \star {}^t\mathbf{k} * \mathbf{f} &= \sum_{i,k=1}^{3} {}^t\mathbf{h}^{(i,k)} \star {}^t\mathbf{k}^{(i,k)} * \mathbf{f} \\ &= \sum_{i,k=1}^{3} \int_\Omega \int_\Omega {}^t\mathbf{h}^{(i,k)}(\cdot,\zeta)\, {}^t\mathbf{k}^{(i,k)}(\eta,\zeta) \cdot \mathbf{f}(\eta)\, d\omega(\eta)\, d\omega(\zeta) \\ &= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} {}^t\mathbf{h}^{(i,k)\wedge}(n)\, {}^t\mathbf{k}^{(i,k)\wedge}(n) \sum_{m=1}^{2n+1} \mathbf{f}^{(i,k)\wedge}(n,m)\, \mathbf{y}_{n,m}^{(i,k)}. \end{aligned} \tag{9.48} $$

In connection with (9.44) and (9.45), we obtain the desired result. □

9.5 Dirac Families of Zonal Tensor Kernel Functions

Starting once more from a scalar Dirac family $\{\Phi_\rho\}_{\rho\in(0,\infty)}$, we are able to construct a Dirac family $\{\mathbf{\Phi}_\rho\}_{\rho\in(0,\infty)}$ of zonal rank-4 tensorial kernel functions as follows.

$$\mathbf{\Phi}_\rho(\xi,\eta) = \sum_{i=1}^{3}\sum_{k=1}^{3} \mathbf{\Phi}_\rho^{(i,k,i,k)}(\xi,\eta), \quad \xi,\eta \in \Omega, \tag{9.49}$$

with

$$\mathbf{\Phi}_\rho^{(i,k,i,k)}(\xi,\eta) = \sum_{n=0_{ik}}^{\infty} (\Phi_\rho)^\wedge(n) \sum_{m=1}^{2n+1} \mathbf{y}_{n,m}^{(i,k)}(\xi) \otimes \mathbf{y}_{n,m}^{(i,k)}(\eta), \tag{9.50}$$

$\xi,\eta\in\Omega$. Correspondingly, a Dirac family $\{\boldsymbol{\varphi}_\rho\}_{\rho\in(0,\infty)}$ of zonal rank-2 tensor kernel functions reads as follows

$$^t\boldsymbol{\varphi}_\rho(\xi,\eta) = \sum_{i=1}^{3}\sum_{k=1}^{3} {^t\boldsymbol{\varphi}}_\rho^{(i,k)}(\xi,\eta) \tag{9.51}$$

with

$$^t\boldsymbol{\varphi}_\rho^{(i,k)}(\xi,\eta) = \sum_{n=0_{i,k}}^{\infty} (\Phi_\rho)^\wedge(n) \sum_{m=1}^{2n+1} Y_{n,m}(\xi)\mathbf{y}_{n,m}^{(i,k)}(\eta), \tag{9.52}$$

$\xi,\eta\in\Omega$.

From our consideration, it is clear that the following theorem holds true.

Theorem 9.19. *Let $\{\Phi_\rho\}_{\rho\in(0,\infty)}$ be a scalar function as defined by (9.49). Then*

$$\lim_{\rho\to 0} \|\mathbf{f} - \mathbf{\Phi}_\rho * \mathbf{f}\|_{\mathbf{l}^2(\Omega)} = 0, \tag{9.53}$$

$$\lim_{\rho\to 0} \|\mathbf{f} - \mathbf{\Phi}_\rho * \mathbf{\Phi}_\rho * \mathbf{f}\|_{\mathbf{l}^2(\Omega)} = 0 \tag{9.54}$$

and

$$\lim_{\rho\to 0} \|\mathbf{f} - \boldsymbol{\varphi}_\rho \star \boldsymbol{\varphi}_\rho * \mathbf{f}\|_{\mathbf{l}^2(\Omega)} = 0. \tag{9.55}$$

Theorem 9.19 extends the notion of an approximate identity to tensor spherical fields.

Obviously,

$$\mathbf{\Phi}_\rho^{(1,1,1,1)}(\xi,\eta) = o_\xi^{(1,1)} o_\eta^{(1,1)} \Phi_\rho(\xi\cdot\eta). \tag{9.56}$$

$\xi, \eta \in \Omega$. For $i \in \{2,3\}$ we have

$$\mathbf{\Phi}_{\rho}^{(i,i,i,i)}(\xi,\eta) = \frac{1}{2} o_{\xi}^{(i,i)} o_{\eta}^{(i,i)} \Phi_{\rho}(\xi \cdot \eta), \tag{9.57}$$

$\xi, \eta \in \Omega$, while, for $(i,k) \in \{(1,2),(1,3),(2,1),(3,1)\}$,

$$\mathbf{\Phi}_{\rho}^{(i,k,i,k)}(\xi,\eta) = -o_{\xi}^{(i,k)} o_{\eta}^{(i,k)} \int_{\Omega} G(\Delta^*; \xi \cdot \zeta) \Phi_{\rho}(\zeta \cdot \eta) \; d\omega(\zeta), \tag{9.58}$$

$\xi, \eta \in \Omega$. Finally, for $(i,k) \in \{(2,3),(3,2)\}$,

$$\mathbf{\Phi}_{\rho}^{(i,k,i,k)}(\xi,\eta) = \frac{1}{n(n+1)(n(n+1)-2)} \\ o_{\xi}^{(i,k)} o_{\eta}^{(i,k)} \int_{\Omega} G(\Delta^*(\Delta^*-2); \xi \cdot \zeta) \Phi_{\rho}(\zeta \cdot \eta) \; d\omega(\zeta), \tag{9.59}$$

$\xi, \eta \in \Omega$.

9.6 Bibliographical Notes

The generalization of zonal kernel functions to the tensor context as proposed here is essentially based on M. Schreiner (1994), W. Freeden et al. (1994), W. Freeden et al. (1998), and H. Nutz (2002).

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

10 Zonal Function Modeling of Earth's Mass Distribution

There is a growing public concern about the future of our planet, its climate, its environment and about expected shortage of natural resources. Any consistent and efficient strategy of protection against these threats depends on a profound understanding of the Earth system. In particular, the knowledge of the Earth mass distribution is of crucial importance for the exploration of processes driving deformation of the Earth surface and influencing ocean surface topography. Closely interrelated with mass transport and mass anomalies is the Earth's gravity field and its constituting ingredients (see Table 10.1).

Table 10.1: Scientific uses of gravity field observables.

Solid Earth	**Oceanography**	**Glaciology**	**Geodesy**	**Climate**
Crustal density	Dynamic topography	Bedrock topography	Leveling (GPS)	Sea level changes
Post glacial rebound	Heat transport	Flux	Height systems	Coastal zones
	Mass transport		Orbit determination	

In what follows, we deal with a *spherical approach* to the so-called gravity quantities, i.e., the geomathematically relevant functions on the sphere characterizing the observables of the Earth's gravity potential. Spherical harmonics and zonal kernel functions are shown to be the essential tools for the determination of mass anomalies and mass distribution between essential Earth system components, viz. gravity field, elastic field and oceanic flow field. Our particular interest in this chapter is a systematic framework of the gravity observables by the principles of spectral theory in terms of spherical harmonics. Moreover, all representers of the observables can be described by convolution against zonal kernels.

It should be pointed out that our framework is not constructed in such a way as to consist only of scalar ingredients. Indeed, two different choices are viable, namely either as composition by scalar but anisotropic components of the vectorial and tensorial building elements, or as composition by isotropic vectorial and/or tensorial building blocks in their original nature. Clearly, this work is concerned with the structural advantages of the second variant (e.g., orthogonal invariance of fields and isotropy of operators) avoiding decompositions into component ingredients thereby knowing that vectorial/tensorial constituting elements are simpler in structure but larger in dimension.

10.1 Key Observables

If the Earth had a perfectly spherical shape and if the mass inside the Earth were distributed homogeneously or rotationally symmetric, then the line along which a test mass fell would be a straight line, directed radially and going exactly through the Earth's center of mass. The gravitational field obtained in this way would be spherically symmetric. In reality, however, the situation is more complex. The topographic features, mountains and valleys, are very irregular. The actual gravitational field is influenced by strong irregularities in density within the Earth. As a result, the gravitational force deviates from one place to the other from that of a homogeneous sphere.

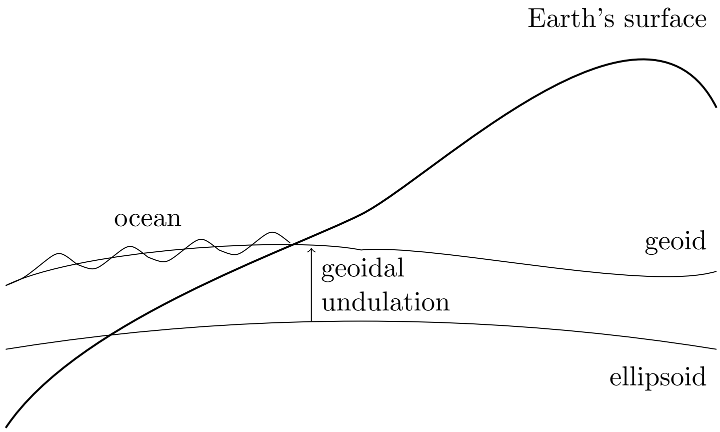


Fig. 10.1: Earth's surface, geoid, ellipsoid

The knowledge of the gravitational field of the global Earth is of great importance for many applications from which we only mention a few significant examples, for example, geodesy, civil engineering, solid Earth physics, oceanography. A particular role is played for aspects of global 'climate change' in the Earth system: Indeed, there is a growing awareness of global environmental problems (e.g., the CO_2-question, the rapid decrease of rain forests, global sea level changes, etc.). What is the role of the future airborne methods and satellite missions in this context? They do not tell us the reasons for physical processes, but it is essential to bring the phenomena into one system (e.g., to make sea level records comparable in different parts of the world). In other words, equipotential surfaces such as the geoid (see Figs. 10.1 and 10.2) are viewed as an almost static reference for many rapidly changing processes and at the same time as a 'frozen picture' of tectonic processes that evolved over geological time spans.

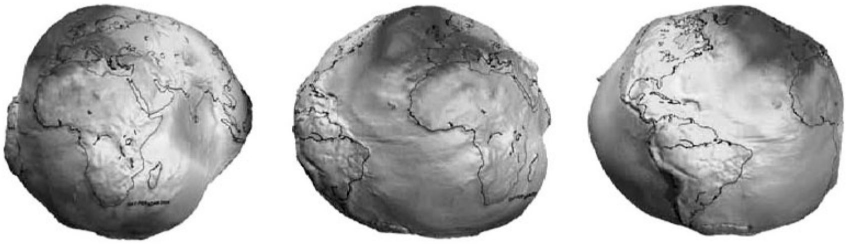


Fig. 10.2: Geoidal surface (GFZ-EIGEN-CG01C geoid (2005)).

Indeed, the gravity field plays a peculiar dual role in Earth sciences. On the one hand, by comparing the actual field with that of an idealized Earth body (e.g., an idealized Earth in hydrostatic equilibrium) their deviations, called *gravity anomalies*, are derivable. The gravity anomalies indicate the state of mass imbalance in the Earth's interior. On the other hand, the *geoid*, i.e., the equipotential surface at (mean) sea-level of a hypothetical ocean at rest, serves as the reference surface for all topographical features (for more details see, e.g., ESA (1999)).

Internal density signatures of the Earth are reflected by gravitational field signatures, and gravitational field signatures smooth out exponentially with increasing distance from the Earth's body. As a consequence, positioning systems are ideally located as far as possible from the Earth, whereas gravity field sensors are ideally located as close as possible to the Earth. Following these basic principles, various positioning and gravity field determination techniques have been designed. Sensors may be sensitive to local or global features of the gravity field. Considering the spatial location of the data, we may differentiate between terrestrial (surface), airborne, and spaceborne methods. Regarding the data type we have various measurement principles of the gravity field leading to different types of data:

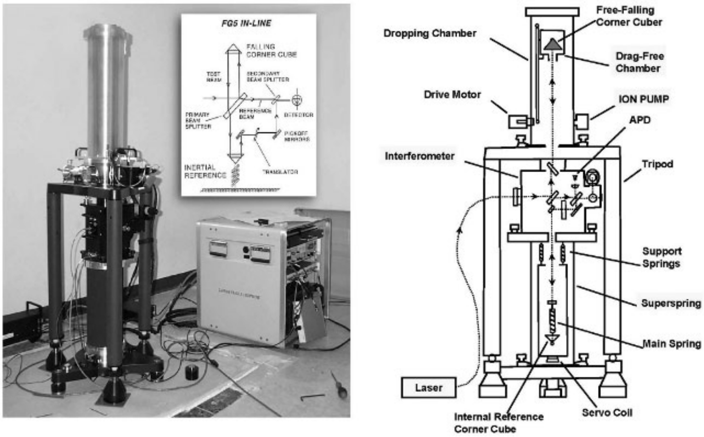


Fig. 10.3: Absolute gravimeter

To be more precise, the *force of gravity* provides a directional structure to the space above the Earth's surface. It is tangential to the vertical plumb lines and perpendicular to all (level) equipotential surfaces (see also Fig. 10.12). Any water surface at rest is part of a level surface. Level (equipotential) surfaces are ideal reference surfaces, for example, for heights. As already mentioned, the geoid is defined as that level surface of the gravity field which best fits the mean sea level. Gravity can be measured by absolute or relative gravimeters.

Absolute gravimeters are based directly on measuring the acceleration of free fall (e.g., of a test mass in a vacuum tube (see Fig. 10.3, right). Most common relative gravimeters are spring-based (see Fig. 10.4). By determining the amount by which the weight stretches the spring, gravity becomes available. The highest accuracy relative gravity measurements are conducted at the Earth's surface. Measurements on ships and in aircraft deliver reasonably good data only after the removal of inertial noise. Gravity data can be converted into gravity anomalies by subtracting a corresponding reference potential derived from a simple gravity field model associated to an, e.g., ellipsoidal surface. Gravity anomalies are furthermore converted into mean gravity anomalies by a proper averaging process over well defined areas. In future, *gravity disturbances* will become more important than gravity anomalies, because the Global Positioning System (GPS) determines the ellipsoidal coordinates directly at the surface point, so that the gravity disturbances can be considered observational data instead of the gravity anomalies. Classical spirit leveling measuring (via the height difference between two points) *potential differences* is a very time-consuming procedure. GPS leveling has introduced a revolution here. If the *ellipsoidal height* (above the reference ellipsoid) is measured by GPS, and if there exists a reliable geoidal map, then the so-called *orthometric height* (above the geoid) can be obtained immediately. In other words, geocentric positions can be determined in a purely geometric way.

The direction of the gravity vector can be obtained by astronomical positioning. Measurements are possible on the Earth's surface only. Observations of the gravity vector are converted into so-called *vertical deflections* by subtracting a corresponding reference direction derived from a simple gravity field model, e.g., associated to an ellipsoidal surface. Vertical deflections are surface-curl free tangential fields generated by the surface gradient applied to the disturbing potential (in a spherical Earth model). Due to the high measurement effort required to acquire these types of data compared to a gravity measurement, the data density of vertical deflections is much less than that of gravity anomalies. Gravitational field determination based on the observation of vertical deflections and combined with gravity is feasible in smaller areas with good data coverage.

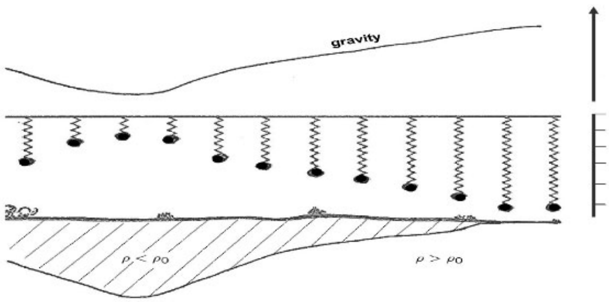


Fig. 10.4: The principle of a relative gravimeter

Concerning gravity, however, it should be pointed out that the terrestrial distribution of Earth's gravity data on a global scale is far from being homogeneous with large gaps, in particular over oceans but also over land. In addition, the quality of the data is very distinct. Thus, terrestrial gravity data coverage now and in the foreseeable future is far from being satisfactory. This is the reason why spaceborne measurements have to come into play.

Airborne gravimetry is a highly sensitive detection method of the gravitational potential of the Earth by a gravity accelerometer. Proposals to implement airborne gravimetry go back to the late fifties of the last century, and first flight experiments were already done in the early 1960s. A major obstacle of such techniques at that time was the inaccuracy of navigational information (e.g., velocity and acceleration of the space vehicle) which is needed to obtain the desired precision. Although at an appropriate level of accuracy, airborne gravimetry is vastly superior in economy and efficiency to pointwise terrestrial methods, there were serious doubts in the seventies and eighties of ever achieving useful results. In the early 1990s, however, great advances in GPS technology opened new ways to resolve the navigational problems. More explicitly, attitude, position, and velocity of the airborne gravity system become sufficiently computable from the inertial measurements updated by GPS carrier phase and Doppler observations for GPS leveling). Vehicle accelerations are derivable from GPS data only, so that in a third step, the airborne gravity disturbance is determinable from the difference between the force vector and the GPS-derived acceleration vector. Nowadays, some industrial companies are perfecting their system concepts by paying careful attention to the operational conditions under which an airborne gravimeter works best. Major advances in airborne gravimetry will be expected in the coming years.

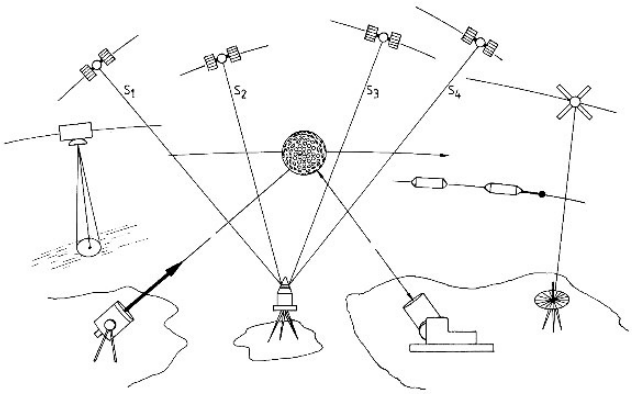


Fig. 10.5: Essential satellite methods (due to G. Seeber (1984)).

A variety of observational techniques exploiting satellites from the ground (see Fig. 10.5) have been used to determine the Earth's geoid. Two of them provide essential inputs to the recent elaboration of global gravity models: *Satellite laser ranging* (SLR) delivers the distance of a satellite from a ground station with accuracy depending on the quality of the SLR station. Worldwide, there exist a large number of operational SLR systems. SLR data contain information about the orbit of the satellite, the position of the measurement site and Earth's rotation and plate tectonic parameters. It remains today the most accurate technique (in the absolute sense) to which others can be compared and calibrated. *Range rate measurements* are based on the observation of the Doppler effect by which the frequency of a transmitted signal is observed with a modified value proportional to the line-of-sight velocity between the transmitter and the observer. Plenty of such measurements have been collected between satellite borne transmitters and ground stations but to a limited precision. The system can also be inverted, where transmitters are at the stations and the receiver is onboard a satellite.

Satellite radar altimetry has demonstrated an impressive capability of mapping the surface of the oceans. As already pointed out, the ocean surface is a good approximation of an equipotential surface and, as such, its offset from the geoid at mean sea level (mean in terms of time) is called sea surface topography. This offset reflects many effects including the variables salinity, ocean temperature, ocean currents, variable atmospheric conditions such as wind and air pressure perturbations, tides, etc. Since the sea surface topography refers to the geoid, the precise and sufficiently detailed knowledge of the geoid is mandatory. In a geostrophic approach (divergence-free), surface flow and sea surface topography are related by virtue of the surface curl gradient. In fact, satellite altimetry has revolutionized the understanding of ocean variability and dynamics.

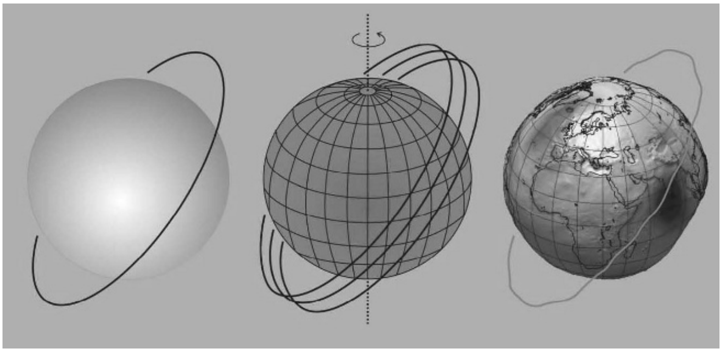


Fig. 10.6: Orbit illustration: Homogeneous spherical Earth's model (*right*) and space fixed ellipse, ellipsoidal Earth's model and spirals, real Earth (*left*) and modulation by the gravity signal (due to R. Rummel, IAPG Munich).

The three satellite concepts under present operation are satellite-to-satellite tracking in the high-low mode (SST hi-lo), satellite-to-satellite tracking in the low-low mode (SST lo-lo), and satellite gravity gradiometry (SGG). Representatives of these three concepts (see Figs. 10.7 and 10.8) are CHAMP (SST hi-lo), GRACE (SST lo-lo combined with SST hi-lo), GOCE (SGG combined with SST hi-lo). Common to all three concepts is that the determination of the Earth's gravity field is based on the measurement of the relative motion (in the Earth's gravity field) of test masses.

The concept of *satellite-to-satellite tracking* (SST) goes back almost three decades. The original idea was to fly two satellites in an identical low orbit with a separation of a few hundred kilometers between the spacecrafts

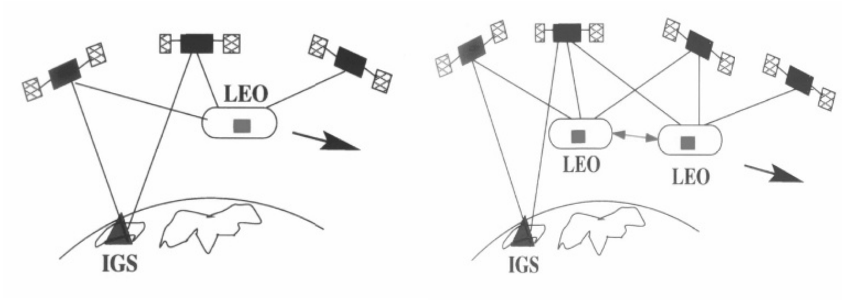


Fig. 10.7: The CHAMP concept (*left*) and the GRACE concept (*right*)(cf. ESA (1998)).

(low–low SST). Between the satellites, the distance and the Doppler frequency shift can be measured. As such, the data represent admittedly, to some degree of approximation, first order tangential derivatives of the gravitational potential. The alternative to low-low SST is high-low SST: Nowadays GPS is fully operational with a number of satellites in space which can track a Low Earth Orbiter (LEO). From continuous carrier phase measurements of all visible GPS satellites, the orbit can be determined to an accuracy of a few centimeters (cf. Fig. 10.6). Such data, when collected by a dedicated gravity field satellite over a period of several months, can deliver estimates of the long wavelength part of the global gravity field, represented by the geoid.

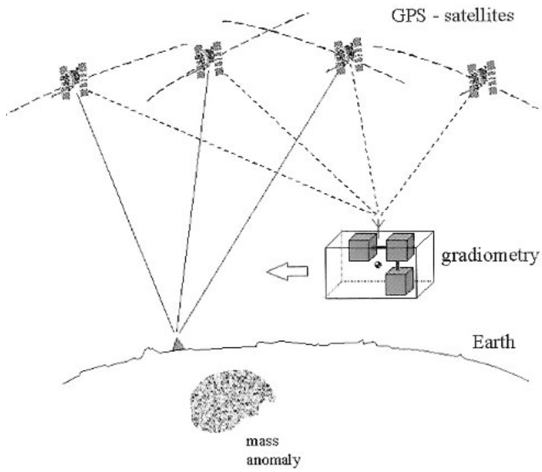


Fig. 10.8: The GOCE concept (cf. ESA (1998)).

In the case of SST hi-lo, the low flying test mass is a low earth orbiter (LEO) and the high flying test masses are the satellites of the Global Positioning System (GPS). As the GPS receiver mounted on the LEO always 'contacts' four or even more of the GPS satellites, the relative motion of the LEO can be monitored three-dimensionally, i.e. in all three coordinate directions. The lower the orbit of the LEO, the higher is its sensitivity with respect to the spatial variations of the gravitational forces but by skin forces as well (atmospheric drag, solar radiation, albedo, etc.), the latter have either to be compensated for by a drag-free mechanism or, as for CHAMP, be measured by a three axis accelerometer. Also the high orbiters, the GPS satellites, are affected by non-gravitational forces. However, the latter can be modeled quite well. They affect mainly the very long spatial scales, and to a large extent, their effect averages out. In addition, the ephemerides of the GPS satellites are determined very accurately by the large network of ground stations that constitute the International Geodynamic Service (IGS). In the case of SST lo-lo, the relative motion between two LEO's, chasing each other, is measured with highest precision. The quantity of interest is the relative motion of the center of mass of the two satellites. Again, the effect of non-gravitational forces on the two spacecrafts either has to be compensated actively or measured (GRACE). Over land, it is for the first time demonstrated with GRACE, that satellites are able to globally probe the Earth for largely unknown soil moisture and aquifer changes on seasonal and interannual time scales. Being important for the understanding of the global water cycle, a GRACE-based system shall continue to trace global hydrology.

Satellite gravity gradiometry (SGG) is a technique of measuring the relative acceleration, not between free falling test masses like satellites, but of measuring test masses at different locations inside one satellite (see Fig. 10.8). Each test mass is enclosed in a housing and kept levitated (floating, without ever touching the walls) by a capacitive or inductive feedback mechanism. The difference in feedback signals between two test masses is proportional to their relative acceleration and exerted purely by the differential gravitational field. Non-gravitational acceleration of the spacecraft affects all accelerometers inside the satellite in the same manner and so ideally drops out during differencing. The rotational motion of the satellite affects the measured differences. However, the rotational signal (angular velocities and accelerations) can be separated from the gravitational signal, if acceleration differences are taken in all possible (spatial) combinations (= full tensor gradiometer). Again, low orbit means high sensitivity. The GOCE mission (see Fig. 10.8) opens a completely new range of spatial scales to research.

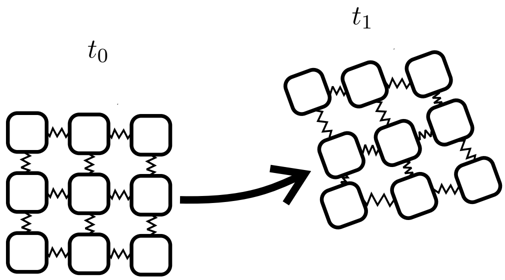


Fig. 10.9: The principle of a gradiometer.

One can argue that the basic observable in all three cases (SST hi-lo, SST lo-lo, SGG) is the gravitational acceleration. In the case of SST hi-lo, with the motion of the high orbiting GPS satellites assumed to be perfectly known, this corresponds to an in situ 3-D acceleration measurement in the LEO. For the case of SST lo-lo, it is the measurement of acceleration difference over the intersatellite distance and in the line-of-sight (LOS) of the LEOs. Finally, in the case of gradiometry, it is the measurement of acceleration differences in 3-D over the tiny baseline of the gradiometer. In short, we are confronted with the following situation:

SST hi–lo:	3–D acceleration	= gravitational gradient,
SST lo–lo:	acceleration difference	= difference in gradient,
SGG:	differential	= gradient of gradient ('tensor').

Thus, in the mathematical sense, it is a transition from the first derivative of the gravitational potential via a difference in the first derivative to the second derivative. The guiding parameter that determines sensitivity with respect to the spatial scales of the Earth's gravitational potential is the distance between the test masses, being almost infinity for SST hi-lo and almost zero for gradiometry (cf. Fig. 10.9).

Summarizing our introductory remarks on gravity quantities, we come to following conclusion: Over the years, geoscientists have realized the great complexity of the Earth and its environment. In particular, the knowledge of the gravity potential and its level (equipotential) surfaces giving information about mass distribution and mass transport in the Earth's system has become an important issue. In this respect, the gravity field is the key component of future investigation. Seen from numerical point of view, it must be remarked for future work that combining data from different sensors and sources is the way forward. Only coordinated research between geodesy, geophysics, and geomathematics will provide a breakthrough in understanding and modeling of important processes in the Earth system.

An overview of gravitational quantities (GQ) involved in the modeling of Earth's mass distribution is given in Table 10.2.

Table 10.2: Gravity quantities (actual situation) for determining the Earth's gravitational potential.

GPS–leveling ($\rightarrow$ positions x, y, heights N, H)	High altitude
Conventional satellite techniques Laser, Doppler, etc. ($\rightarrow$ positions $x, y, x \pm y$) satellite altimetry ($\rightarrow$ dynamic ocean topography $\Xi(x)$, gravitational potential $V(x)$ at ocean positions x)	Medium altitude
Satellite-to-satellite tracking (high-low) ($\rightarrow$ gravitational gradient $\nabla V(x)$ at satellite positions x)	Medium altitude
Satellite-to-satellite tracking (low-low) ($\rightarrow$ difference $\nabla V(x) - \nabla V(y)$ of gradients at satellite positions x, y)	Medium altitude
Satellite–gravity–gradiometry ($\rightarrow$ gravitational tensor $\nabla^{(2)} V(x)$ at satellite positions x)	Low altitude
Gravimetry, astrogeodesy ($\rightarrow$ gravity anomalies $A(x)$, gravity disturbances $D(x)$, vertical deflections $\Theta(x)$, gravitational magnitude $\lvert\nabla V(x)\rvert$, gravitational direction $\nabla V(x)/\lvert\nabla V(x)\rvert$, torsion balance $\nabla^{(2)} V(x)$)	Ground level

10.2 Gravity Potential

Gravity, as observed on the Earth's surface, is the combined effect of the gravitational mass attraction and the centrifugal force due to the Earth's rotation. The force of gravity provides a directional structure to the space above the Earth's surface. It is tangential to the vertical plumb lines and perpendicular to all level surfaces. Any water surface at rest is part of a level surface. As if the Earth were a homogeneous, spherical body, gravity turns out to be constant all over the Earth's surface, the well-known quantity 9.8 ms^{-2}. The plumb lines are directed toward the Earth's center of mass, and this implies that all level surfaces are nearly spherical, too.

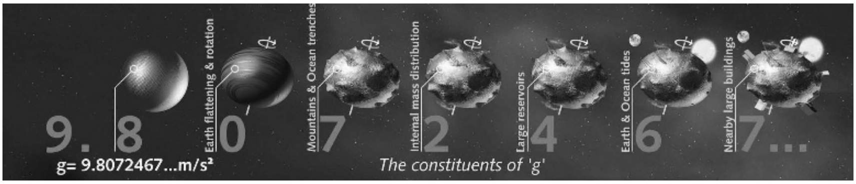


Fig. 10.10: Illustration of the components of the gravity acceleration (ESA medialab, ESA communication production SP–1314)

First, the gravity decreases from the poles to the equator by about 0.05 ms^{-2} (see Fig. 10.10). This is caused by the flattening of the Earth's figure and the negative effect of the centrifugal force, which is maximal at the equator. Second, high mountains and deep ocean trenches cause the gravity to vary. Third, materials within the Earth's interior are not uniformly distributed. The irregular gravity field shapes as virtual surface, the geoid. The level surfaces ideal reference surfaces, for example, for heights.

In more detail, the *gravity acceleration (gravity)* w is the resultant of gravitation v and centrifugal acceleration c :

$$w = v + c. \tag{10.1}$$

The centrifugal force c arises as a result of the rotation of the Earth about its axis. We assume here a rotation of constant angular velocity ω_0 about the rotational axis x_3, which is further assumed to be fixed with respect to the Earth. The centrifugal acceleration acting on a unit mass is directed outward perpendicular to the spin axis (see Fig. 10.11).

If the ε^3-axis of an Earth-fixed coordinate system coincides with the axis of rotation, then we have

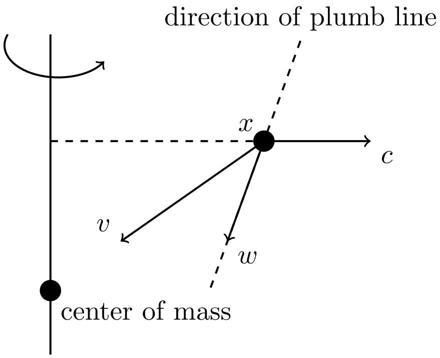


Fig. 10.11: Gravitation v, centrifugal acceleration c, gravity acceleration w.

$$c(x) = -\omega_0^2 \varepsilon^3 \wedge (\varepsilon^3 \wedge x). \tag{10.2}$$

Using the so–called *centrifugal potential*

$$C(x) = \frac{\omega_0^2}{2} |\varepsilon^3 \wedge (\varepsilon^3 \wedge x)| = \frac{\omega_0^2}{2} \left((x \cdot \varepsilon^1)^2 + (x \cdot \varepsilon^2)^2 \right) = \frac{\omega_0^2}{2} (x_1^2 + x_2^2) \tag{10.3}$$

we can write $c = \nabla C$. Applying the Laplace operator gives us $\Delta C = 2\omega_0^2$, thus, the function C is *not* harmonic.

The direction of the gravity w is known as the direction of the *plumb line*, the quantity $|w|$ is called the *gravity intensity* (often just *gravity*). The *gravity potential of the Earth* can be expressed in the form:

$$W = V + C. \tag{10.4}$$

The gravity acceleration w is given by

$$w = \nabla W = \nabla V + \nabla C. \tag{10.5}$$

The surfaces of constant gravity potential $W(x) = \text{const}$, $x \in \mathbb{R}^3$, are designated as *equipotential (level,) or geopotential surfaces of gravity* (for more details see, e.g., E. Groten (1979), W.A. Heiskanen, H. Moritz (1967), W. Torge (1991)).

The *gravity potential* W of the Earth is the sum of the *gravitational potential* V and the *centrifugal potential* C, i.e., $W = V + C$. In the Earth's fixed coordinate system, the centrifugal potential C is explicitly known. Hence, the determination of equipotential surfaces of the potential W is strongly related to the knowledge of the potential V. The gravity vector w given by $w(x) = \nabla_x W(x)$ where the point $x \in \mathbb{R}^3$ is located outside and on a sphere around the origin with Earth's radius R (see Fig. 10.12), is normal to the equipotential surface passing through the same point (for the specification of the (mean) Earth's radius R see, e.g., E. Groten (1979), W.A. Heiskanen, H. Moritz (1967), W. Torge (1991)). Thus, equipotential surfaces intuitively express the notion of tangential surfaces, as they are normal to the plumb lines given by the direction of the gravity vector.

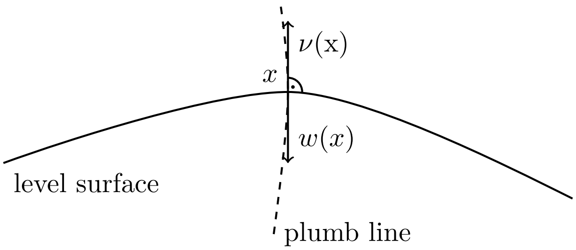


Fig. 10.12: Level surface and plumb line.

According to the classical Newton Law of Gravitation (1687), knowing the density distribution F of a body, the gravitational potential can be computed everywhere in $\mathbb{R}^3$. More explicitly, the gravitational potential V of the Earth's exterior is given by

$$V(x) = G \int_{\text{Earth}} \frac{F(y)}{|x-y|} \, dV(y), \qquad x \in \mathbb{R}^3 \backslash \text{Earth}, \tag{10.6}$$

where G is the gravitational constant ($G = 6.6742 \cdot 10^{-11} \text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$).

The properties of the gravitational potential (10.6) in the Earth's exterior are easily described as follows:

$$\Delta V(x) = 0, \; x \in \mathbb{R}^3 \backslash \text{Earth}. \tag{10.7}$$

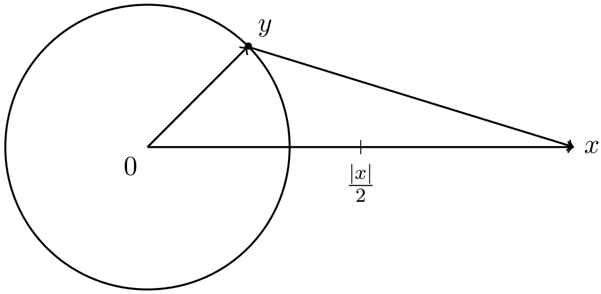


Fig. 10.13: Regularity at infinity.

Moreover, the gravitational potential V is *regular at infinity*, i.e.,

$$|V(x)| = O\left(\frac{1}{|x|}\right) \quad , \quad |x| \to \infty, \tag{10.8}$$

$$|\nabla V(x)| = O\left(\frac{1}{|x|^2}\right) \quad , \quad |x| \to \infty. \tag{10.9}$$

Note that, for suitably large values $|x|$ (see Fig. 10.13), we have $|y| \leq \frac{1}{2}|x|$, hence, $|x - y| \geq ||x| - |y|| \geq \frac{1}{2}|x|$.

Clearly, the gravitational field $v = \nabla V$ fulfills the following identities:

$$\mathrm{L} \cdot \nabla V(x) = 0, \tag{10.10}$$

$$\nabla \cdot \nabla V(x) = \Delta V(x) = 0, \tag{10.11}$$

$x \in \mathbb{R}^3 \backslash \mathrm{Earth}$.

However, the problem is that in reality the density distribution is very irregular and known only for parts of the upper crust of the Earth. It is actually so that geoscientists would like to know it from measuring the gravitational field. Even if the Earth is supposed to be spherical, the determination of the gravitational potential by integrating Newton's potential is not achievable. This is the reason why, in spherical nomenclature, we first expand the gravitational potential of the spherical Earth $\overline{\Omega_R^{\mathrm{int}}}$ into a series of spherical harmonics. In doing so, we observe that the so-called reciprocal distance can be expressed as a Legendre series as follows:

$$\frac{1}{|x - y|} = \frac{1}{|x|} \sum_{n=0}^{\infty} \left(\frac{|y|}{|x|}\right)^n P_n\left(\frac{x}{|x|} \cdot \frac{y}{|x|}\right), \tag{10.12}$$

$y \in \overline{\Omega_R^{\mathrm{int}}}$, $x \in \Omega_R^{\mathrm{ext}}$, i.e., $|y| \leq R < |x|$.

Relating (10.12) to the radius R, we obtain

$$\frac{1}{|x-y|} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{4\pi R}{2n+1} H^R_{-n-1,k}(x) H^R_{n,k}(y), \tag{10.13}$$

where $H^R_{n,k}$ is an *inner harmonic* of degree n and order k given by

$$H^R_{n,k}(x) = \frac{1}{R}\left(\frac{|x|}{R}\right)^n Y_{n,k}(\xi), \quad x = |x|\xi, \xi \in \Omega, \tag{10.14}$$

and $H^R_{-n-1,k}$ is an *outer harmonic* of degree n and order k given by

$$H^R_{-n-1,k}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1} Y_{n,k}(\xi), \quad x = |x|\xi, \xi \in \Omega. \tag{10.15}$$

Note that $\{Y_{n,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ is an $\mathrm{L}^2(\Omega)$-orthonormal system of scalar spherical harmonics.

Insertion of the series expansion (10.13) into the Newton formula for the gravitational potential yields for $x \in \Omega^{\mathrm{ext}}_R$:

$$V(x) = G \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{4\pi R}{2n+1} \int_{\overline{\Omega^{\mathrm{int}}_R}} F(y)\ H^R_{n,k}(y)\ dV(y)\ H^R_{-n-1,k}(x). \tag{10.16}$$

At first sight, we might conclude that we end up with an infinite series of integrals, where we have only one integral in the beginning. However, the integrals involving inner/outer harmonics are regular, and a closer look at the individual terms reveals their geophysical relevance:

The zero term gives the potential with mass equal to that of the gravitating mass distribution of the spherical Earth's body $\overline{\Omega^{\mathrm{int}}_R}$. The first order term relates to dipole mass moments. The quadrupole moments obtained by the second order term reflect the oblateness of the mass distribution.

As already pointed out, the expansion coefficients of the series (10.16)

$$\frac{4\pi RG}{2n+1} \int_{\overline{\Omega^{\mathrm{int}}_R}} F(y)\ H^R_{n,k}(y)\ dV(y) \tag{10.17}$$

are not computable, since their determination requires the knowledge of the density function F in the Earth's interior $\overline{\Omega^{\mathrm{int}}_R}$. In fact, it turns out that there are infinitely many mass distributions, which have the given gravitational potential of the Earth as exterior potential. To overcome the difficulties, the solution of the (Dirichlet) boundary-value problem $\Delta V(x) = 0, \quad x \in \overline{\Omega_R^{\mathrm{int}}}$, corresponding to the boundary condition $V|\Omega_R \in \mathrm{C}(\Omega_R)$ would suffice for purposes of determining the exterior gravitational potential, in principle, from geophysical point of view, the expansion coefficients (10.17) can be expressed by the 'boundary function' $V|\Omega_R$. However, the comparison of the spherical harmonic coefficients leads to an infinite number of equations relating $V|\Omega_R$ on the (spherical) Earth's surface Ω_R to the density distribution F inside the (spherical) Earth Ω_R^{int}. In other words, the knowledge of the density function inside the Earth allows the Fourier (orthogonal) expansion in terms of the potential coefficients. Inversely, given the potential coefficients as derived from the terrestrial potential, $V|\Omega_R$ does not suffice to determine the density distribution. In geophysics, this ambiguity is known as the gravimetry problem of determining Earth's density distribution.

Table 10.3: Gravimetric units.

Physical quantity	**SI units**	**Traditional units**
Gravity	$10^{-2}\mathrm{ms}^{-2}$	1 Gal
Gravity	$10^{-5}\mathrm{ms}^{-2}$	1 mGal
Gravity	$10^{-8}\mathrm{ms}^{-2}$	1μGal
Gravity potential	$10^{8}\mathrm{m}^2\mathrm{s}^{-2}$	1kGal$\cdot$ m
Gravity gradients	$10^{-9}\mathrm{s}^{-2}$	1 E

Collecting the results on the Earth's gravitational field v for the outer space of the Earth (in spherical approximation, of course, Ω_R^{ext}), we are confronted with the following (mathematical) characterization: v is an infinitely often differentiable vector field in the exterior of the Earth such that

(v1)

$$\mathrm{div}\ v = \nabla \cdot v = 0, \qquad \mathrm{curl}\ v = \mathrm{L} \cdot v = 0 \tag{10.18}$$

in the Earth's exterior,

(v2) v is regular at infinity:

$$|v(x)| = O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty. \tag{10.19}$$

Seen from mathematical point of view, the properties (v1) and (v2) imply that the Earth's gravitational field v in the exterior of the Earth (see, e.g., O.D. Kellogg (1929), M.E. Gurtin (1971), A. Wangerin (1921)) is a gradient field

$$v = \nabla V, \tag{10.20}$$

where the gravitational potential V fulfills the properties: V is an infinitely often differentiable scalar field in the exterior of the Earth such that

(V1) V is harmonic in the Earth's exterior, i.e., $\Delta V = 0$,

(V2) V is regular at infinity, i.e.,

$$|V(x)| = O\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{10.21}$$

$$|\nabla V(x)| = O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty, \tag{10.22}$$

and vice versa.

Moreover, the gradient field of the Earth's gravitational field (i.e., the *Jacobi matrix field*)

$$\mathbf{v} = \nabla v, \tag{10.23}$$

obeys the following properties: $\mathbf{v}$ is an infinitely often differentiable tensor field in the exterior of the Earth such that

($\mathbf{v}$1)

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = \mathrm{L} \cdot \mathbf{v} = 0 \tag{10.24}$$

in the Earth's exterior,

($\mathbf{v}$2) $\mathbf{v}$ is regular at infinity:

$$|\mathbf{v}(x)| = O\left(\frac{1}{|x|^3}\right), \qquad |x| \to \infty \ , \tag{10.25}$$

and vice versa.

Combining (10.24) with (10.20), we see that $\mathbf{v}$ can be represented as the *Hesse tensor of the scalar field* V, i.e.,

$$\mathbf{v} = (\nabla \otimes \nabla)\, V = \nabla^{(2)} V. \tag{10.26}$$

10.3 Inner/Outer Harmonics

As preparation for the theory of boundary-value problems in terms of outer harmonics, some results known from potential theory should be recapitulated briefly. More explicitly, we are interested in essential ingredients of potential theory in their specific formulation for the outer space Ω_R^{ext} of the sphere around the origin with radius R.

Let $V : \Omega_R^{\text{ext}} \to \mathbb{R}$, $v : \Omega_R^{\text{ext}} \to \mathbb{R}^3$, and $\mathbf{v} : \Omega_R^{\text{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, be a scalar, vector, and tensor field on the set Ω_R^{ext}. We say that $V, v, \mathbf{v}$, respectively, are *harmonic* on Ω_R^{ext} if $V, v, \mathbf{v}$ are twice continuously differentiable on Ω_R^{ext} and $\Delta V = 0$, $\Delta v = 0$, $\Delta \mathbf{v} = 0$ on Ω_R^{ext}.

Without proof, we mention some well-known theorems concerning harmonic fields on Ω_R^{ext} (for the proofs see, for example, M.E. Gurtin (1971), O.D. Kellogg (1929)):

(1) Every harmonic field in Ω_R^{ext} is analytic in Ω_R^{ext}, i.e., every harmonic field is determined by its local properties .

(2) *Harnack's convergence theorem:* Let $V_\delta : \Omega_R^{\text{ext}} \to \mathbb{R}$, $v_\delta : \Omega_R^{\text{ext}} \to \mathbb{R}^3$, and $\mathbf{v}_\delta : \Omega_R^{\text{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, be harmonic on Ω_R^{ext} for each value δ $(0 < \delta < \delta_0)$, and regular at infinity. Moreover, let

$$\begin{aligned} V_\delta \to V \quad &, \quad \delta \to 0,\ \delta > 0, \\ v_\delta \to v \quad &, \quad \delta \to 0,\ \delta > 0, \\ \mathbf{v}_\delta \to \mathbf{v} \quad &, \quad \delta \to 0,\ \delta > 0, \end{aligned}$$

uniformly on each subset K of Ω_R^{ext} with $\operatorname{dist}(\overline{K}, \partial\Omega_R^{\text{ext}}) > 0$. Then $V : \Omega_R^{\text{ext}} \to \mathbb{R}$, $v : \Omega_R^{\text{ext}} \to \mathbb{R}^3$, and $\mathbf{v} : \Omega_R^{\text{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, is harmonic on Ω_R^{ext} and regular at infinity. Furthermore, for each fixed integer n

$$\begin{aligned} \nabla^{(n)} V_\delta &\to \nabla^{(n)} V \ ,\ \delta \to 0,\ \delta > 0, \\ \nabla^{(n)} v_\delta &\to \nabla^{(n)} v \ ,\ \delta \to 0,\ \delta > 0, \\ \nabla^{(n)} \mathbf{v}_\delta &\to \nabla^{(n)} \mathbf{v} \ ,\ \delta \to 0,\ \delta > 0, \end{aligned}$$

holds uniformly on each subset K of Ω_R^{ext} with $\operatorname{dist}(\overline{K}, \partial\Omega_R^{\text{ext}}) > 0$.

(3) Let $V : \overline{\Omega_R^{\text{ext}}} \to \mathbb{R}$ be twice continuously differentiable on Ω_R^{ext} and continuous on $\overline{\Omega_R^{\text{ext}}}$, i.e., $V \in \mathrm{C}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$, harmonic on Ω_R^{ext}, and regular at infinity. Then, the *maximum/minimum principle* tells us that

$$\sup_{x \in \overline{\Omega_R^{\text{ext}}}} |V(x)| \le \sup_{x \in \Omega_R} |V(x)| \quad . \tag{10.27}$$

(4) There is a so-called fundamental solution (singularity function) $S : x \mapsto |x-y|^{-1}$, $x \neq y$ with respect to the Laplace operator Δ such that *the fundamental theorem of potential theory* (see Theorem 2.4)

$$\int_{\partial\Omega_R^{\text{ext}}} \left(\frac{1}{|x-y|} \frac{\partial V}{\partial \nu}(y) - V(y) \frac{\partial}{\partial \nu_y} \frac{1}{|x-y|} \right) d\omega(y) \tag{10.28}$$
$$= \begin{cases} -4\pi V(x) & , \quad x \in \Omega_R^{\text{ext}}, \\ -2\pi V(x) & , \quad x \in \partial\Omega_R^{\text{ext}}, \\ 0 & , \quad x \notin \overline{\Omega_R^{\text{ext}}} \end{cases}$$

holds true.

Consider the sphere $\Omega_R \subset \mathbb{R}^3$ around the origin with radius $R > 0$. As usual, Ω_R^{int} is the inner space of Ω_R, and Ω_R^{ext} is the outer space. By virtue of the isomorphism $\Omega \ni \xi \mapsto R\xi \in \Omega_R$, we assume functions $F : \Omega_R \to \mathbb{R}$ to be defined on Ω. It is clear that the function spaces defined on Ω admit their natural generalizations as spaces of functions defined on Ω_R. We have, for example, $\mathrm{C}^{(\infty)}(\Omega_R)$, $\mathrm{L}^p(\Omega_R)$, etc. Obviously, an $\mathrm{L}^2(\Omega)$-orthonormal system of spherical harmonics forms an orthogonal system on Ω_R (with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Omega_R)}$). More explicitly, we have

$$(Y_{n,k}, Y_{p,q})_{\mathrm{L}^2(\Omega_R)} = \int_{\Omega_R} Y_{n,k}\left(\frac{x}{|x|}\right) Y_{p,q}\left(\frac{x}{|x|}\right) d\omega(x) = R^2 \delta_{np}\delta_{kq}. \tag{10.29}$$

With the relationship $\xi \leftrightarrow R\xi$, the *surface gradient* $\nabla^{*;R}$ and the *Beltrami operator* $\Delta^{*;R}$ on Ω_R, respectively, have the representation $\nabla^{*;R} = (1/R)\nabla^{*;1} = (1/R)\nabla^*$, $\Delta^{*;R} = (1/R^2)\Delta^{*;1} = (1/R^2)\Delta^*$, where ∇^*, Δ^* are the surface gradient and the Beltrami operator of the unit sphere Ω. For $Y_n \in \mathrm{Harm}_n(\Omega)$, we have $\Delta^{*;R} Y_n = (1/R^2)(\Delta^*)^\wedge(n) Y_n$.

We now introduce the system $\{Y_{n,k}^R\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ by letting

$$Y_{n,k}^R(x) = \frac{1}{R} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \Omega_R \ . \tag{10.30}$$

Due to (10.29), the system $\{Y_{n,k}^R\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ is an orthonormal basis in $\mathrm{L}^2(\Omega_R)$:

$$\mathrm{L}^2(\Omega_R) = \overline{\mathrm{span}_{\substack{n=0,1,\ldots, \\ k=1,\ldots,2n+1}} (Y_{n,k}^R)}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}} \ . \tag{10.31}$$

The system $\{H_{n,k}^R\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ of *inner harmonics* $H_{n,k}^R$ *of degree* n *and order* k can be written as

$$H_{n,k}^R(x) = \left(\frac{|x|}{R}\right)^n Y_{n,k}^R(x), \qquad x \in \mathbb{R}^3. \tag{10.32}$$

It satisfies the following properties:

- $H_{n,k}^R$ is of class $\mathrm{C}^{(\infty)}(\mathbb{R}^3)$
- $H_{n,k}^R$ satisfies Laplace's equation in $\mathbb{R}^3$:

$$\Delta_x H_{n,k}^R(x) = 0, \quad x \in \mathbb{R}^3$$

- $H_{n,k}^R|_{\Omega_R} = Y_{n,k}^R = \frac{1}{R} Y_{n,k}$
- $(H_{n,k}^R, H_{p,q}^R)_{\mathrm{L}^2(\Omega_R)} = \int_{\Omega_R} Y_{n,k}^R(x)\, Y_{p,q}^R(x)\; d\omega(x) = \; \delta_{np}\delta_{kq}$

(note that in the case of $\Omega_R = \Omega$, we have $H_{n,k}^R|_{R=1} = H_{n,k}^1 = Y_{n,k}$ for all $n = 0, 1, \ldots; \; k = 1, \ldots, 2n+1$).

From the addition theorem of spherical harmonics, we obtain

$$\sum_{k=1}^{2n+1} H_{n,k}^R(x) H_{n,k}^R(y) = \frac{2n+1}{4\pi R^2} \left(\frac{|x||y|}{R^2} \right)^n P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right) \tag{10.33}$$

for all $(x, y) \in \overline{\Omega_R^{\mathrm{int}}} \times \overline{\Omega_R^{\mathrm{int}}}$, which is known as the *addition theorem of inner harmonics* (see (3.26)).

In accordance with our notation, $\mathrm{Harm}_n(\overline{\Omega_R^{\mathrm{int}}})$ denotes the space of all inner harmonics of degree n on $\overline{\Omega_R^{\mathrm{int}}}$, i.e., $\mathrm{Harm}_n(\overline{\Omega_R^{\mathrm{int}}})$ is equal to the space of all linear combinations of the $2n+1$ elements $H_{n,1}^R, \ldots H_{n,2n+1}^R$. Consequently, $d(\mathrm{Harm}_n(\overline{\Omega_R^{\mathrm{int}}})) = 2n+1$. We let

$$\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}}) = \bigoplus_{n=p}^{q} \mathrm{Harm}_n(\overline{\Omega_R^{\mathrm{int}}}), \quad 0 \leq p \leq q \;. \tag{10.34}$$

The kernel $K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(\cdot,\cdot) : \overline{\Omega_R^{\mathrm{int}}} \times \overline{\Omega_R^{\mathrm{int}}} \to \mathbb{R}$ given by

$$K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(x, y) = \sum_{n=p}^{q} \sum_{k=1}^{2n+1} H_{n,k}^R(x) H_{n,k}^R(y), \;\; (x, y) \in \overline{\Omega_R^{\mathrm{int}}} \times \overline{\Omega_R^{\mathrm{int}}}, \tag{10.35}$$

is the reproducing kernel of the space $\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})$ with respect to $\|\cdot\|_{\mathrm{L}^2(\Omega_R)}$, i.e.:

(i) For every $y \in \overline{\Omega_R^{\mathrm{int}}}$, the functions $K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(y, \cdot)$ as well as $K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(\cdot, y)$ belong to $\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})$

(ii) For any $H \in \mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})$ and any $x \in \overline{\Omega_R^{\mathrm{int}}}$, the reproducing property

$$\begin{aligned} H(x) &= \left(K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(\cdot, x), H\right)_{\mathrm{L}^2(\Omega_R)} \\ &= \left(K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{int}}})}(x, \cdot), H\right)_{\mathrm{L}^2(\Omega_R)} \end{aligned}$$

holds true.

The system $\{H^R_{-n-1,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ of *outer harmonics* $H^R_{-n-1,k}$ *of degree* n *and order* k defined by

$$H^R_{-n-1,k}(x) = \left(\frac{R}{|x|}\right)^{n+1} Y^R_{n,k}(x), \qquad x \in \mathbb{R}^3 \backslash \{0\}, \tag{10.36}$$

satisfies the following properties:

- $H^R_{-n-1,k}$ is of class $\mathrm{C}^{(\infty)}(\mathbb{R}^3 \backslash \{0\})$
- $H^R_{-n-1,k}$ satisfies Laplace's equation in $\mathbb{R}^3 \backslash \{0\}$:

$$\Delta_x H^R_{-n-1,k}(x) = 0, \quad x \in \mathbb{R}^3 \backslash \{0\}$$

- H^R_{-n-1} is regular at infinity, i.e.

$$\left|H^R_{-n-1,k}(x)\right| = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty \tag{10.37}$$

and

$$\left|\nabla H^R_{-n-1,k}(x)\right| = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty$$

- $H^R_{-n-1,k}\Big|\Omega_R = Y^R_{n,k} = \frac{1}{R} Y_{n,k}$
- $\left(H^R_{-n-1,k}, H^R_{-p-1,q}\right)_{\mathrm{L}^2(\Omega_R)} = \delta_{np}\delta_{kq}$.

The addition theorem of spherical harmonics now yields

$$\sum_{k=1}^{2n+1} H^R_{-n-1,k}(x) H^R_{-n-1,k}(y) = \frac{2n+1}{4\pi R^2} \left(\frac{R^2}{|x|\ |y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \tag{10.38}$$

for all $(x, y) \in \overline{\Omega_R^{\mathrm{ext}}} \times \overline{\Omega_R^{\mathrm{ext}}}$, which is known as the *addition theorem of outer harmonics.*

We let

$$\mathrm{Harm}_n\left(\overline{\Omega_R^{\mathrm{ext}}}\right) = \operatorname*{span}_{k=1,\ldots,2n+1}\left(H^R_{-n-1,k}|\overline{\Omega_R^{\mathrm{ext}}}\right) \tag{10.39}$$

and

$$\mathrm{Harm}_{p,\ldots,q}\left(\overline{\Omega_R^{\mathrm{ext}}}\right) = \bigoplus_{n=p}^{q}\mathrm{Harm}_n\left(\overline{\Omega_R^{\mathrm{ext}}}\right) \ . \tag{10.40}$$

The kernel $K_{\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{ext}}})}(\cdot,\cdot)\colon \overline{\Omega_R^{\mathrm{ext}}}\times\overline{\Omega_R^{\mathrm{ext}}}\to\mathbb{R}$ given by

$$K_{\mathrm{Harm}_{p,\ldots,q(\overline{\Omega_R^{\mathrm{ext}}})}}(x,y) = \sum_{n=p}^{q}\sum_{k=1}^{2n+1} H^R_{-n-1,k}(x)H^R_{-n-1,k}(y), \tag{10.41}$$

$(x,y)\in\overline{\Omega_R^{\mathrm{ext}}}\times\overline{\Omega_R^{\mathrm{ext}}}$, is the reproducing kernel of the space $\mathrm{Harm}_{p,\ldots,q}(\overline{\Omega_R^{\mathrm{ext}}})$ with respect to $\|\cdot\|_{\mathrm{L}^2(\Omega_R)}$.

For brevity, we set

$$\mathrm{Harm}_{p,\ldots,q}(K) = \mathrm{Harm}_{p,\ldots,q}\left(\overline{\Omega_R^{\mathrm{ext}}}\right)|K \tag{10.42}$$

for every subset K of $\overline{\Omega_R^{\mathrm{ext}}}$.

It should be noted that an inner harmonic $H^R_{n,k}$ is related to the corresponding outer harmonic $H^R_{-n-1,k}$ in the following way:

$$H^R_{-n-1,k}(x) = \left(\frac{R}{|x|}\right)^{2n+1} H^R_{n,k}(x) = \frac{R}{|x|}H^R_{n,k}\left(\frac{R^2}{|x|^2}x\right) \ . \tag{10.43}$$

In other words, the outer harmonic is obtainable by the '*Kelvin transform*' K^R relative to the sphere Ω_R from its inner counterpart as follows:

$$H^R_{-n-1,k}(x) = K^R\left(H^R_{n,k}\right)(x) = \frac{R}{|x|}H^R_{n,k}(\overline{x}), \tag{10.44}$$

where the map $x\mapsto\overline{x}$ defined by

$$\overline{x} = \frac{R^2}{|x|^2}x, \quad x\neq 0 \tag{10.45}$$

is called the *inversion of* $\mathbb{R}^3$ *relative to the sphere* Ω_R. Note that $\overline{x}$ lies on the ray from the origin determined by x, with

$$\frac{|\overline{x}|}{R} = \frac{R}{|x|} \ . \tag{10.46}$$

It is well known (see, for example, W. Walter (1971)) that the inversion map of $\mathbb{R}^3$ relative to the sphere Ω_R is continuous and its own inverse. Moreover, it is the identity on Ω_R. Furthermore, it is easily seen that

$$H^R_{n,k}(x) = \frac{R}{|x|} H^R_{-n-1,k}(\overline{x}) = K^R\left(H^R_{-n-1,k}\right)(x), \tag{10.47}$$

which demonstrates that it is reasonable to introduce the Kelvin transform for the compactification $\mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$ (by additionally letting $\overline{x} = \infty$ for $x = 0$ and $\overline{x} = 0$ for $x = \infty$).

Next, we discuss the representations of outer harmonics on spheres of different altitudes. By convention, throughout this work, R is the height of the ground level, while S describes the satellite level such that $S > R > 0$.

By virtue of (10.36), we are immediately able to deduce that

$$H^R_{-n-1,k} = \left(\frac{R}{r}\right)^n H^r_{-n-1,k} \tag{10.48}$$

for all $r \geq R$. Moreover, the radial derivative ∂_r admits the following representations

$$\partial_r H^R_{-n-1,k} = \frac{\partial H^R_{-n-1,k}}{\partial r} = -\frac{n+1}{r} H^R_{-n-1,k} \tag{10.49}$$

$$= -\frac{n+1}{r}\left(\frac{R}{r}\right)^n H^r_{-n-1,k} \tag{10.50}$$

$$= -\frac{n+1}{R}\left(\frac{R}{r}\right)^{n+1} H^r_{-n-1,k}.$$

Furthermore, for all $r \geq R$, we have

$$(\partial_r)^2 H^R_{-n-1,k} = \left(-\frac{n+1}{r}\right)\left(-\frac{n+2}{r}\right)\left(\frac{R}{r}\right)^n H^r_{-n-1,k}$$

$$= \frac{(n+1)(n+2)}{R^2}\left(\frac{R}{r}\right)^{n+2} H^r_{-n-1,k}. \tag{10.51}$$

These results about 'upward continuation' can be arranged in a scheme as shown in Table 10.4.

Table 10.4: Outer harmonics characterizing ‘upward continuation’.

$$\begin{array}{lccc}
\Omega_S\text{–level:} & H^S_{-n-1,k} & \xrightarrow{-\frac{n+1}{S}} & \partial_r H^S_{-n-1,k} \\
 & \uparrow \cdot \left(\frac{R}{S}\right)^n & & \uparrow \cdot \left(\frac{R}{S}\right)^{n+1} \\
\Omega_R\text{–level:} & H^R_{-n-1,k} & \xrightarrow{-\frac{n+1}{R}} & \partial_r H^R_{-n-1,k}
\end{array}$$

The concise scheme in Table 10.4 connects the outer harmonics and their derivatives at the altitudes R (ground level) and S (satellite level), respectively. This scheme applies per degree and order. The vertical arrows characterize ‘upward continuation’, while the horizontal arrows describe transition from the function to its radial derivative.

Finding the solution of the Laplace equation subject to certain boundary conditions (see, e.g., O.D. Kellogg (1929), F. Neumann (1887)) is what we call a boundary-value problem (BVP). Of particular importance in classical potential theory is the Dirichlet and Neumann boundary-value problem, i.e., the determination of a potential from given potential values and normal derivatives, respectively. Our considerations are restricted to (the geophysically relevant) exterior boundary-value problems (note that the interior boundary-value problems can be discussed analogously). If the boundary is a sphere Ω_R around the origin, then it is well known (see, for example, O.D. Kellogg (1929), F. Neumann (1887)) that the solutions of the classical boundary-value problems can be given in explicit integral form.

Exterior Dirichlet Problem (EDP): Given $F \in \mathrm{C}(\Omega_R)$. Then the function $U : \overline{\Omega_R^{\text{ext}}} \to \mathbb{R}$ given by

$$U(x) = \int_{\Omega_R} D(x,y) F(y) \, d\omega(y) \tag{10.52}$$

with the *Abel–Poisson kernel function (briefly called Abel–Poisson kernel)*

$$D(x,y) = \frac{1}{4\pi R} \frac{|x|^2 - R^2}{|x-y|^3}, \quad x \in \Omega_R^{\text{ext}}, \tag{10.53}$$

is the unique solution of the exterior Dirichlet boundary–value problem:

(i) U is continuous in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $U \in \mathrm{C}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$.

(ii) U is harmonic on Ω_R^{ext}, i.e., $\Delta U = 0$ in Ω_R^{ext}.

(iii) U is regular at infinity, i.e., $|U(x)| = O(\frac{1}{|x|})$, $|\nabla U(x)| = O(\frac{1}{|x|^2})$ as $|x| \to \infty$.

(iv) $U|\Omega_R = F$.

Furthermore, U can be represented by a Fourier series expansion in terms of outer harmonics

$$U = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} F^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k} \ , \tag{10.54}$$

where the Fourier coefficients are given by

$$F^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) = \int_{\Omega_R} F(y) H^R_{-n-1,k}(y) \ d\omega(y), \tag{10.55}$$

$n = 0, 1, \ldots; \ k = 1, \ldots, 2n+1$, and the series expansion is absolutely and uniformly convergent on each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(K, \Omega_R) > 0$.

Exterior Neumann Problem (ENP): Given $F \in \mathrm{C}(\Omega_R)$. Then the funtion $U : \overline{\Omega_R^{\mathrm{ext}}} \to \mathbb{R}$ given by

$$U(x) = -\frac{R}{4\pi} \int_{\Omega_R} N(x,y) F(y) d\omega(\eta)$$

with the *Neumann kernel function (Neumann kernel)*

$$N(x,y) = \frac{2R}{|x-y|} + \ln\left(\frac{|x| + |x-y| - R}{|x| + |x-y| + R}\right), \quad x \in \Omega_R^{\mathrm{ext}},$$

is the unique solution of the exterior Neumann boundary-value problem:

(i) U is continuously differentiable in $\overline{\Omega_R^{\mathrm{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $U \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$

(ii) U is harmonic on Ω_R^{ext}, i.e., $\Delta U = 0$ in Ω_R^{ext}

(iii) U is regular at infinity, i.e., $|U(x)| = O(\frac{1}{|x|})$, $|\nabla U(x)| = O(\frac{1}{|x|^2})$ as $|x| \to \infty$

(iv) $\partial_r U|_{r=R} = \nu \cdot (\nabla U)|\Omega_R = F$, where $\nu : \Omega_R \to \mathbb{R}^3$ (more explicitly, $\nu_{\Omega_R} : \Omega_R \to \mathbb{R}^3$) is the (unit) normal field pointing into the exterior space Ω_R^{ext}.

Furthermore, U can be represented by a Fourier series expansion in terms of outer harmonics

$$U = -\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{R}{n+1} F^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k}, \tag{10.56}$$

where the Fourier coefficients are given by (10.55), and the series expansion is absolutely and uniformly convergent on each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(K, \Omega_R) > 0$.

The solution of the classical boundary-value problems leads us to the schemes of Tables 10.5 and 10.6 (characterizing 'upward continuation').

Table 10.5: (Frequency) Meissl scheme for 'upward continuation' (involving outer harmonics).

$$
\begin{array}{lccc}
\Omega_S\text{–level:} & V^{\wedge \mathrm{L}^2(\Omega_S)}(n,k) & \xrightarrow{-\frac{n+1}{S}} & (\partial_r V)^{\wedge \mathrm{L}^2(\Omega_S)}(n,k) \\
 & \uparrow \cdot \left(\frac{R}{S}\right)^n & & \uparrow \cdot \left(\frac{R}{S}\right)^{n+1} \\
\Omega_R\text{–level:} & V^{\wedge \mathrm{L}^2(\Omega_R)}(n,k) & \xrightarrow{-\frac{n+1}{R}} & (\partial_r V)^{\wedge \mathrm{L}^2(\Omega_R)}(n,k)
\end{array}
$$

The vertical arrows characterizing 'upward continuation' amount to an attenuation by the factor $\left(\frac{R}{S}\right)^n$. The opposite directions characterizing 'downward continuation' amount to an amplification by the factor $\left(\frac{S}{R}\right)^n$.

Table 10.6: (Space) Meissl scheme for 'upward continuation' (involving zonal kernel functions).

$$
\begin{array}{lccccc}
\Omega_S\text{–level:} & & V(S\eta) & \xrightarrow{*N(S\xi,S\eta)} & \partial_r V(S,\xi) & \\
 & *D(S\eta,R\zeta) & \uparrow & & \uparrow & *D(S\xi,R\alpha) \\
\Omega_R\text{–level:} & & V(R\zeta) & \xrightarrow{*N(R\alpha,R\zeta)} & \partial_r V(R\alpha) &
\end{array}
$$

The vertical arrows characterizing 'upward continuation' describe the convolution with the (zonal) Abel-Poisson kernel, while the transition to the Neumann problem amounts to the convolution with the (zonal) Neumann kernel function.

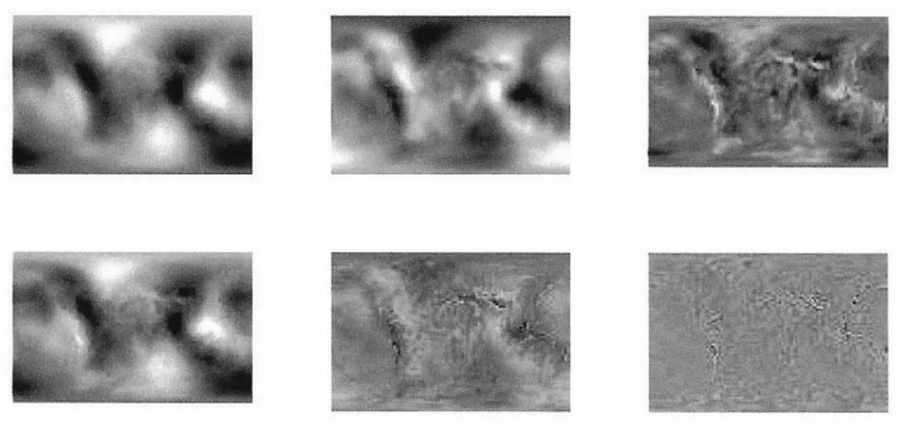


Fig. 10.14: The disturbing potential EGM96 and its first and second radial derivatives $\partial_r T$ and $\partial^2_{rr} T$, respectively, at Earth's surface [0km] and at altitude [200km] (illustrated in the *upper row*), Geomathematics Group, TU Kaiserslautern, W. Freeden (1999).

A system $\{\Phi_n\}_{n=0,1,\ldots}, \Phi_n \in \mathrm{L}^2(\Omega_R)$, is called *complete* in the Hilbert space $\mathrm{L}^2(\Omega_R)$ if it satisfies the following property: For every $\Phi \in \mathrm{L}^2(\Omega_R)$, the condition

$$(\Phi, \Phi_n)_{\mathrm{L}^2(\Omega_R)} = \int_{\Omega_R} \Phi(x)\Phi_n(x)\, d\omega(x) = 0 \tag{10.57}$$

for all $n = 0, 1, \ldots$ implies $\Phi = 0$ (in the sense of $\mathrm{L}^2(\Omega_R)$).

In classical potential theory (see e.g. O.D. Kellogg (1929)), a large number of systems $\{\tilde{\Phi}_n\}_{n=0,1,\ldots}$, $\tilde{\Phi}_n : \overline{\Omega_R^{\mathrm{ext}}} \to \mathbb{R}$, is known satisfying the following properties:

(i) $\tilde{\Phi}_n$ is continuous on $\overline{\Omega_R^{\mathrm{ext}}}$ and twice continuously differentiable on Ω_R^{ext} for $n = 0, 1, \ldots$

(ii) $\tilde{\Phi}_n$ is harmonic on Ω_R^{ext}, i.e. $\Delta_x \tilde{\Phi}_n(x) = 0$ for all $x \in \Omega_R^{\mathrm{ext}}$ and $n = 0, 1, \ldots$

(iii) $\{\Phi_n\}_{n=0,1,\ldots}$ with $\Phi_n = \tilde{\Phi}_n | \Omega_R$, $n = 0, 1, \ldots$, is complete in $\mathrm{L}^2(\Omega_R)$.

The most important system (e.g., in geosciences) is the already known system of outer harmonics (i.e., multipoles). A proof can be found, for example, in W. Freeden (1979a), C. Müller (1998).

Lemma 10.1. *Let* $\{H^R_{-n-1,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ *be a system of outer harmonics.*

Then

$$\left\{ H^R_{-n-1,k} | \Omega_R \right\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}} \tag{10.58}$$

is a linearly independent complete system in $\mathrm{L}^2(\Omega_R)$*.*

In order to illustrate the role of single poles, we use the concept of fundamental systems in Ω_R^{int}.

Definition 10.2. A system $Y = \{y_n\}_{n=0,1,\ldots} \subset \Omega_R^{\mathrm{int}}$ ($y_n \neq y_k$ for all $n \neq k$) with $\sup_{n=0,1,\ldots} |y_n| = \rho < R$ is called a fundamental system in Ω_R^{int} if the conditions

(i) F is twice continuously differentiable in Ω_R^{int},
(ii) F is harmonic on Ω_R^{int}, i.e. $\Delta F = 0$ in Ω_R^{int},
(iii) $F(y_n) = 0$ for $n = 0, 1, \ldots$

imply the property

$$F = 0$$

in Ω_R^{int}.

Analogously, a system $Y = \{y_n\}_{n=0,1,\ldots} \subset \Omega_R^{\mathrm{ext}}$ ($y_n \neq y_k$ for all $n \neq k$) with $\inf_{n=0,1,\ldots} |y_n| = \overline{\rho} > R$ is called a fundamental system in Ω_R^{ext} if the conditions

(i) F is twice continuously differentiable in Ω_R^{ext},
(ii) F is harmonic on Ω_R^{ext}, i.e. $\Delta F = 0$ in Ω_R^{ext},
(iii) F is regular at infinity, i.e.

$$\begin{aligned} |F(x)| &= O\left(\frac{1}{|x|}\right), |x| \to \infty, \\ \left| (\nabla F)(x) \right| &= O\left(\frac{1}{|x|^2}\right), |x| \to \infty, \end{aligned}$$

(iv) $F(y_n) = 0$ for $n = 0, 1, \ldots$

imply the property

$$F = 0$$

in Ω_R^{ext}.

Observing this definition, we are able to formulate the following lemma.

Lemma 10.3. *Suppose that* $Y = \{y_n\}_{n=0,1,\ldots}$ *is a fundamental system in* Ω_R^{int} *with*

$$\sup_{n=0,1,\ldots} |y_n| = \rho < R \ . \tag{10.59}$$

Denote by

$$x \mapsto M(x, y_n) = \frac{1}{|x - y_n|}, \quad x \in \overline{\Omega_R^{\text{ext}}} \ . \tag{10.60}$$

the single poles (mass points) at $y_n \in Y$, $n = 0, 1, \ldots$. *Then*

$$\{M\,(x, y_n)\,|_{x \in \Omega_R}\}_{n=0,1,\ldots} \tag{10.61}$$

is a linearly independent complete system in $\mathrm{L}^2(\Omega_R)$.

Proof. Provided that $y_n \neq y_k$ for all $n \neq k$, we are immediately able to verify the linear independence.

Our aim is to prove the completeness. Consider a function $\Phi \in \mathrm{L}^2(\Omega_R)$ and require that

$$(\Phi, M\,(\cdot, y_n))_{\mathrm{L}^2(\Omega_R)} = \int_{\Omega_R} M(y, y_n)\Phi(y)\; d\omega(y) = 0 \tag{10.62}$$

for $n = 0, 1, \ldots$. Then, the (single-layer) potential U defined by

$$U(x) = \int_{\Omega_R} M(y, x)\Phi(y)\; d\omega(y) \tag{10.63}$$

vanishes at all points $y_n \in Y$. Since Y is a fundamental system in Ω_R^{int}, this fact shows us that $U = 0$ in Ω_R^{int}. Observing the fact that

$$M(y, x) = \frac{1}{|y|} \sum_{n=0}^{\infty} \left(\frac{|x|}{|y|}\right)^n P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \quad y \in \Omega_R, \tag{10.64}$$

we obtain for all $x \in \Omega_R^{\text{int}}$ with $|x| = \rho$

$$\begin{aligned} U(x) &= \frac{1}{R} \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{4\pi R^2}{2n+1} \left(\frac{\rho}{R}\right)^n Y_{n,k}^R\,(x) \int_{\Omega_R} \Phi(y) Y_{n,k}^R\,(y)\; d\omega(y) \\ &= 0. \end{aligned} \tag{10.65}$$

This tells us that

$$\left(\Phi, H_{n,k}^R\right)_{\mathrm{L}^2(\Omega_R)} = \int_{\Omega_R} \Phi(y) H_{n,k}^R(y)\; d\omega(y) = 0 \tag{10.66}$$

for $n = 0, 1, \ldots;\ k = 1, \ldots, 2n+1$. Thus, the completeness of the system $\{H_{n,k}^R\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$ shows us that $\Phi = 0$ in $\mathrm{L}^2(\Omega_R)$, as required. □

Some examples of fundamental systems in Ω_R^{int} should be listed below:

(i) If Y is a countable dense set of points on a closed surface $\Xi \subset \Omega_R^{\text{int}}$ with $\text{dist}(\Xi, \Omega_R) > 0$, then Y is a fundamental system in Ω_R^{int}.

(ii) If Y is a countable dense set of points in the inner space Ξ_{int} of a closed surface Ξ with $\text{dist}(\Xi, \Omega_R) > 0$, then Y is a fundamental system in Ω_R^{int}.

(iii) Let w_0 be a point in Ω_R^{int}. Let $\{x_n\}_{n=0,1,\ldots} \subset \Omega_R^{\text{int}}$ be an infinite system of points (with $x_n \neq x_k$ for all $n \neq k$) converging to w_0. For the set

$$S = \left\{ \left(x_p \cdot \varepsilon^1, x_q \cdot \varepsilon^2, x_r \cdot \varepsilon^3 \right)^T \middle| p, q, r \in \mathbb{N}_0 \right\} \tag{10.67}$$

we assume $S \subset \Omega_R^{\text{int}}$. Let $Y = \{y_k\}_{k=0,1,\ldots}$ be an enumeration of S. Then Y is a fundamental system in Ω_R^{int}.

Further, complete systems which are of relevance in potential theory can be obtained by using $\{K(x, y_n)\}_{n=0,1,\ldots}$ with

$$K(x,y) = \frac{1}{|x|} \sum_{k=0}^{\infty} \frac{2k+1}{4\pi R^2} K^\wedge(k) \left(\frac{|y|}{|x|} \right)^k P_k \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \tag{10.68}$$

$$x \in \overline{\Omega_R^{\text{ext}}}, \qquad y \in Y \subset \Omega_R^{\text{int}},$$

instead of the system $\{M(x, y_n)\}_{n=0,1,\ldots}$ provided that Y is a fundamental system in Ω_R^{int} with $\rho = \sup_{y \in Y} |y| < R$, and the coefficients $K^\wedge(k)$, $K^\wedge(k) \neq 0$ for $k = 0, 1, \ldots$, have to be chosen in such a way that

$$\sum_{k=0}^{\infty} (2k+1) \left| K^\wedge(k) \right| \left(\frac{\rho}{R} \right)^k < \infty \; . \tag{10.69}$$

Lemma 10.4. *Suppose that* $Y = \{y_n\}_{n=0,1,\ldots}$ *is a fundamental system in* Ω_R^{int} *satisfying* $\sup_{y \in Y} |y| = \rho < R$. *Let* $K(x, y_n)$ *be given by (10.68) (with coefficients* $K^\wedge(k) \neq 0$ *for* $k = 0, 1, \ldots$, *satisfying the condition (10.69)). Then*

$$\left\{ K(x, y_n) \big|_{x \in \Omega_R} \right\}_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(\Omega_R)$.

The proof of the completeness for the system $\{K(\cdot, y_n)\}_{n=0,1,\ldots}$ in $\mathrm{L}^2(\Omega_R)$ again follows from the completeness of the system of spherical harmonics.

Of numerical significance are series expansions (10.68) with explicit (i.e. elementary) representations (as, for example, the single poles).

Lemma 10.5. *Let y_0 be a fixed point in Ω_R^{int} and K be given by (10.68). Denote by $\mathrm{P}_\beta^{y_0}(x)$, $n = 0, 1, \ldots$, the expression given by*

$$\left(\frac{\partial}{\partial y_0}\right)^\beta K(x, y_0),$$

($\beta \in {\mathbb{N}_0}^3$: multi-index, $[\beta] = \beta_1 + \beta_2 + \beta_3$, $\left(\frac{\partial}{\partial y_0}\right)^\beta = \frac{\partial^{[\beta]}}{\partial_{y_1}^{\beta_1} \partial_{y_2}^{\beta_2} \partial_{y_3}^{\beta_3}}|_{y_0}$).

Then

$$\left\{ \left(\frac{\partial}{\partial y_0}\right)^\beta K(x, y_0) \Big|_{x \in \Omega_R} \right\}_{[\beta]=n;\ n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(\Omega_R)$.

The proof follows from Maxwell's representation theorem (cf. W. Freeden et al. (1994)) in connection with the completeness of the system of spherical harmonics.

Applying the Kelvin transform with respect to the sphere Ω_R with radius R around the origin (cf. O.D. Kellogg (1929)), we are led to systems

$$\left\{ \overline{K}(x, \overline{y}_n) \Big|_{x \in \overline{\Omega_R^{\mathrm{ext}}}} \right\}_{n=0,1,\ldots}$$

with

$$\overline{K}(x, \overline{y}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi R^2} K^\wedge(k) \left(\frac{R^2}{|x|\,|\overline{y}|}\right)^{k+1} P_k\left(\frac{x}{|x|} \cdot \frac{\overline{y}}{|\overline{y}|}\right), \quad x \in \overline{\Omega_R^{\mathrm{ext}}}, \quad \overline{y} \in \overline{Y} \subset \Omega_R^{\mathrm{ext}}, \tag{10.70}$$

where $\overline{Y} = \{\overline{y}_n\}_{n=0,1,\ldots}$ is the point system generated by Y by letting

$$\overline{y}_n = \frac{R^2}{|y_n|^2} y_n, \quad n = 0, 1, \ldots \tag{10.71}$$

(thereby assuming $0 \notin Y$). Note that our above assumptions imply the estimate

$$\sum_{k=0}^{\infty} (2n+1) \left| K^\wedge(k) \right| \left(\frac{R}{\overline{\rho}}\right)^k < \infty, \tag{10.72}$$

where $\overline{\rho}$ is given by

$$\overline{\rho} = \inf_{\overline{y} \in \overline{Y}} |\overline{y}| > R\ . \tag{10.73}$$

Therefore, we are able to formulate the following result.

Lemma 10.6. *Suppose that* $\overline{Y} = \{\overline{y}_n\}_{n=0,1,\ldots}$ *is given as described above. Let* $\overline{K}(x, \overline{y}_n)$ *be given as above (with coefficients* $K^\wedge(k) \neq 0$ *for* $k = 0, 1, \ldots,$ *satisfying (10.72)). Then*

$$\left\{\overline{K}\,(x, \overline{y}_n)\,|_{x\in\Omega_R}\right\}_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(\Omega_R)$.

The kernels of the form (10.70) play a central role in the Sobolev space theory of harmonic functions. Typical examples are as follows:

(i) *Abel–Poisson kernel:*

$$K^\wedge(k) = 1, \quad k = 0, 1, \ldots \ . \tag{10.74}$$

The kernel reads as follows:

$$\overline{K}(x, \overline{y}) = \frac{1}{4\pi}\frac{|x|^2|\overline{y}|^2 - R^2}{(L(x, \overline{y}))^{3/2}}, \quad x \in \overline{\Omega_R^{\mathrm{ext}}}, \quad \overline{y} \in Y \subset \overline{\Omega_R^{\mathrm{ext}}}, \tag{10.75}$$

where we have introduced the abbreviation

$$L\,(x, \overline{y}) = |x|^2|\overline{y}|^2 - 2R^2 x\cdot\overline{y} + R^4 \ . \tag{10.76}$$

(ii) *Singularity kernel:*

$$K^\wedge(k) = \frac{2}{2k+1}, \quad k = 0, 1, \ldots \ . \tag{10.77}$$

The kernel is now given by

$$\overline{K}\,(x, \overline{y}) = \frac{1}{2\pi}\frac{1}{(L(x, \overline{y}))^{1/2}}, \quad x \in \overline{\Omega_R^{\mathrm{ext}}}, \quad \overline{y} \in \overline{Y} \subset \overline{\Omega_R^{\mathrm{ext}}} \ . \tag{10.78}$$

(iii) *Logarithmic kernel:*

$$K^\wedge(k) = \frac{1}{(k+1)(2k+1)}, k = 0, 1, \ldots \ . \tag{10.79}$$

Now we have

$$\overline{K}(x, \overline{y}) \ = \ \frac{1}{4\pi R^2}\ln\left(\frac{R^2 - x\cdot\overline{y} + (L(x, \overline{y}))^{1/2}}{|x|\ |\overline{y}| + x\cdot\overline{y}}\right), \tag{10.80}$$

$$x \in \overline{\Omega_R^{\mathrm{ext}}}, \quad \overline{y} \in \overline{Y} \subset \overline{\Omega_R^{\mathrm{ext}}} \ .$$

Remark 10.7. Choosing (instead of (10.69) and (10.72)) $K^\wedge(k)$ (with $K^\wedge(k) \neq 0$ for $k = 0, 1, \ldots$) in such a way that

$$\sum_{k=0}^{\infty} (2k+1) \left| K^\wedge(k) \right| < \infty, \tag{10.81}$$

i.e. the sequence

$$\left\{ \left| K^\wedge(k) \right|^{-1/2} \right\}_{n=0,1,\ldots} \tag{10.82}$$

is assumed to be *summable* (in the sense of W. Freeden (1998)), ρ and $\overline{\rho}$ are even allowed to satisfy $\rho \leq R$ and $\overline{\rho} \geq R$, respectively.

An equivalent statement to the completeness of a system $\{\Phi_n\}_{n=0,1,\ldots} \subset \mathrm{L}^2(\Omega_R)$ is the *closure* (see e.g. P.J. Davis (1963) for the proof of equivalence): For a given function $\Phi \in \mathrm{L}^2(\Omega_R)$ and an arbitrary value $\varepsilon > 0$, there exist an integer $N (= N(\varepsilon))$ and constants $a_0, \ldots, a_N$ such that

$$\left(\int_{\Omega_R} \left| \Phi(y) - \sum_{n=0}^{N} a_n \Phi_n(y) \right|^2 \, d\omega(y) \right)^{1/2} \leq \varepsilon \ . \tag{10.83}$$

This property means that any $\Phi \in \mathrm{L}^2(\Omega_R)$ can be approximated by a member of the span of $\{\Phi_n\}_{n=0,1,\ldots} \subset \mathrm{L}^2(\Omega_R)$ in the sense of the $\mathrm{L}^2(\Omega_R)$–metric.

The step from approximation on the sphere Ω_R to approximation in the outer space Ω_R^{ext} can be performed as indicated by the following theorem.

Theorem 10.8. *Let K be a (not necessarily compact) subset of the space Ω_R^{ext} satisfying $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. Let $\tilde{\Phi}, \tilde{\Psi}$ be functions of class $\mathrm{C}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$, being harmonic on Ω_R^{ext} and regular at infinity, such that $\tilde{\Phi}\,|\Omega_R = \Phi$ and $\tilde{\Psi}\,|\Omega_R = \Psi$. Then, there exists a positive constant $C (= C(K, \Omega_R))$ such that*

$$\sup_{x \in K} \left| \tilde{\Phi}(x) - \tilde{\Psi}(x) \right|^2 \leq C \left(\int_{\Omega_R} \left(\Phi(y) - \Psi(y) \right)^2 \, d\omega(y) \right)^{1/2} \ . \tag{10.84}$$

Proof. Theorem 10.8 is easily verified by application of the Abel–Poisson integral formula (see Section 3.6)

$$\tilde{\Phi}(x) - \tilde{\Psi}(x) = \int_{\Omega_R} D(x,y) \left(\Phi(y) - \Psi(y) \right) \, d\omega(y), \tag{10.85}$$

where $D(x,y)$ denotes the Abel–Poisson kernel (see (10.53)). Put

$$C = C(K, \Omega_R) = \sup_{x \in K} \left(\int_{\Omega_R} \left(D(x,y) \right)^2 \, d\omega(y) \right)^{1/2} \ . \tag{10.86}$$

Then, for each $x \in K$, the Cauchy–Schwarz inequality yields

$$\left|\tilde{\Phi}(x) - \tilde{\Psi}(x)\right|^2 \leq C^2 \int_{\Omega_R} (\Phi(y) - \Psi(y))^2 \, d\omega(y) \ . \tag{10.87}$$

This is the desired result. □

Let $\tilde{\Phi}$ be the unique solution of the Dirichlet problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\tilde{\Phi}\big|\Omega_R = \Phi$, $\Phi \in \mathrm{C}(\Omega_R)$. If now $\{\tilde{\Phi}_n\}_{n=0,1,\ldots}$ is given such that each $\tilde{\Phi}_n$ is the unique solution of the Dirichlet problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\tilde{\Phi}_n \,|\Omega_R = \Phi_n, n = 0, 1, \ldots,$ and $\{\Phi_n\}_{n=0,1,\ldots} \subset \mathrm{C}(\Omega_R)$ forms a complete system in $\mathrm{L}^2(\Omega_R)$, then, for every $\varepsilon > 0$, there exist an integer $N(= N(\varepsilon))$ and coefficients $a_0, \ldots, a_N$ such that

$$\begin{aligned} & \sup_{x \in K} \left| \tilde{\Phi}(x) - \sum_{n=0}^{N} a_n \tilde{\Phi}_n(x) \right| \qquad\qquad (10.88) \\ \leq \ & C(K, \Omega_R) \left(\int_{\Omega_R} \left(\Phi(y) - \sum_{n=0}^{N} a_n \Phi_n(y) \right)^2 \, d\omega(y) \right)^{1/2} \\ \leq \ & C(K, \Omega_R) \, \varepsilon \end{aligned}$$

for each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. In other words, approximation in quadratic sense on the sphere Ω_R implies uniform approximation for each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. This result is illustrated for typical satellite problems in the scheme of Table 10.7.

Table 10.7: Dirichlet's problem.

Satellite's height

convergence in $\mathrm{C}(\Omega_S)$– topology

$$\lim_{N \to \infty} \sup_{x \in \Omega_S} \left| U(x) - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} U^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(x) \right| = 0$$

$$\Uparrow$$

Earth's height

convergence in $\mathrm{L}^2(\Omega_R)$– topology

$$\lim_{N \to \infty} \left(\int_{\Omega_R} \left| U(x) - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} \underbrace{U^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)}_{= F^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)} H^R_{-n-1,k}(x) \right|^2 d\omega(x) \right)^{1/2} = 0$$

A similar result is true for the Neumann problem:

Theorem 10.9. *Let K be a (not necessarily compact) subset of the space Ω_R^{ext} satisfying $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. Let $\tilde{\Phi}, \tilde{\Psi}$ be functions of class $\mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}})$*

$\cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$, *being harmonic on* Ω_R^{ext} *and regular at infinity, such that* $\partial_r \tilde{\Phi} \,|\Omega_R = F$ *and* $\partial_r \tilde{\Psi} \,|\Omega_R = G$. *Then there exists a positive constant* $C (= C(K, \Omega_R))$ *such that*

$$\sup_{x \in K} \left| \tilde{\Phi}(x) - \tilde{\Psi}(x) \right|^2 \leq C \left(\int_{\Omega_R} (F(y) - G(y))^2 \; d\omega(y) \right)^{1/2} . \tag{10.89}$$

This result enables us to formulate a solution procedure for the exterior Neumann problem: Let $\tilde{\Phi}$ be the unique solution of the Neumann problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\partial_r \tilde{\Phi} \big| \Omega_R = F$, $F \in \mathrm{C}(\Omega_R)$. If now $\{\tilde{\Phi}_n\}_{n=0,1,\ldots}$ is given such that each $\tilde{\Phi}_n$ is the unique solution of the Neumann problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\partial_r \tilde{\Phi}_n \,|\Omega_R = F_n, n = 0, 1, \ldots,$ and $\{F_n\}_{n=0,1,\ldots} \subset \mathrm{C}(\Omega_R)$ forms a complete system in $\mathrm{L}^2(\Omega_R)$, then, for every $\varepsilon > 0$, there exist an integer $N (= N(\varepsilon))$ and coefficients $a_0, \ldots, a_N$ such that

$$\begin{aligned} & \sup_{x \in K} \left| \tilde{\Phi}(x) - \sum_{n=0}^{N} a_n \tilde{\Phi}_n(x) \right| \qquad (10.90) \\ \leq \;& C(K, \Omega_R) \left(\int_{\Omega_R} \left(F(y) - \sum_{n=0}^{N} a_n F_n(y) \right)^2 \; d\omega(y) \right)^{1/2} \\ \leq \;& C(K, \Omega_R)\, \varepsilon \end{aligned}$$

for each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. In other words, approximation in quadratic sense on the sphere Ω_R of the normal derivative implies uniform approximation for each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$. An illustration of this result for a satellite situation is given in the scheme of Table 10.8.

Table 10.8: Neumann's problem.

Satellite's height

convergence in $\mathrm{C}(\Omega_S)$– topology

$$\lim_{N \to \infty} \sup_{x \in \Omega_S} \left| U(x) - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} \frac{R}{n+1} (\partial_r U)^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(x) \right| = 0$$

$$\Uparrow$$

Earth's height

convergence in $\mathrm{L}^2(\Omega_R)$– topology

$$\lim_{N \to \infty} \left(\int_{\Omega_R} \left| \partial_r U(x) - \sum_{n=0}^{N} \sum_{k=1}^{2n+1} \underbrace{(\partial_r U)^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)}_{= F^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)} H^R_{-n-1,k}(x) \right|^2 d\omega(x) \right)^{1/2} = 0$$

We conclude our considerations with the following extension of Theorem 10.8.

Theorem 10.10. *Under the assumptions of Theorem 10.8, there exists for every $k \in \mathbb{N}$ a constant $D(= D(K, \Omega_R))$ such that*

$$\sup_{x \in K} \left| \left(\nabla^{(k)} \tilde{\Phi} \right)(x) - \left(\nabla^{(k)} \tilde{\Psi} \right)(x) \right| \leq D \left(\int_{\Omega_R} \left(\Phi(y) - \Psi(y) \right)^2 \, d\omega(y) \right)^{1/2} . \tag{10.91}$$

Proof. From (10.85) it follows that for $k \in \mathbb{N}$

$$\left(\nabla^{(k)} \tilde{\Phi} \right)(x) - \left(\nabla^{(k)} \tilde{\Psi} \right)(x) = \int_{\Omega_R} \nabla_x^{(k)} D(x, y) \left(\Phi(y) - \Psi(y) \right) \, d\omega(\eta) \tag{10.92}$$

for all $x \in K$. Hence, we get from the Cauchy–Schwarz inequality

$$\left| \left(\nabla^{(k)} \tilde{\Phi} \right)(x) - \left(\nabla^{(k)} \tilde{\Psi} \right)(x) \right| \leq D^2 \int_{\Omega_R} \left(\Phi(y) - \Psi(y) \right)^2 \, d\omega(\eta),$$

where

$$D = D\left(K, \Omega_R\right) = \sup_{x \in K} \left(\int_{\Omega_R} \left(\nabla_x^{(k)} D(x, y) \right)^2 \, d\omega(\eta) \right)^{1/2} . \tag{10.93}$$

This proves Theorem 10.10. □

Let $\tilde{\Phi}$ be the unique solution of the Dirichlet problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\tilde{\Phi}\big|\Omega_R = \Phi$, $\Phi \in \mathrm{C}(\Omega_R)$. If now $\{\tilde{\Phi}_n\}_{n=0,1,\ldots}$ is given such that each $\tilde{\Phi}_n$ is the unique solution of the Dirichlet problem in $\overline{\Omega_R^{\mathrm{ext}}}$ corresponding to the boundary values $\tilde{\Phi}_n\big|\Omega_R = \Phi_n$, $n = 0, 1, \ldots$, and $\{\Phi_n\}_{n=0,1,\ldots} \subset \mathrm{C}(\Omega_R)$ forms a complete system in $\mathrm{L}^2(\Omega_R)$, then, for every $\varepsilon > 0$, there exist an integer $N(= N(\varepsilon))$ and coefficients $a_0, \ldots, a_N$ such that

$$\begin{aligned} &\sup_{x \in K} \left| \left(\nabla^{(k)} \tilde{\Phi} \right)(x) - \sum_{n=0}^{N} a_n \left(\nabla^{(k)} \tilde{\Phi}_n \right)(x) \right| \\ &\leq \; D(K, \Omega_R) \left(\int_{\Omega_R} \left(\Phi(y) - \sum_{n=0}^{N} a_n \Phi_n(y) \right)^2 \, d\omega(y) \right)^{1/2} \\ &\leq \; D(K, \Omega_R)\, \varepsilon \end{aligned}$$

for each subset $K \subset \Omega_R^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \Omega_R) \geq \rho > 0$.

10.4 Limit Formulas and Jump Relations

Let F be a continuous function on the sphere Ω_R. Then the functions $U_n : \mathbb{R}^3 \backslash \Omega_R \to \mathbb{R}$, $n = 1, 2, \ldots$, defined by

$$U_n(x) = \int_{\Omega_R} F(y) \left(\frac{\partial}{\partial \nu(y)} \right)^{n-1} \frac{1}{|x-y|} \, d\omega(y) \tag{10.94}$$

are infinitely often differentiable and satisfy the Laplace equation in Ω_R^{int} and Ω_R^{ext}(ν is the (unit) normal field pointing into the outer space Ω_R^{ext} such that $\nu(x) = x/R$ for all $x \in \Omega_R$). In addition, the functions U_n are regular at infinity. The function U_1 given by

$$U_1(x) = \int_{\Omega_R} F(y) \frac{1}{|x-y|} \, d\omega(y) \tag{10.95}$$

is called the *potential of the single layer* on Ω_R, while U_2 given by

$$U_2(x) = \int_{\Omega_R} F(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} \, d\omega(y) \tag{10.96}$$

is called the *potential of the double layer* on Ω_R.

For $F \in \mathrm{C}(\Omega_R)$, the functions $U_n, n = 1, 2$, can be continued continuously to the surface Ω_R, but the limits depend from which parallel surface (inner or outer) the points x tend to Ω_R. On the other hand, the functions $U_n, n = 1, 2$, also are defined on the surface Ω_R, i.e., the integrals (10.95), (10.96) exist for $x \in \Omega_R$. Furthermore, the integral

$$U_1'(x) = \int_{\Omega_R} F(y) \frac{\partial}{\partial \nu(x)} \frac{1}{|x-y|} \, d\omega(y) \tag{10.97}$$

exists for all $x \in \Omega_R$ and can be continued continuously to Ω_R. However, the integrals do not coincide, in general, with the inner or outer limits of the potentials (see, for example, S.G. Michlin (1975), R. Leis (1967), W. Walter (1971)).

From classical potential theory (see, for example, O.D. Kellogg (1929), W. Walter (1971)) and the references therein), it is known that for all $x \in \Omega_R$ and $F \in \mathrm{C}(\Omega_R)$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} U_1(x \pm \tau \nu(x)) = U_1(x), \tag{10.98}$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} \frac{\partial U_1}{\partial \nu(x)}(x \pm \tau \nu(x)) = \mp 2\pi F(x) + U_1'(x), \tag{10.99}$$

$$\lim_{\substack{\tau \to 0 \\ \tau > 0}} U_2(x \pm \tau \nu(x)) = \pm 2\pi F(x) + U_2(x), \tag{10.100}$$

(limit relations)

$$\lim_{\substack{\tau\to 0\\ \tau>0}} (U_1(x+\tau\nu(x)) - U_1(x-\tau\nu(x))) = 0, \tag{10.101}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\frac{\partial U_1}{\partial \nu(x)}(x+\tau\nu(x)) - \frac{\partial U_1}{\partial \nu(x)}(x-\tau\nu(x)) \right) = -4\pi F(x), \tag{10.102}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} (U_2(x+\tau\nu(x) - U_2(x-\tau\nu(x)) = 4\pi F(x), \tag{10.103}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\frac{\partial U_2}{\partial \nu(x)}(x+\tau\nu(x)) - \frac{\partial U_2}{\partial \nu(x)}(x-\tau\nu(x)) \right) = 0 \tag{10.104}$$

(jump relations).

In addition, it was shown by O.D. Kellogg (1929) that the preceding relations hold uniformly with respect to all $x \in \Omega$. This means that

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} |U_1(x \pm \tau\nu(x)) - U_1(x)| = 0, \tag{10.105}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| \frac{\partial U_1}{\partial \nu(x)}(x \pm \tau\nu(x)) \pm 2\pi F(x) - U_1'(x) \right| = 0, \tag{10.106}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| U_2(x \pm \tau\nu(x)) \mp 2\pi F(x) - U_2(x) \right| = 0 \tag{10.107}$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| U_1(x+\tau\nu(x)) - U_1(x-\tau\nu(x)) \right| = 0, \tag{10.108}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| \frac{\partial U_1}{\partial \nu(x)}(x+\tau\nu(x)) - \frac{\partial U_1}{\partial \nu(x)}(x-\tau\nu(x)) + 4\pi F(x) \right| = 0, \tag{10.109}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| U_2(x+\tau\nu(x)) - U_2(x-\tau\nu(x)) - 4\pi F(x) \right| = 0, \tag{10.110}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \sup_{x\in\Omega_R} \left| \frac{\partial U_2}{\partial \nu(x)}(x+\tau\nu(x)) - \frac{\partial U_2}{\partial \nu(x)}(x-\tau\nu(x)) \right| = 0\ . \tag{10.111}$$

Here we have written, as usual,

$$\frac{\partial U}{\partial \nu(x)}(x \pm \tau\nu(x)) = \frac{x}{R} \cdot (\nabla U)(x \pm \tau\nu(x))\ . \tag{10.112}$$

Furthermore, by means of functional analysis, W. Freeden (1980a) (see also W. Freeden, C. Mayer (2003)) was able to show that the limit and jump relations also hold true in L^2-topology. In more detail,

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega |U_1(x \pm \tau\nu(x)) - U_1(x)|^2 \ d\omega(x) \right)^{1/2} = 0, \tag{10.113}$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| \frac{\partial U_1}{\partial \nu(x)}(x \pm \tau\nu(x)) \pm 2\pi F(x) - U_1'(x) \right|^2 \, d\omega(x) \right)^{1/2} = 0, \quad (10.114)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| U_2(x \pm \tau\nu(x)) \mp 2\pi F(x) - U_2(x) \right|^2 \, d\omega(x) \right)^{1/2} = 0 \qquad (10.115)$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| U_1(x + \tau\nu(x)) - U_1(x - \tau\nu(x)) \right|^2 \, d\omega(x) \right)^{1/2} = 0, \qquad (10.116)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| \frac{\partial U_1}{\partial \nu(x)}(x + \tau\nu(x)) - \frac{\partial U_1}{\partial \nu(x)}(x - \tau\nu(x)) + 4\pi F(x) \right|^2 \, d\omega(x) \right)^{1/2} = 0, \qquad (10.117)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| U_2(x + \tau\nu(x)) - U_2(x - \tau\nu(x)) - 4\pi F(x) \right|^2 \, d\omega(x) \right)^{1/2} = 0, \qquad (10.118)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \left(\int_\Omega \left| \frac{\partial U_2}{\partial \nu(x)}(x + \tau\nu(x)) - \frac{\partial U_2}{\partial \nu(x)}(x - \tau\nu(x)) \right|^2 \, d\omega(x) \right)^{1/2} = 0 \,. \qquad (10.119)$$

The classical boundary-value problems can be solved in terms of layer potentials. We recapitulate the essential results for the Dirichlet and Neumann problem (for more details the reader is referred, e.g., to S.G. Michlin (1975), W. Walter (1971) and the references therein).

Exterior Dirichlet Problem (EDP): Given $F \in \mathrm{C}(\Omega_R)$, find a function $U \in \mathrm{C}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$ which is harmonic in Ω_R^{ext} and regular at infinity such that

$$U^+_{\Omega_R}(x) = \lim_{\substack{\tau\to 0\\ \tau>0}} U(x + \tau\nu(x)) = F(x), \quad x \in \Omega_R. \qquad (10.120)$$

Exterior Neumann Problem (ENP): Given a function $F \in \mathrm{C}(\Omega_R)$, find $U \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$ which is harmonic in Ω_R^{ext} and regular at infinity such that

$$\frac{\partial U^+}{\partial \nu_{\Omega_R}}(x) = \lim_{\substack{\tau\to 0\\ \tau>0}} \nu(x) \cdot (\nabla U)(x + \tau\nu(x)) = F(x), \quad x \in \Omega_R. \qquad (10.121)$$

Existence and Uniqueness: We recall the role of layer potentials in the aforementioned boundary-value problems.

(EDP) Let D^+ (more accurately, $D^+_{\Omega_R}$) denote the set consisting of all $H^+_{\Omega_R}$, where H is of class $\mathrm{C}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$, harmonic in Ω_R^{ext}, and regular at infinity.

By virtue of the maximum/minimum principle, the solution of (EDP) is uniquely determined, hence,

$$D^+ = \mathrm{C}(\Omega_R). \tag{10.122}$$

It can be formulated in terms of a potential of the form

$$\begin{aligned} U(x) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10.123) \\ = & \int_{\Omega_R} Q(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|}\ d\omega(y) + \frac{1}{|x|} \int_{\Omega_R} Q(y)\ d\omega(y), \quad Q \in \mathrm{C}(\Omega_R), \end{aligned}$$

such that Q satisfies the integral equation

$$F = U^+_{\Omega_R} = \left(2\pi I + P + P_{|2}(0,0)\right), \quad Q \in \mathrm{C}(\Omega_R), \tag{10.124}$$

where

$$P(Q) : x \mapsto \frac{1}{|x|} \int_{\Omega_R} Q(y)\ d\omega(y). \tag{10.125}$$

and

$$P_{|2}(0,0)Q(x) = \int_{\Omega_R} Q(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|}\ d\omega(y), \tag{10.126}$$

Setting

$$T = 2\pi I + P + P_{|2}(0,0) \tag{10.127}$$

we obtain

$$\mathrm{kern}(T^*) = \{0\}, \tag{10.128}$$

$$T\left(\mathrm{C}(\Omega_R)\right) = D^+. \tag{10.129}$$

By completion,

$$\mathrm{L}^2(\Omega_R) = \overline{D^+}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}} = \overline{\mathrm{C}(\Omega_R)}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}}. \tag{10.130}$$

(ENP) Let N^+ (more accurately, $N^+_{\Omega_R}$) denote the set consisting of all $\frac{\partial H^+}{\partial \nu_{\Omega_R}}$, where H is of class $\mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$, harmonic in Ω_R^{ext}, and regular at infinity.

By virtue of the first Green theorem (cf. Theorem 2.2), the solution of (ENP) can be shown to be uniquely determined, hence,

$$N^+ = \mathrm{C}(\Omega_R). \tag{10.131}$$

It can be formulated in terms of a single-layer potential

$$U(x) = \int_{\Omega_R} Q(y) \frac{1}{|x-y|} \, d\omega(y), \quad Q \in \mathrm{C}(\Omega_R), \tag{10.132}$$

such that Q satisfies the integral equations

$$F = \frac{\partial U^+}{\partial \nu_{\Omega_R}} = \left(-2\pi I + P_{|1}(0,0)\right) Q, \tag{10.133}$$

where

$$P_{|1}(0,0)Q(x) = \frac{\partial}{\partial \nu(x)} \int_{\Omega} Q(y) \frac{1}{|x-y|} \, d\omega(y). \tag{10.134}$$

Setting

$$T = \left(-2\pi I + P_{|1}(0,0)\right) \tag{10.135}$$

we obtain

$$\text{kern } (T^*) = \{0\}, \tag{10.136}$$

$$T\left(\mathrm{C}(\Omega_R)\right) = N^+. \tag{10.137}$$

By completion,

$$\mathrm{L}^2(\Omega_R) = \overline{N^+}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}}. \tag{10.138}$$

Analogous arguments, of course, hold for the inner boundary-value problems. The details are left to the reader. A more comprehensive treatment of classical potential theory may be found in standard textbooks, e.g., O.D. Kellogg (1929), W. Walter (1971).

10.5 Gravity Anomalies and Deflections of the Vertical

The traditional concept of physical geodesy (for more details, the reader is referred to the survey article given by, e.g., R. Rummel (1992)) is based on the assumption that all over the Earth, the position (e.g., latitude and longitude) and scalar gravity g are available. Moreover, it is common practice that the gravitational effects of the sun and moon and of the Earth's atmosphere are accounted for by means of corrections. The gravitational part of the gravity potential can then be regarded as a harmonic function. A classical approach to gravity field modeling was conceived by G.G. Stokes (1849). He proposed reducing the given gravity accelerations from the Earth's surface to the geoid. As the geoid is a level surface, its potential value is constant. The difference between the reduced gravity on the geoid and the reference gravity on the so-called normal ellipsoid is called the gravity anomaly. The disturbing potential, i.e., the difference between the actual and the reference potential, can be obtained from a (third) boundary value problem of potential theory. Its solution is representable in integral form, i.e., by the Stokes integral. The disadvantage of the Stokes approach is that the reduction to the geoid requires the introduction of assumptions concerning the unknown mass distribution between the Earth's surface and the geoid.

In this paper, we briefly recapitulate the classical approach to global gravity field determination due to Stokes (1849), Bruns (1878), and Neumann (1887) by formulating the differential/integral relations between gravity disturbance, gravity anomaly, vertical deflections on the one hand, and the disturbing potential and the geoidal undulations on the other hand. The representation of the disturbing potential in terms of gravity disturbances, gravity anomalies, and deflections of the vertical are written in terms of well-known integral representations over the geoid. For practical purposes, the integrals are replaced by approximate formulas using certain integration weights and knots within a spherical framework.

Equipotential surfaces of the gravity potential W allow, in general, no simple representation. This is the reason why a reference surface, in physical geodesy usually an ellipsoid of revolution, is chosen for the (approximate) construction of the geoid. As a matter of fact, the deviations of the gravity field of the Earth from the normal field of such an ellipsoid are small. The remaining parts of the gravity field are gathered in a so-called *disturbing gravity field* ∇T corresponding to the *disturbing potential* T. Knowing the gravity potential, all equipotential surfaces – including the geoid – are given by an equation of the form $W(x) = \text{const}$. By introducing U as the normal gravity potential corresponding to the ellipsoidal field and T as the disturbing potential (for details see, e.g., E. Groten (1979), W.A. Heiskanen, H. Moritz (1967), W. Torge (1991)) we are led to a decomposition of the gravity potential in the form

$$W = U + T \tag{10.139}$$

such that

(1) the center of the ellipsoid coincides with the center of gravity of the Earth,

(2) the difference of the mass of the Earth and the mass of the ellipsoid is zero.

Consequently, in accordance with the classical approach (see, e.g., E. Groten (1979); W.A. Heiskanen, H. Moritz (1967); W. Torge (1991)), T is given in such a way that

(1)

$$\int_{\Omega_R} T(y) H^R_{-1,0}(y)\, d\omega(y) = 0, \tag{10.140}$$

(2)

$$\int_{\Omega_R} T(y) H^R_{-2,k}(y)\, d\omega(y) = 0,\ k = 1, 2, 3. \tag{10.141}$$

The series expansion of T in terms of scalar (outer) harmonics (see Figs. 10.14 and 10.15) is given by

$$T(x) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} T^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(x), \quad x \in \Omega_R^{\mathrm{ext}}, \tag{10.142}$$

where $T^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)$ is given by

$$T^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) = \int_{\Omega_R} T(y) H^R_{-n-1,k}(y)\, d\omega(y). \tag{10.143}$$

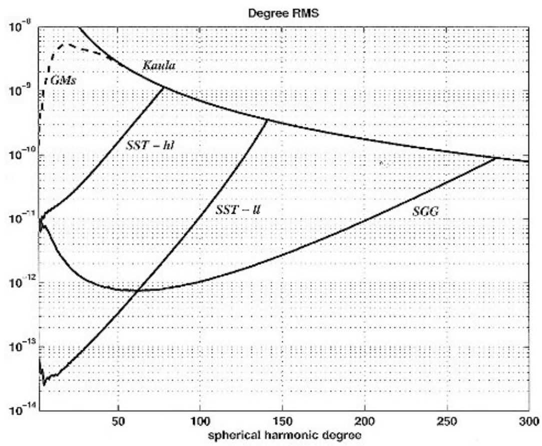


Fig. 10.15: Degree variances $\sum_{n=1}^{2n+1} T^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)$ for the anomalous potential derived from satellite data (see ESA (1998)).

A point x of the geoid is projected onto the point y of the ellipsoid by means of the ellipsoidal normal (see Fig. 10.16). The distance between x and y is called the *geoidal height*, or *geoidal undulation*.

The *gravity anomaly vector* is defined as the difference between the gravity vector $w(x)$ and the normal gravity vector $u(y)$, $u = \nabla U$, i.e.,

$$\alpha(x) = w(x) - u(y) \tag{10.144}$$

(see Fig. 10.16). It is also possible to subtract the vectors w and u at the same point x to get the *gravity disturbance vector*

$$\delta(x) = w(x) - u(x). \tag{10.145}$$

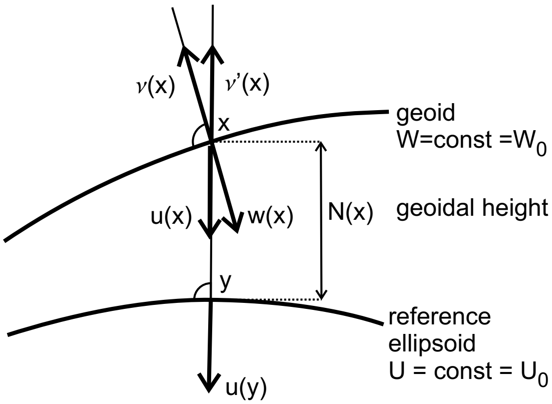


Fig. 10.16: Illustration of the definition of the gravity anomaly vector $\alpha(x) = w(x) - u(y)$ and the gravity disturbance vector $\delta(x) = w(x) - u(x)$.

Of course, several basic mathematical relations between the quantities just mentioned are known. In what follows, we only describe heuristically the fundamental relations (in spherical nomenclature). We start by observing that the gravity disturbance vector at the point x can be written as

$$\delta(x) = w(x) - u(x) = \nabla(W(x) - U(x)) = \nabla T(x). \tag{10.146}$$

Expanding the potential U at x according to Taylor's theorem and truncating the series at the linear term, we get

$$U(x) \doteq U(y) + \frac{\partial U}{\partial \nu'}(y) N(x) \tag{10.147}$$

($\doteq$ means approximation in linearized sense). Here, $\nu'(y)$ is the ellipsoidal normal at y, i.e., $\nu'(y) = -u(y)/\gamma(y)$, $\gamma(y) = |u(y)|$, and the geoid undulation $N(x)$, as indicated in Fig. 10.16, is the aforementioned distance between x and y, i.e., between the geoid and the reference ellipsoid. Using

$$\begin{aligned} \gamma(y) &= |u(y)| = -\nu'(y) \cdot u(y) \\ &= -\nu'(y) \cdot \nabla U(y) = -\frac{\partial U}{\partial \nu'}(y) \end{aligned} \tag{10.148}$$

we arrive at

$$\begin{aligned} N(x) &= \frac{T(x) - (W(x) - U(y))}{|u(y)|} \\ &= \frac{T(x) - (W(x) - U(y))}{\gamma(y)}. \end{aligned} \tag{10.149}$$

Letting $U(y) = W(x) = \text{const} = W_0$ we obtain the so-called *Bruns' formula* (cf. E.H. Bruns (1878))

$$N(x) = \frac{T(x)}{\gamma(y)}. \tag{10.150}$$

It should be noted that Bruns' formula (10.150) relates the physical quantity T to the geometric quantity N.

In what follows, we are interested in introducing the deflections of the vertical of the gravity disturbing potential T. For this purpose, let us consider the vector field $\nu(x) = -w(x)/|w(x)|$. This gives us the identity (with $g(x) = |w(x)|$ and $\gamma(x) = |u(x)|$)

$$w(x) = \nabla W(x) = -|w(x)|\, \nu(x) = -g(x)\nu(x). \tag{10.151}$$

Furthermore, we have

$$u(x) = \nabla U(x) = -|u(x)|\, \nu'(x) = -\gamma(x)\nu'(x). \tag{10.152}$$

The *deflection of the vertical* $\Theta(x)$ at the point x on the geoid is defined to be the angular (i.e., tangential) difference between the directions $\nu(x)$ and $\nu'(x)$, i.e., the plumb line and the ellipsoidal normal through the same point:

$$\Theta(x) = \nu(x) - \nu'(x) - ((\nu(x) - \nu'(x)) \cdot \nu(x))\, \nu(x). \tag{10.153}$$

Clearly, because of (10.153), $\Theta(x)$ is orthogonal to $\nu(x)$, i.e., $\Theta(x) \cdot \nu(x) = 0$. Since the plumb lines are orthogonal to the level surfaces of the geoid and the ellipsoid, respectively, the deflections of the vertical give, briefly spoken, a measure of the gradient of the level surfaces. This aspect will be described in more detail below: From (10.151) we obtain, in connection with (10.153),

$$\begin{aligned} w(x) &= \nabla W(x) \qquad (10.154) \\ &= -|w(x)| \left(\Theta(x) + \nu'(x) + ((\nu(x) - \nu'(x)) \cdot \nu(x))\nu(x)\right). \end{aligned}$$

Altogether, we get for the gravity disturbance vector

$$\begin{aligned} w(x) - u(x) &= \nabla T(x) && (10.155) \\ &= -|w(x)| \left(\Theta(x) + \left((\nu(x) - \nu'(x)) \cdot \nu(x)\right) \nu(x)\right) \\ &\quad - (|w(x)| - |u(x)|)\, \nu'(x). \end{aligned}$$

The magnitude

$$D(x) = |w(x)| - |u(x)| = g(x) - \gamma(x) \quad (10.156)$$

is called the *gravity disturbance*, while

$$A(x) = |w(x)| - |u(y)| = g(x) - \gamma(y) \quad (10.157)$$

is called the *gravity anomaly*.

Since the vector $\nu(x) - \nu'(x)$ is (almost) orthogonal to $\nu'(x)$, physical geodesy tells us that it can be neglected in (10.155). Hence, it follows that

$$\begin{aligned} w(x) - u(x) &= \nabla T(x) && (10.158) \\ &\doteq -|w(x)|\Theta(x) - (|w(x)| - |u(x)|)\, \nu'(x). \end{aligned}$$

The gradient $\nabla T(x)$ can be split into a normal part (pointing into the direction of $\nu(x)$) and an angular (tangential) part (characterized by the surface gradient ∇^*). It follows that

$$\nabla T(x) = \frac{\partial T}{\partial \nu}(x)\nu(x) + \frac{1}{|x|}\nabla^* T(x). \quad (10.159)$$

By comparison of (10.158) and (10.159), we therefore obtain

$$D(x) = g(x) - \gamma(x) = |w(x)| - |u(x)| = -\frac{\partial T}{\partial \nu'}(x), \quad (10.160)$$

i.e., the gravity disturbance, beside being the difference in magnitude of the actual and the normal gravity vector, is also the normal component of the gravity disturbance vector. In addition, we are led to the angular, i.e., (tangential) differential equation

$$\frac{1}{|x|}\nabla^* T(x) = -|w(x)|\; \Theta(x). \quad (10.161)$$

Since $|\Theta(x)|$ is a small quantity, it may be (without loss of precision) multiplied either by $-|w(x)|$ or by $-|u(x)|$, i.e., by $-g(x)$ or by $-\gamma(x)$.

The reference ellipsoid deviates from a sphere only by quantities of the order of the flattening. Therefore, in numerical calculations, if we treat the reference ellipsoid as a sphere Ω_R (with mean radius R as defined by, e.g., W.A. Heiskanen, H. Moritz (1967), B. Hofmann–Wellenhof, H. Moritz (2005)), this may cause a relative error of the same order (for more details, the reader is referred to standard textbooks of physical geodesy (e.g., W.A. Heiskanen, H. Moritz (1967)). If this error is permissible, we are allowed to replace $|u(R\xi)|$ by its spherical approximation GM/R^2 such that

$$\nabla^*_\xi T(R\xi) = -\frac{GM}{R}\Theta(R\xi), \tag{10.162}$$

where G is the gravitational constant and M is the constant of the mass.

By virtue of Bruns' formula, we finally find the relation between geoidal undulations and deflections of the vertical

$$\frac{GM}{R^2}\nabla^*_\xi N(R\xi) = -\frac{GM}{R}\Theta(R\xi), \quad \xi \in \Omega, \tag{10.163}$$

i.e.,

$$\nabla^*_\xi N(R\xi) = -R\,\Theta(R\xi), \quad \xi \in \Omega. \tag{10.164}$$

In other words, the knowledge of the geoid undulations allows the determination of the deflections of the vertical by taking the surface gradient on the unit sphere.

From the identity (10.160), it follows that

$$\begin{aligned} -\frac{\partial T}{\partial \nu'}(x) = D(x) &= |w(x)| - |\gamma(x)| \\ &\doteq |w(x)| - |\gamma(y)| - \frac{\partial \gamma}{\partial \nu'}(y)\, N(x) \\ &= A(x) - \frac{\partial \gamma}{\partial \nu'}(y)\, N(x), \end{aligned} \tag{10.165}$$

where A represents the scalar gravity anomaly as defined by (10.157). Observing Bruns' formula we get

$$A(x) = -\frac{\partial T}{\partial \nu'}(x) + \frac{1}{\gamma(y)}\frac{\partial \gamma}{\partial \nu'}(y)\, T(x). \tag{10.166}$$

In well-known *spherical approximation*, we have (see, e.g., W.A. Heiskanen, H. Moritz (1967))

$$\gamma(y) = |u(y)| = \frac{GM}{|y|^2}, \tag{10.167}$$

$$\frac{\partial \gamma}{\partial \nu'}(y) = \frac{y}{|y|}\cdot \nabla_y \gamma(y) = -2\frac{GM}{|y|^3} \tag{10.168}$$

and

$$\frac{1}{\gamma(y)}\frac{\partial \gamma}{\partial \nu'}(y) = -\frac{2}{|y|}. \tag{10.169}$$

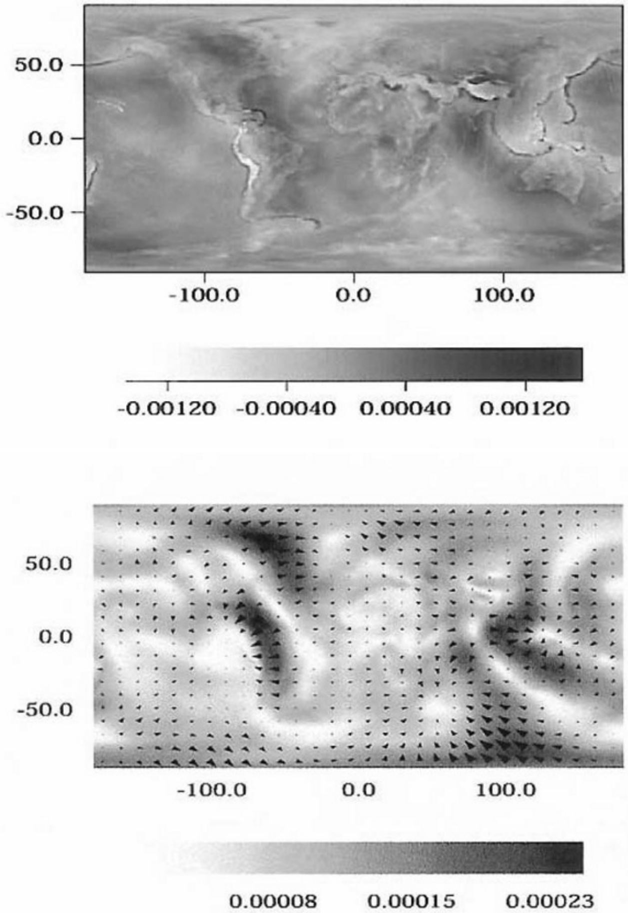


Fig. 10.17: The normal derivative [in 100 Gal] and the surface gradient illustrated for EGM96 (in 100 Gal), Geomathematics Group, TU Kaiserslautern, S. Beth (2000).

This leads us to the basic relations (cf. Figs. 10.17, 10.20 and 10.21)

$$-D(x) = \frac{x}{|x|} \cdot \nabla T(x), \quad x \in \Omega_R, \tag{10.170}$$

and

$$-A(x) = \frac{x}{|x|} \cdot \nabla T(x) + \frac{2}{|x|} T(x), \quad x \in \Omega_R, \tag{10.171}$$

as so-called *fundamental equations of physical geodesy.*

In the sense of physical geodesy (cf., e.g., A.A. Aardalan, E.W. Grafarend, G. Finn (2006), W.A. Heiskanen, H. Moritz (1967)), the meaning of the spherical approximation should be carefully kept in mind. It is used only for expressions relating to small quantities of the disturbing potential, the geoidal undulations, the gravity disturbances, the gravity anomalies, etc. Actually, in all geodetic approaches, the reference surface will never be understood to be a sphere in any geometrical sense, but it always is an ellipsoid. However, as the flattening of this ellipsoid is very small, the ellipsoidal formulas are expandable into Taylor series in terms of the flattening, and then all terms containing higher order expressions of the flattening may be neglected. In this way, together with suitable pre-reduction processes of gravity, formulas are obtained that are rigorously valid for the sphere.

Remark 10.11. In physical geodesy, the deflections of the vertical are usually decomposed into mutually perpendicular scalar components, and the so-called Vening Meinesz' kernel comes into play (see W.A. Heiskanen, H. Moritz (1967)). In fact, there are various distinctions in the introduction of the deflections of the vertical (see, e.g., C. Jekeli (1999), W.E. Featherstone, J.M. Rüeger (2000), E.W. Grafarend (2001), W. Torge (1991)). Recently, an ellipsoidally based approach to gravity field modeling is given by A.A. Aardalan, E.W. Grafarend, G. Finn (2006) (see also the references therein).

Remark 10.12. The advantages of 'zooming-in' techniques for global and/or local approximation in physical geodesy is discussed appropriately, e.g., in E. Groten (2003) (see also the references therein).

Since the disturbing potential T is a harmonic function in Ω_R^{ext}, we are confronted with boundary-value problems of potential theory to determine T in Ω_R^{ext} from prescribed gravity disturbance D or the gravity anomaly A, respectively.

Remark 10.13. It should be noted that, at the present state of practice, much more gravity anomalies are available than gravity disturbances. In future, because of GPS, it may be expected that the gravity disturbances become more important than the gravity anomalies (for more details see, e.g., B. Hofmann–Wellenhof, H. Moritz (2005)). This is the reason why both problems will be discussed here.

As is well known, standard methods for solving boundary–value problems corresponding to a spherical boundary are as follows:

(1) The expansion method in terms of outer harmonics,

(2) The representation by means of layer-potentials.

In what follows, we explain these methods both for the (modified) exterior Neumann problem and for the exterior Stokes problem. Furthermore, we deal with new procedures of regularization for the integral expression of the solution on the boundary Ω_R, respectively.

Expansion method in terms of outer harmonics. The determination of the disturbing potential T in Ω_r^{ext} from known gravity disturbances on Ω_R leads us to the (modified) Neumann boundary-value problem:

(Modified) Exterior Neumann Problem (ENP): We are given $D \in \mathrm{C}(\Omega_R)$ with

$$\int_{\Omega_R} D(y) H^R_{-n-1,k}(y) d\omega(y) = 0$$

$n = 0, 1;$ and $k = 1, \ldots, 2n+1$. Then the function $T : \overline{\Omega_R^{\text{ext}}} \to \mathbb{R}$ given by

$$T(x) = \frac{R}{4\pi} \int_{\Omega_R} N(x,y) D(y) \; d\omega(y) \tag{10.172}$$

with the *Neumann kernel function*

$$N(x,y) = \frac{2R}{|x-y|} + \ln\left(\frac{|x| + |x-y| - R}{|x| + |x-y| + R}\right), \tag{10.173}$$

is the unique solution of the exterior Neumann boundary-value problem:

(i) T is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $T \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$,

(ii) T is harmonic on Ω_R^{ext}, i.e., $\Delta T = 0$ in Ω_R^{ext},

(iii) T is regular at infinity, i.e., $|T(x)| = O\left(\frac{1}{|x|}\right), |\nabla T(x)| = O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$,

(iv) $\int_{\Omega_R} T(y) H^R_{-n-1,k}(y) \; d\omega(y) = 0$, $n = 0, 1;$ and $k = 1, \ldots, 2n+1$.

(v) $-\frac{x}{|x|} \cdot \nabla_x T(x) = D(x), \quad x \in \Omega_R$.

The solution T can be represented by a Fourier series expansion in terms of outer harmonics

$$T = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \frac{R}{n+1} D^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k}, \tag{10.174}$$

where

$$D^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) = \int_{\Omega_R} D(y) H^R_{-n-1,k}(y) \; d\omega(y), \tag{10.175}$$

$n = 2, 3, \ldots$, $k = 1, \ldots, 2n+1$, where the series expansion is absolutely and uniformly convergent on each subset $K \subset \Omega_R^{\text{ext}}$ with $\operatorname{dist}(\overline{K}, \Omega_R) > 0$.

For points $x, y \in \Omega_R$, we (formally) get the so-called *Neumann formula* which is an improper integral over Ω_R

$$T\left(\frac{Rx}{|x|}\right) = \frac{1}{4\pi R}\int_{\Omega_R}\left(\frac{\sqrt{2}}{\sqrt{1-\frac{x}{|x|}\cdot\frac{y}{|y|}}} + \ln\left(\frac{\sqrt{2-2\frac{x}{|x|}\cdot\frac{y}{|y|}}}{2+\sqrt{2-2\frac{x}{|x|}\cdot\frac{y}{|y|}}}\right)\right) D\left(\frac{Ry}{|y|}\right)\, d\omega(y). \tag{10.176}$$

Note that the surface integral (10.176) indeed has to be extended over the whole surface. In accordance with our approach, it is valid under the following assumptions: (i) The mass within the reference ellipsoid is equal to the mass of the Earth, (ii) The potential of the geoid and the reference ellipsoid are equal, (iii) The center of the reference ellipsoid is coincident with the center of the Earth, (iv) There are no masses outside, (v) The approximation is simplified in spherical sense.

The identity (10.176) formulated in an equivalent way over the unit sphere Ω yields

$$T(R\xi) = \frac{R}{4\pi}\int_{\Omega} N(\xi\cdot\eta)D(R\eta)\, d\omega(\eta), \quad \xi\in\Omega, \tag{10.177}$$

where the *Neumann kernel* is given by

$$N(\xi\cdot\eta) = \frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}} - \ln\left(1+\frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}}\right),\ 1-\xi\cdot\eta\neq 0. \tag{10.178}$$

Note that

$$N(R\xi, R\eta) = N(\xi\cdot\eta), \quad \xi,\eta\in\Omega. \tag{10.179}$$

The essential idea now is that the improper integral (10.177) can be regularized, e.g., by replacing the zonal kernel (cf. W. Freeden, K. Wolf (2008))

$$S(\xi\cdot\eta) = \frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}}, \qquad 1-\xi\cdot\eta\neq 0, \tag{10.180}$$

via the space-regularized zonal kernel (see Figs. 10.18 and 10.19)

$$S^{\rho}(\xi\cdot\eta) = \begin{cases} \frac{R}{\rho}\left(3-\frac{2R^2}{\rho^2}(1-\xi\cdot\eta)\right), & 0\le 1-\xi\cdot\eta\le\frac{\rho^2}{2R^2} \\ \frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}}, & \frac{\rho^2}{2R^2} < 1-\xi\cdot\eta\le 2. \end{cases} \tag{10.181}$$

$$
\begin{aligned}
&T^\rho(R\xi) \qquad (10.182)\\
&= \frac{R}{4\pi} \int_{1-\xi\cdot\eta > \frac{\rho^2}{2R^2}} \frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}} D(R\eta)\, d\omega(\eta)\\
&- \frac{R}{4\pi} \int_{1-\xi\cdot\eta > \frac{\rho^2}{2R^2}} \ln\left(1+\frac{\sqrt{2}}{\sqrt{1-\xi\cdot\eta}}\right) D(R\eta)\, d\omega(\eta)\\
&+ \frac{R}{4\pi} \int_{1-\xi\cdot\eta \le \frac{\rho^2}{2R^2}} \frac{R}{\rho}\left(3-\frac{2R^2}{\rho^2}(1-\xi\cdot\eta)\right) D(R\eta)\, d\omega(\eta)\\
&- \frac{R}{4\pi} \int_{1-\xi\cdot\eta \le \frac{\rho^2}{2R^2}} \ln\left(1+\frac{R}{\rho}\left(3-\frac{2R^2}{\rho^2}(1-\xi\cdot\eta)\right)\right) D(R\eta)\, d\omega(\eta).
\end{aligned}
$$

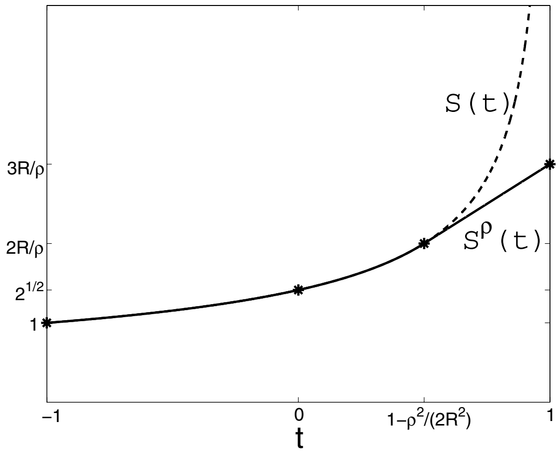


Fig. 10.18: The function S and its ‘regularization’ S^ρ on the intervals $[-1,1)$ and $[-1,1]$, respectively.

In other words, a low-pass filtered version of T is given by

$$T^\rho(R\xi) = \frac{R}{4\pi} \int_\Omega N^\rho(\xi\cdot\eta) D(R\eta)\, d\omega(\eta), \qquad (10.183)$$

where the regularized Neumann kernel reads as follows

$$N^\rho(\xi\cdot\eta) = S^\rho(\xi\cdot\eta) - \ln(1+S^\rho(\xi\cdot\eta)), \quad \xi,\eta\in\Omega. \qquad (10.184)$$

Note (cf. Fig. 10.20) that $t \mapsto S^\rho(t), t \in [-1,1]$, given by

$$S^\rho(t) = \begin{cases} \frac{R}{\rho}\left(3 - \frac{2R^2}{\rho^2}(1-t)\right), & 0 \le 1-t \le \frac{\rho^2}{2R^2} \\ \frac{\sqrt{2}}{\sqrt{1-t}}, & \frac{\rho^2}{2R^2} < 1-t \le 2 \end{cases} \quad (10.185)$$

is continuously differentiable. Moreover, we have (cf. Fig. 10.18)

$$S\left(1 - \frac{\rho^2}{2R^2}\right) = S^\rho\left(1 - \frac{\rho^2}{2R^2}\right) \quad (10.186)$$

and

$$S'\left(1 - \frac{\rho^2}{2R^2}\right) = (S^\rho)'\left(1 - \frac{\rho^2}{2R^2}\right). \quad (10.187)$$

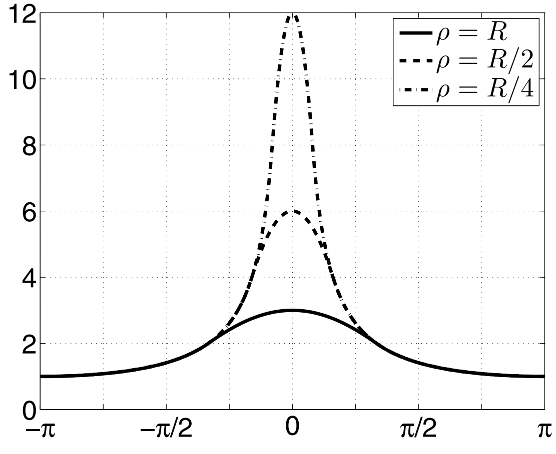


Fig. 10.19: The kernel $\vartheta \mapsto S^\rho(\cos\vartheta)$ (10.181) for several ρ.

Furthermore, S and S^ρ are monotonically increasing with

$$S(t) \ge S^\rho(t) \ge 1 \quad (10.188)$$

for all $t \in [-1,1)$. Thus,

$$S(t) - S^\rho(t) = \begin{cases} \frac{\sqrt{2}}{\sqrt{1-t}} - \frac{R}{\rho}\left(3 - \frac{2R^2}{\rho^2}(1-t)\right), & 0 < 1-t \le \frac{\rho^2}{2R^2} \\ 0, & \frac{\rho^2}{2R^2} < 1-t \le 2. \end{cases} \quad (10.189)$$

Elementary calculations give

$$\begin{aligned}\int_{-1}^{1} |S(t) - S^{\rho}(t)| \ dt &= \int_{1-\frac{\rho^2}{2R^2}}^{1} (S(t) - S^{\rho}(t)) \ dt \\ &= O(\rho) \end{aligned} \quad (10.190)$$

as $\rho \to 0$, hence, it follows that

$$\lim_{j\to\infty} \int_{-1}^{1} |S(t) - S^{\rho_j}(t)| \ dt = 0. \quad (10.191)$$

Observing the properties of the functions S and S^{ρ}, we find

$$|\ln(S(t)) - \ln(S^{\rho}(t))| \leq |S(t) - S^{\rho}(t)| \quad (10.192)$$

and

$$|\ln(1 + S(t)) - \ln(1 + S^{\rho}(t))| \leq \frac{1}{2} |S(t) - S^{\rho}(t)| . \quad (10.193)$$

Consequently, we have

$$\int_{\Omega} |\ln(1 + S(\xi \cdot \eta)) - \ln(1 + S^{\rho}(\xi \cdot \eta))| \ d\omega(\eta) = O(\rho). \quad (10.194)$$

Since $D(R\cdot) : \Omega \to \mathbb{R}$ is continuous and, therefore, uniformly bounded on Ω_R, we finally obtain in connection with (10.194)

$$\begin{aligned}&\lim_{\rho\to 0} \sup_{\xi\in\mathcal{T}} |T(R\xi) - T^{\rho}(R\xi)| \quad (10.195)\\ &= \lim_{\rho\to 0} \sup_{\xi\in\mathcal{T}} \frac{R}{4\pi} \left| \int_{\Omega} (N(\xi \cdot \eta) - N^{\rho}(\xi \cdot \eta)) D(R\eta) \ d\omega(\eta) \right| \\ &= 0\end{aligned}$$

for all subsets $\mathcal{T} \subset \Omega$.

The determination of the disturbing potential T on Ω_R^{ext} from known gravity anomalies on Ω_R leads us to Stokes boundary-value problem.

Exterior Stokes Problem (ESP): We are given $A \in \mathrm{C}(\Omega_R)$ with

$$\int_{\Omega_R} A(y) H_{-n-1,k}(y) \ d\omega(y) = 0, \quad n = 0, 1, \ k = 1, \ldots, 2n + 1. \quad (10.196)$$

Then, the function $T : \overline{\Omega_R^{\mathrm{ext}}} \to \mathbb{R}$ given by

$$T(x) = \frac{R}{4\pi} \int_{\Omega_R} A(y) St(x, y) \ d\omega(y) \quad (10.197)$$

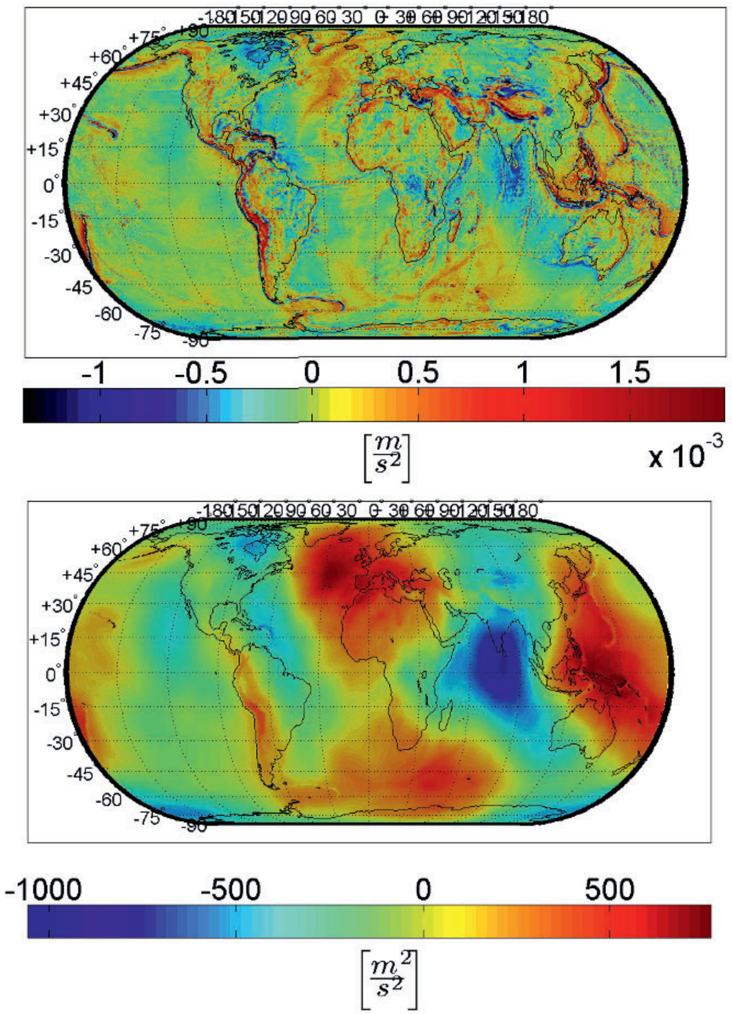


Fig. 10.20: EIGEN-GL04C derived gravity disturbances and disturbing potential (reconstructed by use of smoothed Haar scaling functions, Geomathematics Group, TU Kaiserslautern, D. Mathar (2008)).

with the *Stokes kernel function (briefly called Stokes kernel)*

$$\begin{aligned} St(x,y) \quad = \quad & \frac{R}{|x|} + \frac{2R}{|x-y|} - \frac{5R^2}{|x|^2}\frac{x}{|x|}\cdot\frac{y}{|y|} - \frac{3R}{|x|^2}|x-y| \\ & - 3\frac{R^2}{|x|^2}\frac{x}{|x|}\cdot\frac{y}{|y|}\ln\left(\frac{|x| - R\frac{x}{|x|}\cdot\frac{y}{|y|} + |x-y|}{2|x|}\right) \end{aligned} \tag{10.198}$$

is the unique solution of the exterior Stokes boundary-value problem(see, e.g., W. Freeden (1978a)):

(i) T is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in A_{ext}, i.e., $U \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$,

(ii) T is harmonic in Ω_R^{ext}, i.e., $\Delta T = 0$ in Ω_R^{ext},

(iii) T is regular at infinity,

(iv) $\int_{\Omega_R} T(y) H^R_{-n-1,k}(y)\, d\omega(y) = 0$, $n = 0, 1,\ k = 1, \ldots, 2n+1$,

(v) $-\frac{x}{|x|} \cdot \nabla_x T(x) - \frac{2}{|x|} T(x) = A(x), \quad x \in \Omega_R$.

The potential T can be represented by a Fourier series expansion in terms of outer harmonics

$$T = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \frac{R}{n-1} A^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) H^R_{-n-1,k} \tag{10.199}$$

with

$$A^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) = \int_{\Omega_R} A(y) H^R_{-n-1,k}(y)\, d\omega(y), \tag{10.200}$$

$n = 2, 3, \ldots, k = 1, \ldots, 2n+1$, and the series expansion is absolutely and uniformly convergent on each subset $K \subset \Omega_R^{\text{ext}}$with $\operatorname{dist}(\overline{K}, \Omega_R) > 0$.

For points $x, y \in \Omega_R$, we (formally) get an analogue to the Neumann formula, called *Stokes' formula*, which again represents an improper integral over Ω_R

$$T(R\xi) = \frac{1}{4\pi R} \int_{\Omega_R} St(R\xi, R\eta) A(R\eta)\, d\omega(R\eta). \tag{10.201}$$

Equivalently, we have

$$T(R\xi) = \frac{R}{4\pi} \int_{\Omega} St(\xi \cdot \eta) A(R\eta)\, d\omega(\eta), \tag{10.202}$$

where

$$\begin{aligned} St(\xi \cdot \eta) \quad &= \quad S(\xi \cdot \eta) - 6(S(\xi \cdot \eta))^{-1} + 1 - 5\xi \cdot \eta \\ &\qquad -3\xi \cdot \eta \ln \left(\frac{1}{S(\xi \cdot \eta)} + \frac{1}{(S(\xi \cdot \eta))^2} \right). \end{aligned} \tag{10.203}$$

Note that

$$St(R\xi, R\eta) = St(\xi \cdot \eta), \quad \xi, \eta \in \Omega. \tag{10.204}$$

From Bruns' formula

$$N(R\xi) = T(R\xi) \frac{R^2}{GM} \tag{10.205}$$

we get the geodial undulations (see Fig. 10.21).

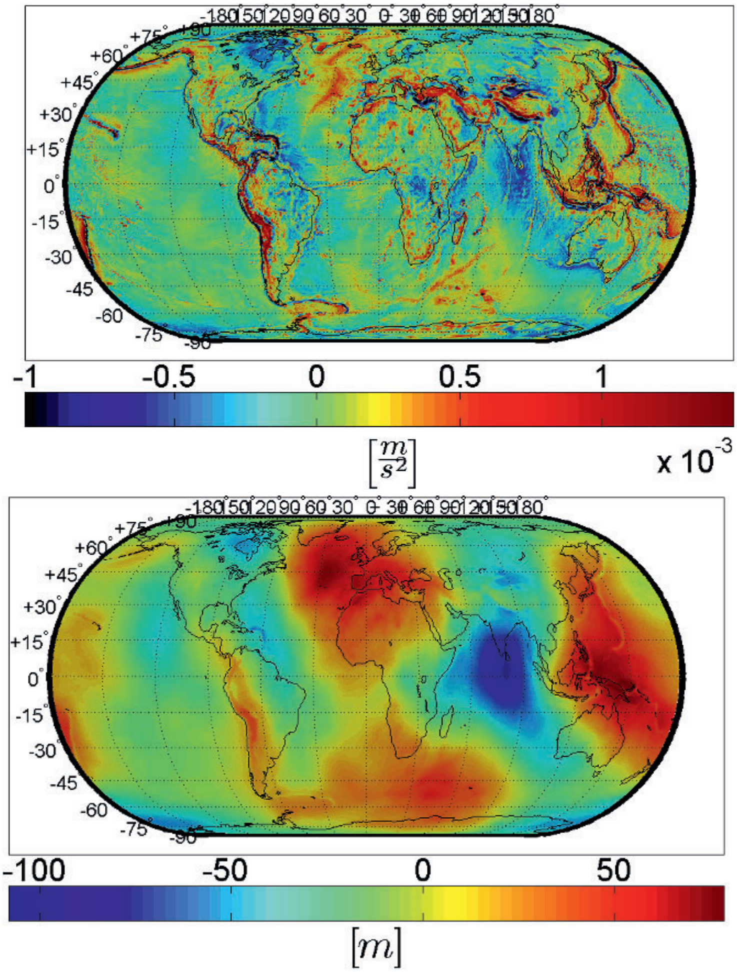


Fig. 10.21: EIGEN-GL04C derived gravity anomalies and geoidal undulations (reconstructed by use of smoothed Haar scaling functions, Geomathematics Group, TU Kaiserslautern, D. Mathar (2008)).

Again, the improper integral (10.202) can be regularized, e.g., by replacing the zonal kernel S (see (10.180) by the space-regularized zonal kernel S^ρ (see (10.181)).

In fact, the regularization (10.181) leads us to the following regularized global representation of the disturbing potential corresponding to gravity anomalies as boundary data (see W. Freeden, K. Wolf (2008))

$$T^\rho(R\xi) = \frac{R}{4\pi} \int_\Omega St^\rho(\xi \cdot \eta) A(R\eta) \, d\omega(\eta) \tag{10.206}$$

with (see Fig. 10.22)

$$St^\rho(\xi\cdot\eta) = S^\rho(\xi\cdot\eta) - 6(S(\xi\cdot\eta))^{-1} + 1 - 5\xi\cdot\eta - 3\xi\cdot\eta\ \ln\left(\frac{1}{S^\rho(\xi\cdot\eta)} + \frac{1}{(S^\rho(\xi\cdot\eta))^2}\right). \quad (10.207)$$

The integrands of $T(R\xi)$ and $T^\rho(R\xi)$ differ only on the spherical cap $\mathcal{S}^\rho(\xi) = \{\eta\in\Omega \big| 1-\xi\cdot\eta \leq \frac{\rho^2}{2R^2}\}$. Here we have

$$\begin{aligned} & St(\xi\cdot\eta) - St^\rho(\xi\cdot\eta) \\ &= (S(\xi\cdot\eta) - S^\rho(\xi\cdot\eta)) \\ &\quad -3\xi\cdot\eta\ \ln\left(\frac{1}{S(\xi\cdot\eta)} + \frac{1}{(S(\xi\cdot\eta))^2}\right) \\ &\quad +3\xi\cdot\eta\ \ln\left(\frac{1}{S^\rho(\xi\cdot\eta)} + \frac{1}{(S^\rho(\xi\cdot\eta))^2}\right). \end{aligned} \quad (10.208)$$

Now it follows that for all $t\in[-1,1)$ with $1-t\leq\frac{\rho^2}{2R^2}$

$$\begin{aligned} & \ln\left(\frac{1}{S(t)} + \frac{1}{(S(t))^2}\right) - \ln\left(\frac{1}{S^\rho(t)} + \frac{1}{(S^\rho(t))^2}\right) \\ &= \ln(1+S(t)) - \ln(1+S^\rho(t)) \\ &\quad -2\left(\ln(S(t)) - \ln(S^\rho(t))\right). \end{aligned} \quad (10.209)$$

Furthermore, by use of the already known properties of the functions S and S^ρ on $[-1,1)$, we get

$$\begin{aligned} & \left|\ln\left(\frac{1}{S(t)} + \frac{1}{(S(t))^2}\right) - \ln\left(\frac{1}{S^\rho(t)} + \frac{1}{(S^\rho(t))^2}\right)\right| \\ &= O(|S(t) - S^\rho(t)|). \end{aligned} \quad (10.210)$$

In connection with (10.189), we therefore find

$$\lim_{\rho\to 0}\sup_{\xi\in\mathcal{T}} |T(R\xi) - T^\rho(R\xi)| = 0 \quad (10.211)$$

for every subset $\mathcal{T}\subset\Omega$.

Remark 10.14. The last identity can be used to guarantee a multiscale approximation by locally supported scalar zonal wavelets (see W. Freeden, K. Wolf (2008)). Hence, a new efficient and economical method has been found for determining geoid undulations from local data, i.e., gravity anomalies.

The gravity anomalies obtained from EGM96 (see F.G. Lemoine et al. (1998)) are shown in Fig. 10.21. In the gravity anomalies, all significantly tectonic processes become visible. In accordance with Newton's law, the

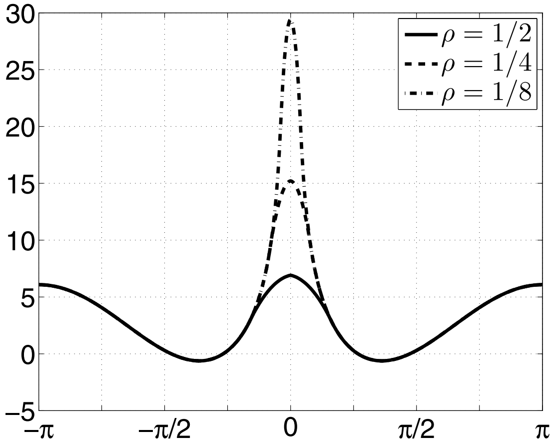


Fig. 10.22: The regularized Stokes kernel $\vartheta \mapsto St^{\rho}(\cos\vartheta)$ for several ρ.

gravity anomalies and gravity disturbances permit the conclusion of an irregular density distribution inside the Earth. Unfortunately, gravity anomalies do not determine uniquely the interior density distribution of the Earth (this point will be made clear later on, when both density and gravity by virtue of the Poisson integral representation will be explained in more detail). Geoid undulations (see Fig. 10.21) are the measure of the pertubations in the hydrostatic equilibrium. They do not show essential correlations to the distribution of the continents.

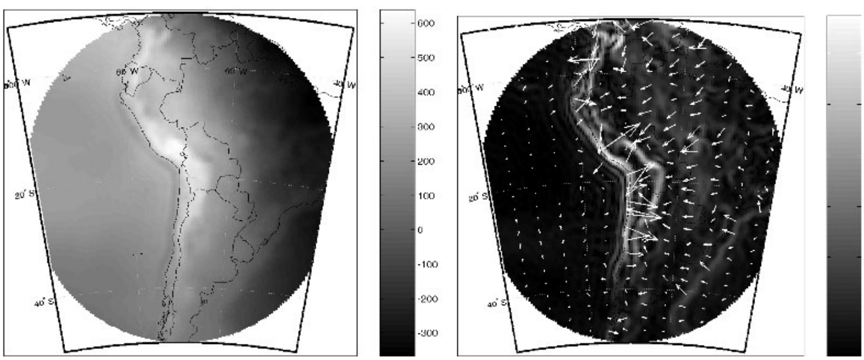


Fig. 10.23: Plot of the regularized EGM-geopotential in $[\mathrm{m}^2/\mathrm{s}^2]$ (*left*), the deflections of the vertical (*right*) in $[\mathrm{m}/\mathrm{s}^2]$, Geomathematics Group, TU Kaiserslautern, T. Fehlinger et al. (2007).

The (unique) solution T of the differential equation for the surface gradient (cf. Figs. 10.17 and 10.23)

$$\nabla_\xi^* T(R\xi) = -\frac{GM}{R}\Theta(R\xi), \quad \xi \in \Omega, \tag{10.212}$$

satisfying

$$\int_\Omega T(R\xi)\, d\omega(\xi) = 0, \tag{10.213}$$

$$\int_\Omega T(R\xi)(\xi \cdot \varepsilon^k)\, d\omega(\xi) = 0, \quad k = 1, 2, 3, \tag{10.214}$$

can be formulated in terms of the Green function with respect to the Beltrami operator given by (cf. Section 4.1)

$$G(\xi \cdot \eta) = 1 + \ln\left(\frac{1}{(S(\xi \cdot \eta))^2}\right), \qquad 1 - \xi \cdot \eta \neq 0, \tag{10.215}$$

as follows

$$T(R\xi) = \frac{GM}{4\pi R}\int_\Omega \nabla_\eta^* G(\xi \cdot \eta) \cdot \Theta(R\eta)\; d\omega(\eta), \quad \xi \in \Omega. \tag{10.216}$$

An easy calculation yields

$$\begin{aligned} \nabla_\eta^* G(\xi \cdot \eta) &= \nabla_\eta^*(1 - 2\ln(S(\xi \cdot \eta))) \\ &= -2\frac{S'(\xi \cdot \eta)}{S(\xi \cdot \eta)}(\xi - (\xi \cdot \eta)\eta) \\ &= -\frac{1}{2}(S(\xi \cdot \eta))^2(\xi - (\xi \cdot \eta)\eta). \end{aligned} \tag{10.217}$$

Thus, it follows that

$$T(R\xi) = \frac{R}{4\pi}\int_\Omega g(\xi, \eta) \cdot \Theta(R\eta)\; d\omega(\eta), \tag{10.218}$$

where

$$g(\xi, \eta) = -\frac{GM}{2R^2}(S(\xi \cdot \eta))^2\,(\xi - (\xi \cdot \eta)\eta)\,, \quad \xi, \eta \in \Omega. \tag{10.219}$$

Replacing S by S^ρ, we get as regularization T^ρ of T corresponding to deflections of the vertical as data set

$$T^\rho(R\xi) = \frac{R}{4\pi}\int_\Omega g^\rho(\xi, \eta) \cdot \Theta(R\eta)\; d\omega(\eta), \tag{10.220}$$

where

$$g^\rho(\xi, \eta) = -\frac{GM}{2R^2}(S^\rho(\xi \cdot \eta))^2(\xi - (\xi \cdot \eta)\eta), \quad \xi, \eta \in \Omega. \tag{10.221}$$

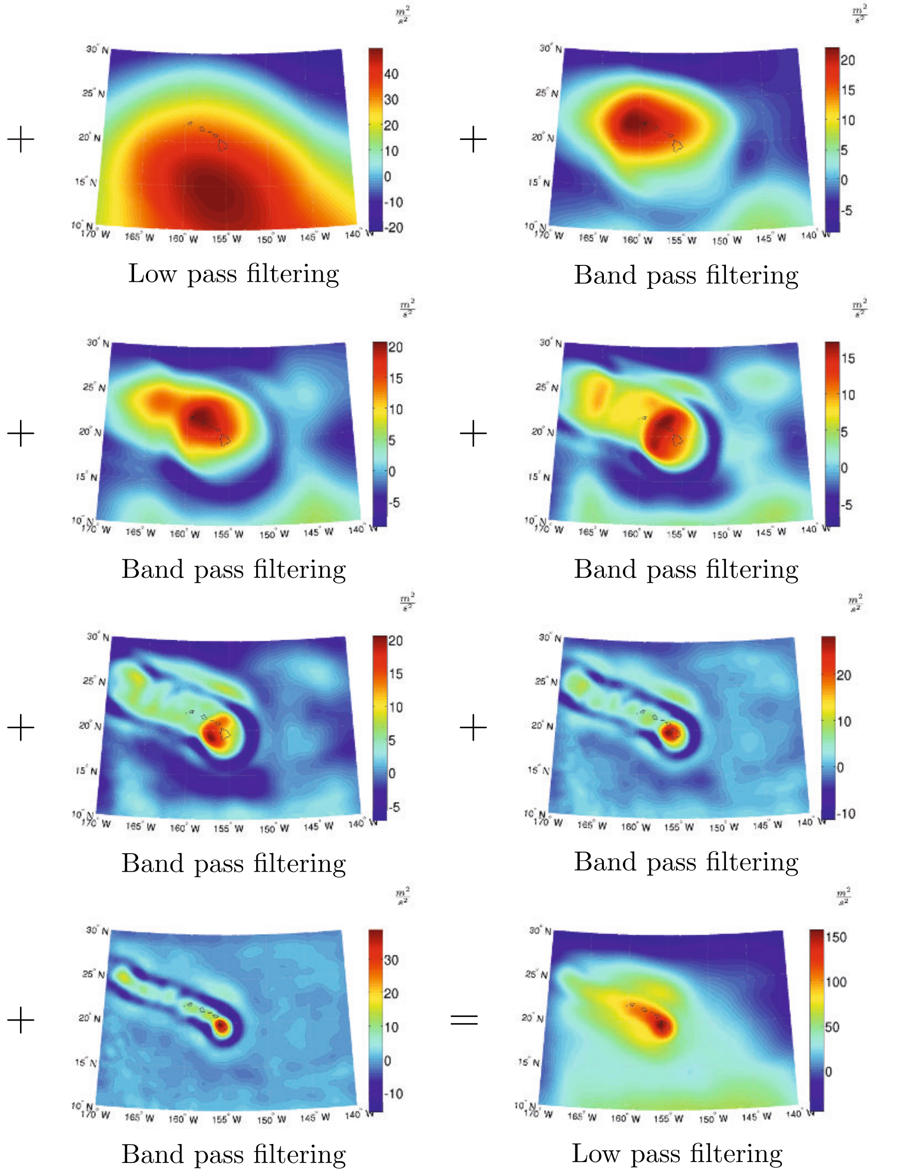


Fig. 10.24: Multiscale reconstruction of the regularized anomalous potential T in $[\mathrm{m}^2\mathrm{s}^{-2}]$ from vertical deflections for the Hawaiian (*plume*) area using regularized vector Green functions. (A rough low pass filtering at scale 6 is improved with several band pass filters of scale 6,...,11. The last illustration shows the approximation of T at scale $J = 12$), Geomathematics Group, TU Kaiserslautern, T. Fehlinger (2008).

From the properties known for S and S^ρ, we are able to derive that

$$\begin{aligned} &\int_\Omega \left((S(\xi\cdot\eta))^2 - (S^\rho(\xi\cdot\eta))^2\right)(\xi - (\xi\cdot\eta)\eta)\cdot\Theta(R\eta)\,d\omega(\eta) \\ &\quad= \int_{1-\frac{\rho^2}{2R^2}\le\xi\cdot\eta\le 1} \left((S(\xi\cdot\eta))^2 - (S^\rho(\xi\cdot\eta))^2\right) \\ &\qquad\times \sqrt{1-(\xi\cdot\eta)^2}\cdot\frac{\xi-(\xi\cdot\eta)\eta}{|\xi-(\xi\cdot\eta)\eta|}\cdot\Theta(R\eta)\,d\omega(\eta) \\ &\quad= O(\rho) \end{aligned} \tag{10.222}$$

provided that $\Theta(R\cdot)$ is a continuous vector field on Ω. Consequently, we have

$$\lim_{\rho\to 0}\sup_{\xi\in\mathcal{T}} |T(R\xi) - T^\rho(R\xi)| = 0 \tag{10.223}$$

for all subsets $\mathcal{T}\subset\Omega$.

Representation by layer potentials. We again begin with the (modified) Neumann problem.

(Modified) Exterior Neumann Problem (ENP): We are given $D\in \mathrm{C}(\Omega_R)$ with

$$\int_{\Omega_R} D(y)H^R_{-n-1,k}(y)d\omega(y) = 0$$

$n = 0, 1, k = 1, \ldots, 2n+1$.

Let N_2^+ denote the set consisting of all $\frac{\partial H^+}{\partial\nu_{\Omega_R}}$ corresponding to the functions $H : \overline{\Omega_R^{\mathrm{ext}}} \to \mathbb{R}$ satisfying the following conditions:

(i) H is continuously differentiable in $\overline{\Omega_R^{\mathrm{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $H\in \mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}})\cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$,
(ii) H is harmonic in Ω_R^{ext}, i.e., $\Delta H = 0$ in Ω_R^{ext},
(iii) H is regular at infinity,
(iv) $\int_{\Omega_R} H(y)H^R_{-n-1,k}(y)\ d\omega(y) = 0$, $n = 0, 1,\ k = 1,\ldots,\ 2n+1$.

Obviously, the potential T satisfying the properties

(i) T is continuously differentiable in $\overline{\Omega_R^{\mathrm{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $T\in \mathrm{C}^{(1)}(\overline{\Omega_R^{\mathrm{ext}}})\cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$,
(ii) T is harmonic on Ω_R^{ext}, i.e., $\Delta T = 0$ in Ω_R^{ext},
(iii) T is regular at infinity, i.e., $|T(x)| = O\left(\frac{1}{|x|}\right), |\nabla T(x)| = O\left(\frac{1}{|x|^2}\right)$ as $|x|\to\infty$,

(iv) $\int_{\Omega_R} T(y) H^R_{-n-1,k}(y) \, d\omega(y) = 0, \ n = 0,1, \ k = 1, \ldots, \ 2n+1.$
(v) $\frac{x}{|x|} \cdot \nabla_x T(x) = -D(x), \quad x \in \Omega_R.$

is uniquely determined, hence,

$$N_2^+ = \mathrm{C}_2(\Omega_R), \tag{10.224}$$

where $\mathrm{C}_2(\Omega_R)$ is the space of all $G \in \mathrm{C}(\Omega_R)$ with

$$\int_{\Omega_R} G(y) H^R_{-n-1,k}(y) \, d\omega(y) = 0, \ n = 0,1, \ k = 1, \ldots, 2n+1. \tag{10.225}$$

T can be formulated in terms of a single-layer potential

$$T(x) = \int_{\Omega_R} Q(y) \frac{1}{|x-y|} \, d\omega(y), \quad Q \in \mathrm{C}_2(\Omega_R), \tag{10.226}$$

such that $Q \in \mathrm{C}_2(\Omega_R)$ satisfies the integral equations

$$-D = \frac{\partial T^+}{\partial \nu_{\Omega_R}} = \left(-2\pi I + P_{|1}(0,0)\right) Q, \tag{10.227}$$

where (cf. (10.134))

$$P_{|1}(0,0) Q(x) = \frac{\partial}{\partial \nu(x)} \int_{\Omega_R} Q(y) \frac{1}{|x-y|} \, d\omega(y). \tag{10.228}$$

Setting

$$T = \left(-2\pi I + P_{|1}(0,0)\right) \tag{10.229}$$

we obtain

$$\mathrm{kern}\ (T^*) = \{0\}, \tag{10.230}$$

$$T\left(\mathrm{C}_2(\Omega_R)\right) = N_2^+. \tag{10.231}$$

By completion,

$$\mathrm{L}^2_2(\Omega_R) = \overline{N_2^+}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}}, \tag{10.232}$$

where $\mathrm{L}^2_2(\Omega_R)$ is the space of all $G \in \mathrm{L}^2(\Omega_R)$ with

$$\int_{\Omega_R} G(y) H^R_{-n-1,k}(y) \, d\omega(y) = 0, \ n = 0,1, \ k = 1, \ldots, 2n+1. \tag{10.233}$$

Exterior Stokes Problem (ESP): We are given $A \in \mathrm{C}_2(\Omega_R)$.

Let S_2^+ denote the set consisting of all $\frac{\partial H^+}{\partial \nu_{\Omega_R}} + \frac{2}{R} H|\Omega_R$ corresponding to functions $H : \overline{\Omega_R^{\mathrm{ext}}} \to \mathbb{R}$ satisfying the following conditions:

(i) H is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in Ω_R^{ext}, i.e., $H \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$,

(ii) H is harmonic in Ω_R^{ext}, i.e., $\Delta H = 0$ in Ω_R^{ext},

(iii) H is regular at infinity,

(iv) $\int_{\Omega_R} H(y) H^R_{-n-1,k}(y)\ d\omega(y) = 0$, $n = 0, 1,\ k = 1, \ldots, 2n+1$.

Obviously, the potential T satisfying the properties

(i) T is continuously differentiable in $\overline{\Omega_R^{\text{ext}}}$ and twice continuously differentiable in A_{ext}, i.e., $U \in \mathrm{C}^{(1)}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$,

(ii) T is harmonic in Ω_R^{ext}, i.e., $\Delta T = 0$ in Ω_R^{ext},

(iii) T is regular at infinity,

(iv) $\int_{\Omega_R} T(y) H^R_{-n-1,k}(y)\ d\omega(y) = 0$, $n = 0, 1,\ k = 1, \ldots, 2n+1$,

(v) $\frac{x}{|x|} \cdot \nabla_x T(x) + \frac{2}{|x|} T(x) = -A(x), \quad x \in \Omega_R$.

is uniquely determined, hence,

$$S_2^+ = \mathrm{C}_2(\Omega_R). \tag{10.234}$$

T can be formulated in terms of a single-layer potential

$$T(x) = \int_{\Omega_R} Q(y) \frac{1}{|x-y|}\ d\omega(y), \quad Q \in \mathrm{C}_2(\Omega_R), \tag{10.235}$$

such that $Q \in \mathrm{C}_2(\Omega_R)$ satisfies the integral equations

$$-A = \frac{\partial T^+}{\partial \nu_{\Omega_R}} + \frac{2}{R} T = \left(-2\pi I + P_{|1}(0,0) + \frac{2}{R} P(0,0) \right) Q, \tag{10.236}$$

where (see (10.134))

$$P(0,0)Q(x) = \int_{\Omega} Q(y) \frac{1}{|x-y|}\ d\omega(y) \tag{10.237}$$

and

$$P_{|1}(0,0)Q(x) = \frac{\partial}{\partial \nu(x)} \int_{\Omega} Q(y) \frac{1}{|x-y|}\ d\omega(y). \tag{10.238}$$

Setting

$$T = \left(-2\pi I + P_{|1}(0,0) + \frac{2}{R} P(0,0) \right) \tag{10.239}$$

we obtain

$$\text{kern}\ (T^*) = \{0\}, \tag{10.240}$$

$$T\left(\mathrm{C}_2(\Omega_R)\right) = S_2^+. \tag{10.241}$$

By completion,

$$\mathrm{L}_2^2(\Omega_R) = \overline{S_2^+}^{\|\cdot\|_{\mathrm{L}^2(\Omega_R)}}. \tag{10.242}$$

Remark 10.15. It should be remarked that, in all integral expressions both for the (modified) exterior Neumann problem and for the exterior Stokes problem, the kernel $|x-y|^{-1}$ can be replaced by

$$\frac{1}{|x-y|} - \frac{1}{|x|} - \frac{1}{|x|}\left(\frac{|y|}{|x|}\right)\frac{|x|}{|x|}\cdot\frac{|y|}{|y|} \tag{10.243}$$

(cf. (10.12)).

On the sphere Ω_R, the (improper) layer-integrals occurring in the solution process yielding the disturbing potential from known (discretely given) gravity disturbances and gravity anomalies, respectively, can be regularized by use of the kernel function (10.181). The resulting regularized integral equations (10.227), (10.236), respectively, can be solved, e.g., by collocation, least squares techniques, or Galerkin approximation (note that S as well as S^ρ as defined by (10.180) and (10.181) are scalar zonal functions). Even more, the whole discretization process corresponding to discrete data can be formulated as multiscale procedure (in analogy to the algorithm proposed by W. Freeden, C. Mayer (2003)).

10.6 Geostrophic Ocean Flow and Dynamic Ocean Topography

First, we are interested in deriving the known set of equations describing the dynamic of a fluid from the physical laws of conservation. For discussing the local and total time derivation, we start from a field u, which depends on a space and time variable and which is assumed to be differentiable with respect to each variable. Applying the Taylor expansion up to the first order, we obtain

$$u(x+\delta x, t+\delta t) = u(x,t) + (\delta x \cdot \nabla_x)u(x,t) + (\frac{\partial}{\partial t}u(x,t))\delta t, \tag{10.244}$$

where, as usual, δx and δt are infinitesimal displacements in space and time. Setting $\delta x = x(t+\delta t) - x(t)$ and observing the limit

$$\lim_{\delta t \to 0} \frac{\partial x}{\partial t} = v(x,t), \tag{10.245}$$

where $v(x,t)$ is the velocity of the fluid under consideration, we are led to the total time derivative of u

$$\begin{aligned} \frac{d}{dt}u(x,t) &= \lim_{\delta t \to 0} \frac{u(x(t+\delta t), t+\delta t) - u(x,t)}{\delta t} \\ &= \lim_{\delta t \to 0} (\frac{\partial x}{\partial t} \cdot \nabla_x) u(x,t) + \frac{\partial}{\partial t} u(x,t) \\ &= (v(x,t) \cdot \nabla_x) u(x,t) + \frac{\partial}{\partial t} u(x,t). \end{aligned} \tag{10.246}$$

In fluid mechanics, the term $\frac{du}{dt}$ characterizes the rate of change of u following a particle of the fluid. It is called the Lagrangian time derivative of u. The term $\frac{\partial u}{\partial t}$ is called the Eulerian time derivative. It indicates the rate of change of u at a fixed point in a coordinate frame.

Let us assume that the inner space Ω_R^{int} of the sphere Ω_R with radius R around the origin is occupied by a fluid. If $\rho : (x,t) \to \rho(x,t)$, $\rho(x,t)$ is the time and space dependent density of the fluid at position $x \in \Omega_R^{\mathrm{int}}$ and time $t \geq 0$, then the mass of the fluid enclosed by the sphere Ω_R at the time t is $\int_{\Omega_R^{\mathrm{int}}} \rho(x,t) dV(x)$ and the rate of mass across the sphere is given by $\int_{\Omega_R} \rho(x,t) v(x,t) \cdot \nu(x) d\omega(x)$, where $v(x,t)$ is the velocity of the fluid at the point x and time t (ν is the unit normal vector field on Ω_R pointing into the outer space Ω_R^{ext}).

The *conservation of mass* of the fluid is guaranteed by the balance equation

$$\frac{\partial}{\partial t} \int_{\Omega_R^{\mathrm{int}}} \rho(x,t)\; dV(x) = - \int_{\Omega_R} \rho(x,t)\; v(x,t) \cdot \nu(x)\; d\omega(x). \tag{10.247}$$

By observing the time derivative in the Eulerian framework

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v(x,t) \cdot \nabla_x \tag{10.248}$$

we consequently find via the Theorem of Gauss

$$\frac{\partial}{\partial t} \rho(x,t) + \nabla \cdot (\rho(x,t) v(x,t)) = 0. \tag{10.249}$$

or

$$\frac{d}{dt} \rho(x,t) + \nabla \cdot (\rho(x,t) v(x,t)) - v(x,t) \cdot \nabla_x \rho(x,t) = 0. \tag{10.250}$$

The *conservation of momentum* states that the rate of change in the momentum is equal to the sum of the total volume force and the total surface force acting on the fluid. In detail,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_R^{\mathrm{int}}} \rho(x,t) v(x,t)\; dV(x) &= \int_{\Omega_R^{\mathrm{int}}} f(x,t)\; \rho(x,t)\; dV(x) \\ &\quad + \int_{\Omega_R} \mathbf{f}(x,t) \nu(x)\; d\omega(x), \end{aligned} \tag{10.251}$$

where $\mathbf{f}$ is the stress tensor. Assuming the incompressibility of the fluid, i.e. that the time changes of the density $\frac{d}{dt}\rho(x,t)$ are negligible, we find in connection with the Theorem of Gauss

$$\begin{aligned}\int_{\Omega_R^{\mathrm{int}}} \rho(x,t)\frac{d}{dt}v(x,t)\, dV(x) &= -\int_{\Omega_R^{\mathrm{int}}} f(x,t)\rho(x,t)\, dV(x) \\ &+ \int_{\Omega_R^{\mathrm{int}}} \nabla_x \cdot \mathbf{f}(x,t)\, dV(x).\end{aligned} \tag{10.252}$$

Since the last equation is also valid for all subsets in $\overline{\Omega_R^{\mathrm{ext}}}$, we are able to deduce that

$$\rho(x,t)\frac{d}{dt}v(x,t) = \rho(x,t)f(x,t) + \nabla_x \cdot \mathbf{f}(x,t). \tag{10.253}$$

This differential equation connects the acceleration of a fluid with the volume force and the stress tensor.

Note that the application of the divergence to a tensor of rank 2 is understood rowwise, i.e., for $\mathbf{f}(x,t) = (f_{ij}(x,t))_{i,j=1,2,3}$,

$$\nabla_x \cdot \mathbf{f}(x,t) = \left(\sum_{j=1}^{3} \frac{\partial}{\partial x_j} f_{ij}(x,t)\right)_{i=1,2,3} \in \mathbb{R}^3. \tag{10.254}$$

The third fundamental physical axiom, *the conservation of angular momentum*, implies the symmetry of the Cauchy stress tensor $\mathbf{f}$, i.e.,

$$\mathbf{f}(x,t) = (\mathbf{f}(x,t))^T. \tag{10.255}$$

Equivalently,

$$f_{ij}(x,t) = f_{ji}(x,t) \tag{10.256}$$

for all $i,j = 1,2,3$.

It should be remarked that the fourth fundamental axiom of physics, the conservation of energy, does not play a role in this context. Some aspects of the behavior of the medium in motion are characterizable by the properties of the Cauchy stress tensor.

In a fluid at rest, the principal stresses are all the same and equal at all points of the fluid, i.e., the stress tensor of a fluid at rest is isotropic and only normal stresses act on the fluid. The fluid is normally in a state of compression and therefore the stress tensor can be written as

$$\mathbf{f}(x,t) = -P(x,t)\mathbf{i}, \tag{10.257}$$

where $P(x,t) = -1/3(f_{11}(x,t) + f_{22}(x,t) + f_{33}(x,t))$ is the *hydrostatic pressure* (which is generally a function of the variable x).

In a fluid motion, the Cauchy stress tensor $\mathbf{f}$ is usually expressed as a sum of the isotropic part $-P(x,t)\mathbf{i}$ and the remaining non-isotropic part $\mathbf{d}(x,t)$ as follows

$$\mathbf{f}(x,t) = -P(x,t)\mathbf{i} + \mathbf{d}(x,t). \tag{10.258}$$

The expression of the non-isotropic stress field is usually written as

$$\mathbf{d}(x,t) = 2\mu\left(\mathbf{e}(x,t) - \frac{1}{3}(\nabla_x \cdot v(x,t))\mathbf{i}\right), \tag{10.259}$$

where μ is the viscosity of the fluid. The first part in (10.259) is called the rate of strain tensor and the second part is called the rate of expansion tensor. The rate of strain reads

$$\mathbf{e}(x,t) = \frac{1}{2}(\nabla_x \otimes v(x,t) + (\nabla_x \otimes v(x,t))^T). \tag{10.260}$$

Using the expression (10.259) for $\mathbf{d}$, the total stress tensor becomes

$$\begin{aligned} &\mathbf{f}(x,t) \\ &= -P(x,t)\mathbf{i} + 2\mu(x,t)\left(\mathbf{e}(x,t) - \frac{1}{3}(\nabla_x \cdot v(x,t))\mathbf{i}\right). \end{aligned} \tag{10.261}$$

Substituting this equation for the stress tensor into the equation of motion (10.253), we get

$$\begin{aligned} &\rho(x,t)\frac{d}{dt}v(x,t) \\ &= \rho(x,t)f(x,t) - \nabla_x P(x,t) + \nabla\cdot\left(2\mu(x,t)\left(\mathbf{e}(x,t) - \frac{1}{3}(\nabla_x \cdot v(x,t))\mathbf{i}\right)\right). \end{aligned}$$

The last equation is called the *Navier–Stokes equation of motion.* If the viscosity μ is supposed to be uniform over the fluid and constant in time, (10.262) can be formulated as follows

$$\begin{aligned} &\rho(x,t)\frac{d}{dt}v(x,t) \\ &= f(x,t)\rho(x,t) - \nabla_x P(x,t) + \mu\left(\Delta v(x,t) + \frac{1}{3}\nabla_x(\nabla_x \cdot v(x,t))\right). \end{aligned}$$

If incompressible flow is assumed, the equation of mass conservation leads to $\nabla \cdot v(x,t) = 0$ for all x and t. The Navier–Stokes equation then takes the form

$$\rho(x,t)\frac{d}{dt}v(x,t) = \rho(x,t)f(x,t) - \nabla_x P(x,t) + \mu\Delta_x v(x,t). \tag{10.262}$$

By introducing the kinematic viscosity $\nu = \mu\ \rho$, the last equation is representable in the form

$$\frac{d}{dt}v(x,t) = f(x,t) - \frac{1}{\rho(x,t)}\nabla_x P(x,t) + \nu \Delta_x v(x,t). \tag{10.263}$$

For incompressible fluids (with $\frac{d}{dt}\rho(x,t) = 0$), it is sufficient to determine v and P, if additional boundary and initial conditions are given (see, e.g., R. Temam (1979)). For a spherical approach and its numerical realization, cf. M.J. Fengler, W. Freeden (2005), M.J. Fengler (2005)).

Remark 10.16. In order to compare the different magnitudes in (10.263), the equation is written in dimensionless variables. For a representative length L and a representative velocity V, we define

$$v' = \frac{v}{V}, \quad x' = \frac{x}{L}, \quad t' = t\frac{V}{L}, \quad P' = \frac{P - P_0}{\rho V^2}, \tag{10.264}$$

where P_0 is a representative value of the pressure of the fluid. The *Reynolds number* is defined by

$$\mathrm{Re} = \frac{\rho L V}{\mu} = \frac{LV}{\nu}. \tag{10.265}$$

In the dimensionless coordinates, the Navier–Stokes equation can be formulated as

$$\frac{d}{dt}v'(x',t') = -\nabla'_x P(x',t') + \frac{1}{\mathrm{Re}}\Delta'_x v(x',t'), \quad \nabla'_x \cdot v'(x',t') = 0. \tag{10.266}$$

After renaming the variables, this equation becomes in the Eulerian framework

$$\frac{\partial}{\partial t}v(x,t) + (v(x,t)\cdot\nabla_x)v(x,t) = -\nabla_x P(x,t) + \frac{1}{\mathrm{Re}}\Delta_x v(x,t), \quad \nabla_x \cdot v(x,t) = 0. \tag{10.267}$$

In the context of the representative magnitudes, the convection term $(v(x,t)\cdot \nabla_x)^T v(x,t)$ is of magnitude V^2/L and the diffusion term $\Delta_x v(x,t)$ is of magnitude V/L^2. Consequently, the Reynolds number is a measure for the relative magnitude of the convective and the viscous forces. More explicitly,

$$\mathrm{Re} \approx \frac{(v(x,t)\cdot\nabla_x)v(x,t)}{\nu\Delta_x v(x,t)}. \tag{10.268}$$

Re essentially smaller than 1, i.e., $\mathrm{Re} \ll 1$ means that the inertia force is much smaller than the viscous force, such that the viscous force and the pressure force are dominant in (10.267), whereas Re essentially larger than 1, i.e., $\mathrm{Re} \gg 1$ tells us that the inertia force is dominant.

The *Stokes flow* problem is characterized by very small velocities, respectively, by very low Reynolds numbers Re such that the non linear convection term can be neglected. In the case, the equation (10.267) takes the simplified form

$$\frac{\partial}{\partial t} v(x,t) = -\nabla_x P(x,t) + \frac{1}{\mathrm{Re}} \Delta_x v(x,t), \quad \nabla_x \cdot v(x,t) = 0. \tag{10.269}$$

If, in addition, the flow is steady, i.e., v and P do not depend on t, the whole inertia flow is small in magnitude compared to pressure forces and to viscous forces. Thus, the inertia force is negligible and (10.269) simplifies to

$$\Delta_x v(x) = -\nabla_x P(x), \quad \nabla_x \cdot v(x) = 0, \tag{10.270}$$

which is the so-called Stokes system of equations (for more details, the reader is referred to C. Mayer (2007)).

After these general preparations, we now come to the particular problem of *modeling the ocean flow*:

The point of departure in the Eulerian framework is the so-called *Euler equation of motion* of a fluid in the form

$$\rho(x,t) \left(\frac{\partial}{\partial t} v(x,t) + (v(x,t) \cdot \nabla_x) v(x,t) \right) = \rho(x,t) f(x,t) + \nabla_x \cdot \mathbf{f}(x,t), \tag{10.271}$$

where

- ρ is the scalar mass density of the medium,
- v is the velocity of the medium,
- f is the (body) force field acting on the medium,
- $\mathbf{f}$ is the Cauchy stress tensor.

Under the assumption of a perfect (ideal) fluid, we are allowed to require the property that $\mathbf{f}(x,t)x$ is parallel to x for all $x \in \mathbb{R}^3 \backslash \{0\}$. In fact, for a perfect fluid, we have

$$\mathbf{f}(x,t) = -P(x,t)\mathbf{i}. \tag{10.272}$$

In addition, the relevant exterior forces may be assumed to consist of the following ingredients:

- the *gravity force*, $\nabla_x W(x,t)$, where, as usual, W is the sum of the gravitational and the centrifugal potential,
- the *Coriolis force*, $-2\rho\ \omega \wedge v(x,t)$, where ω is the angular velocity of the Earth's rotation,

- the *frictional force*, f_{fric}, is due to the geometry, especially the boundary and the internal forces.

Thus, we arrive at the following equation

$$\begin{aligned}\rho(x,t)\left(\frac{\partial}{\partial t}v(x,t)+(v(x,t)\cdot\nabla_x)v(x,t)+2\omega\wedge v(x,t)\right)\\ = \quad -\rho(x,t)\nabla_x W(x,t)-\nabla_x P(x,t). \qquad (10.273)\end{aligned}$$

Even more, for purposes of modeling the ocean flow, the last equation may be reduced by further approximations:

Large scale approximation: For large scale ocean flow, the non-linearity does not play a significant role, i.e., the term $(v(x,t)\cdot\nabla_x)v(x,t)$ can be neglected.

Hydrostatic approximation: The scale of the vertical motion is small compared with the scale of the horizontal motion. The vertical Coriolis acceleration due to the horizontal motion is neglected in the vertical momentum equation, as well as the inertial and frictional forces. The assumption of hydrostatic equilibrium filters out non-hydrostatic gravity waves.

Coriolis force approximation: The horizontal Coriolis acceleration due to the vertical motion is neglected in the horizontal momentum equations. The approximations above leave only the horizontal Coriolis acceleration due to the horizontal motion.

Boussinesq approximation: The vertical scale of the motion is small compared with the scale height. The density variations are neglected except in the vertical momentum equation when coupled to gravity.

With these approximations, the Euler equation changes drastically. Our task below is to extract explicitly the spherical currents out of these assumptions, i.e., to separate horizontal and vertical velocities.

In the sense of the Boussinesq approximation, oceanic water is assumed to be incompressible and homogeneous. In consequence, the density ρ is replaced by a mean density ρ_0. Further, in case of incompressible fluids, the continuity equation $\frac{\partial\rho}{\partial t}+\rho(\nabla\cdot v)=0$ changes to the equation

$$\nabla\cdot v=0 \qquad (10.274)$$

providing divergence–free motions. Under the assumption of incompressibility, observing the well-known splitting of the gradient $\nabla=\xi\frac{\partial}{\partial r}+\frac{1}{r}\nabla^*$, we are able to express vertical velocities by horizontal ones (see, e.g., D. Michel (2007)).

Theorem 10.17. *Let* $v : \Omega_r^{\text{ext}} \to \mathbb{R}$, $r > 0$, *be divergence free. Then*

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)(v(r\xi)\cdot\xi) = -\frac{1}{r}\left(\nabla^* \cdot p_{\tan}(v)\right). \tag{10.275}$$

Proof. The vector $v(r\xi)$, $\xi \in \Omega$, can be written as follows:

$$v(r\xi) = (v(r\xi)\cdot\varepsilon^r(\xi))\varepsilon^r(\xi) + (v(r\xi)\cdot\varepsilon(\xi))\varepsilon(\xi) + (v(r\xi)\cdot\varepsilon^t(\xi))\varepsilon^t(\xi). \tag{10.276}$$

Separating ∇ in radial and tangential parts, we have

$$\begin{aligned}
0 &= \nabla_{r\xi}\cdot v(r\xi) \qquad (10.277)\\
&= \varepsilon^r(\xi)\cdot\frac{\partial}{\partial r}v(r\xi) + \frac{1}{r}\nabla_\xi^*\cdot v(r\xi)\\
&= \varepsilon^r(\xi)\cdot\frac{\partial}{\partial r}\Big((v(r\xi)\cdot\varepsilon^r(\xi))\,\varepsilon^r(\xi) + (v(r\xi)\cdot\varepsilon^\varphi(\xi))\,\varepsilon^\varphi(\xi)\\
&\quad + \big(v(r\xi)\cdot\varepsilon^t(\xi)\big)\,\varepsilon^t(\xi)\Big)\\
&\quad + \frac{1}{r}\nabla_\xi^*\cdot\Big((v(r\xi)\cdot\varepsilon^r(\xi))\,\varepsilon^r(\xi) + (v(r\xi)\cdot\varepsilon^\varphi(\xi))\,\varepsilon^\varphi(\xi)\\
&\quad + \big(v(r\xi)\cdot\varepsilon^t(\xi)\big)\,\varepsilon^t(\xi)\Big).
\end{aligned}$$

The first part of this identity can be rewritten in the form

$$\begin{aligned}
&\varepsilon^r(\xi)\cdot\left(\left(\frac{\partial}{\partial r}v(r\xi)\cdot\varepsilon^r(\xi)\right)\varepsilon^r(\xi) + \left(\frac{\partial}{\partial r}v(r\xi)\cdot\varepsilon^\varphi(\xi)\right)\varepsilon^\varphi(\xi)\right.\\
&\quad + \left.\left(\frac{\partial}{\partial r}v(r\xi)\cdot\varepsilon^t(\xi)\right)\varepsilon^t(\xi)\right) \qquad (10.278)\\
&\quad = \varepsilon^r(\xi)\cdot\left(\frac{\partial}{\partial r}v(r\xi)\cdot\varepsilon^r(\xi)\right)\varepsilon^r(\xi)\\
&\quad = \frac{\partial}{\partial r}v(r\xi)\cdot\varepsilon^r(\xi) = \frac{\partial}{\partial r}\left(v(r\xi)\cdot\varepsilon^r(\xi)\right) \quad .
\end{aligned}$$

The second part allows the reformulation

$$\begin{aligned}
&\nabla_\xi^*\cdot\Big((v(r\xi)\cdot\varepsilon^r(\xi))\,\varepsilon^r(\xi) + (v(r\xi)\cdot\varepsilon^\varphi(\xi))\,\varepsilon^\varphi(\xi) \qquad (10.279)\\
&\quad + \big(v(r\xi)\cdot\varepsilon^t(\xi)\big)\,\varepsilon^t(\xi)\Big)\\
&= \underbrace{\nabla_\xi^*\left(v(r\xi)\cdot\varepsilon^r(\xi)\right)\cdot\varepsilon^r(\xi)}_{=0} + (v(r\xi)\cdot\varepsilon^r(\xi))\underbrace{\nabla_\xi^*\cdot\varepsilon^r(\xi)}_{=2}\\
&\quad + \nabla_\xi^*\cdot\Big(\underbrace{(v(r\xi)\cdot\varepsilon^\varphi(\xi))\,\varepsilon^\varphi(\xi) + \big(v(r\xi)\cdot\varepsilon^t(\xi)\big)\,\varepsilon^t(\xi)}_{=p_{\tan}(v(r\xi))}\Big).
\end{aligned}$$

Summarizing our calculations, we finally obtain

$$0 = \frac{\partial}{\partial r}\left(v(r\xi)\cdot\varepsilon^r(\xi)\right) + 2(v(r\xi)\cdot\varepsilon^r(\xi)) + \frac{1}{r}\nabla^*_\xi \cdot p_{\tan}(v(r\xi)). \qquad (10.280)$$

This is the desired result. □

For the shallow water approximation, the ocean is assumed to be a thin stratified layer with small aspect ratio (see, e.g., J. Pedlovsky (1979)), i.e., the fraction of vertical length scale D to horizontal length scale L satisfies

$$\frac{D}{L} \ll 1\,.$$

In doing so, we can also assume that small variations in the fluid occur mainly in horizontal direction, i.e., that the vertical velocity is by far smaller than its horizontal counterpart. By introducing a characteristic time scale, we obtain immediately characteristic values for tangential and for vertical velocities respectively, where the vertical velocities are considerably smaller than the horizontal ones. This leads us to a separation of the total velocity vector field into a tangential and a normal field in the form

$$v = v_{\tan} + v_{\mathrm{nor}}, \qquad (10.281)$$

where $v_{\tan} = p_{\tan}(v)$ is the tangential part of v and $v_{\mathrm{nor}} = p_{\mathrm{nor}}(v) = (v\cdot\xi)\,\xi$ is the normal part. In accordance with this decomposition, we obtain

$$\frac{d}{dt}v_{\tan}(r\xi,t) = -\frac{1}{\rho_0}\frac{1}{r}\nabla^*_\xi P(r\xi,t) - 2p_{\tan}(\omega\wedge v(r\xi,t)) + p_{\tan}(f_{\mathrm{fric}}), \qquad (10.282)$$

$$\frac{d}{dt}v_{\mathrm{nor}}(r\xi,t) = -\frac{1}{\rho_0}\frac{\partial}{\partial r}P(r\xi,t)\xi - 2p_{\mathrm{nor}}(\omega\wedge v(r\xi,t)) + (w(r\xi)\cdot\xi)\xi \qquad (10.283)$$

where $w = -\nabla W$.

For a decorrelation of these equations, we consider the Coriolis part explicitly. For being energetically consistent, we use the shallow water approximations and simplify this set of equations by stating some additional assumptions. Their detailed motivation can be found in J. Pedlovsky (1979).

The first assumption is based on the fact that $|v_{\mathrm{nor}}| \ll |v_{\tan}|$. Consequently, the expression $p_{\tan}(\omega\wedge v_{\mathrm{nor}})$ is very small. In fact, it follows that

$$\begin{aligned} |p_{\tan}(\omega\wedge v_{\mathrm{nor}})| &= |\omega\wedge v_{\mathrm{nor}} - (\underbrace{(\omega\wedge v_{\mathrm{nor}})\cdot\xi}_{=0})\xi| \qquad (10.284)\\ &\le |\omega||v_{\mathrm{nor}}|\sin\sphericalangle(\varepsilon^3,\xi), \end{aligned}$$

where $\sphericalangle(\varepsilon^3, \xi)$ is the angle between ε^3 and ξ, i.e., between the normalized versions of ω and v_{nor}. Since $p_{\tan}(\omega \wedge v_{\mathrm{nor}})$ is a part of the equation for $|v_{\tan}|$, the factors of the estimation have to be compared to it. We see that all are very small, but especially $|v_{\mathrm{nor}}|$. Hence, we are able to omit this term. Thus, (10.282) can be rewritten as follows

$$\frac{d}{dt} v_{\tan}(r\xi, t) = -\frac{1}{\rho_0}\frac{1}{r}\nabla^*_\xi P(r\xi, t) - 2p_{\tan}(\omega \wedge v_{\tan}(r\xi, t)) + p_{\tan}(f_{\mathrm{fric}}) \quad . \tag{10.285}$$

Moreover, $p_{\mathrm{nor}}(\omega \wedge v)$ is very small when compared to $(w(r\xi) \cdot \xi)\xi$, since we have

$$\begin{aligned} |p_{\mathrm{nor}}(\omega \wedge v)| &= |((\omega \wedge v) \cdot \xi)\xi| \\ &= |-(\omega \wedge \xi) \cdot v| \underbrace{|\xi|}_{=1} \\ &= \Big| - |\omega| |_2 (\varepsilon^3 \wedge \xi) \cdot (\underbrace{v_{\mathrm{nor}}}_{\|\xi} + v_{\tan}) \Big| \\ &= |\omega| \left| (\varepsilon^3 \wedge \xi) \cdot v_{\tan} \right| \\ &\leq |\omega| \sin \sphericalangle(\varepsilon^3, \xi) |v_{\tan}|, \end{aligned} \tag{10.286}$$

where $\sphericalangle(\varepsilon^3, \xi)$ again is the angle between ε^3 and ξ. This means that the vertical component of the Coriolis force is negligible with respect to the term $(w(r\xi) \cdot \xi)\xi$, since the rotation rate and the absolute horizontal velocity are very small. The remaining equation from (10.283) is given by

$$\frac{d}{dt} v_{\mathrm{nor}}(r\xi, t) = -\frac{1}{\rho_0}\frac{\partial}{\partial r} P(r\xi, t)\xi + (w(r\xi) \cdot \xi)\xi \tag{10.287}$$

The only two terms being significant in size within the last equation are gravity (with $w(r\xi) \cdot \xi$ assumed to be constant, W_0, for our purposes) and the radial variation of the pressure field. Therefore, we can assume that these two cancel out each other (see, e.g., J. Pedlovsky (1979)).

In the sense of the hydrostatic approximation, it is assumed that

$$\frac{\partial}{\partial r} P(r\xi) = W_0 \rho_0.$$

This is the reason why we are able to integrate the equation $\partial P / \partial r = W_0 \rho_0$ vertically within a small area slightly below the surface. We understand the ocean height along the ray to be described as follows:

$$P(r\xi) - P(R\xi) = W_0 \rho_0 \left(\int_r^R s \, ds \right) \xi + W_0 \rho_0 \int_R^{R+\Xi(\xi)} \, ds,$$

where $P(R\xi)$ is the atmospheric pressure at the ocean surface. Since we will not model wind-driven circulation here, we can assume that there are no pressure differences on the surface, i.e., that $P(R\xi)$ is constant and will, therefore, vanish within the dynamical equations. The first integral can, at least within the upper ocean, be assumed to be nearly constant, such that its surface gradient is very small compared to $W_0\rho_0\nabla^*_\xi\Xi(\xi)$. Thus, in the upper layer of the ocean, we finally arrive at

$$\nabla^*_\xi P(r\xi) = W_0\rho_0\nabla^*_\xi\Xi(\xi) \quad , \tag{10.288}$$

i.e., the horizontal pressure gradient is given by differences in the water column heights. In more detail, determining the distance $H(\xi)$ of a satellite to the sea surface by satellite altimetry, the difference to the satellite height $H_{\mathrm{sat}}(\xi)$ gives us the height $H_{\mathrm{ocean}}(\xi)$ on the ocean surface: $H_{\mathrm{ocean}}(\xi) = H_{\mathrm{sat}}(\xi) - H(\xi)$. If, in addition, the geoidal height $H_{\mathrm{geoid}}(\xi)$ is known, then the *dynamic topography* (see Fig. 10.25)

$$\Xi(\xi) = H_{\mathrm{ocean}}(\xi) - H_{\mathrm{geoid}}(\xi) \tag{10.289}$$

is obtainable. Consequently, the dynamic topography is understood to be the difference between the sea surface height and the geoidal height (see Fig. 10.25 for the definition and Fig. 10.26 for a graphical illustration).

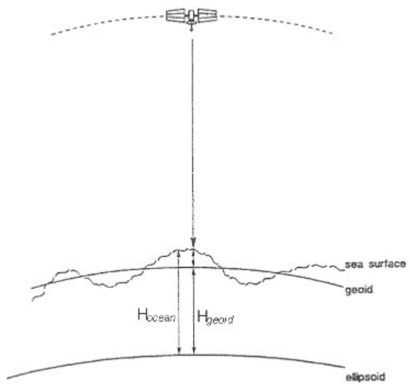


Fig. 10.25: Ocean dynamic topography.

Remark 10.18. In our reduction process of Euler's equation, it remains to consider the viscous friction. Its inclusion, though viscosity is commonly small, is unfortunately a mixture of horizontal and vertical components. Since we use shallow-water approximations, we can neglect certain terms here, too. Viscous friction f_{fric}, as considered here, is given by

$$f_{\mathrm{fric}}(v) = \alpha_{\tan}\frac{1}{r^2}\Delta^* v_{\mathrm{nor}} + \alpha_{\mathrm{lin}} v_{\tan}, \tag{10.290}$$

where α_{lim} is an (artifical) linear friction (to stabilize the solution process). This finally enables us to state the horizontal shallow water equation (10.285) in the form

$$\begin{aligned} \frac{d}{dt} v_{\tan}(r\xi, t) &= -\frac{w(r\xi)\cdot\xi}{r} \nabla^*_\xi \Xi(\xi, t) \\ &\quad + p_{\tan}\left(-2\omega \wedge v_{\tan}(r\xi, t) + \frac{\alpha_{\tan}}{r^2}\Delta^*_\xi v_{\tan}(r\xi)\right) \\ &\quad + \alpha_{\mathrm{lin}} v_{\tan}(r\xi, t). \end{aligned} \tag{10.291}$$

Written explicitly out in polar coordinates and keeping $r \in \mathbb{R}$ fixed, equation (10.291) is identical to the two-dimensional Navier–Stokes equation on the sphere as discussed by R. Temam (1979). Note that one can use the continuity equation (10.275) for the vertical component instead of (10.287). The consideration is omitted, since tangential currents are our main concern in this approach. Summarizing our results, we finally get the following equation

$$\frac{d}{dt} v_{\tan}(r\xi, t) - \mathfrak{t}_{\mathrm{occ}} v_{\tan}(r\xi, t) = -\frac{W_0}{r} \nabla^*_\xi \Xi(\xi, t) \quad , \tag{10.292}$$

where we used the abbreviations

$$\begin{aligned} \mathfrak{t}_{\mathrm{occ}} v_{\tan}(r\xi, t) &= p_{\tan}\left(-2\omega \wedge v_{\tan}(r\xi, t) + \frac{\alpha_{\tan}}{r^2}\Delta^*_\xi v_{\tan}(r\xi, t)\right) \\ &\quad + \alpha_{\mathrm{lin}} v_{\tan}(r\xi, t). \end{aligned} \tag{10.293}$$

In case of linearized, steady state motion ($\frac{d}{dt} v_{\tan}(r\xi, t) = 0$) we have

$$\mathfrak{t}_{\mathrm{occ}} v_{\tan}(r\xi) = \frac{W_0}{r} \nabla^*_\xi \Xi(\xi). \tag{10.294}$$

In case of frictionless currents, the only term left in (10.285) is the Coriolis force balancing the horizontal pressure gradient, i.e., combining (10.285) and (10.294) we find

$$2 p_{\tan}(\omega \wedge v_{\tan}(r\xi)) = -\frac{W_0}{r} \nabla^*_\xi \Xi(\xi). \tag{10.295}$$

This is the so-called geostrophic balance and results in the geostrophic flow assumption. Considerations of this type of oceanic velocity have already been investigated, for example, in S. Levitus (1982), R.S. Nerem, C.J. Koblinsky (1994), R.S. Nerem et al. (1990), S. Beth (2000), W. Freeden et al. (2005), D. Michel (2007).

Altogether, by assuming frictionless motion (far away from coasts, ocean surfaces, and ocean beds) of a homogeneous fluid, neglecting turbulent flows

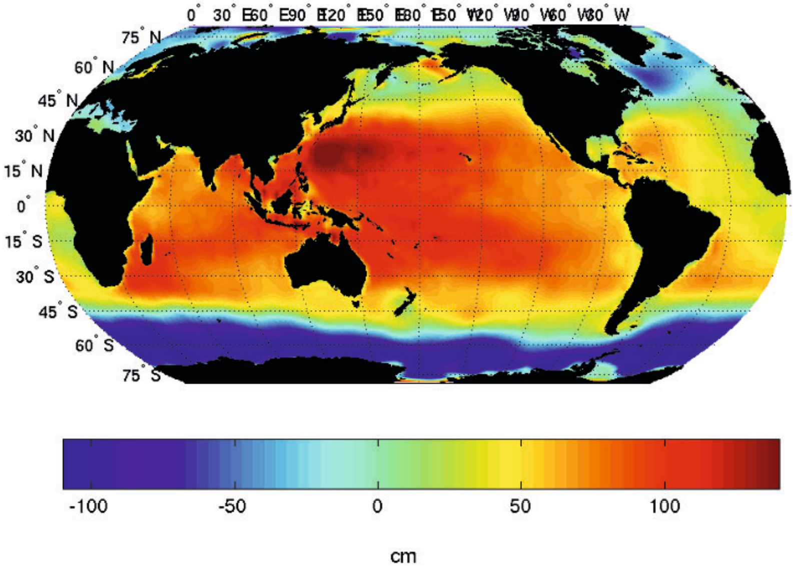


Fig. 10.26: Dynamic topography, as difference between the original altimetric and geoidal data set, Geomathematics Group, TU Kaiserslautern, D. Michel (2005) (see also W. Freeden et al. (2005)).

and vertical velocities, the Euler equations simplify to two common conservations laws, the hydrostatic and the geostrophic balance. In spherical nomenclature, on the Earth's surface Ω_R with $\omega = |\omega|(\xi \cdot \varepsilon^3)\xi$, we are able to relate the horizontal velocity to the dynamic topography Ξ in the following way (cf. R. Coleman (1980))

$$2|\omega|(\xi \cdot \varepsilon^3)(\xi \wedge v_{\tan}(R\xi)) = -\frac{W_0}{R}\nabla^*_\xi \Xi(\xi), \tag{10.296}$$

i.e.,

$$v_{\tan}(R\xi) = \frac{W_0}{2|\omega|(\xi \cdot \varepsilon^3)R} \mathrm{L}^*_\xi \Xi(\xi), \tag{10.297}$$

(note that (10.297) is valid for all $\xi \in \Omega$ with $\varepsilon^3 \cdot \xi \neq 0$, i.e., the equator is excluded). Clearly, for all $\xi \in \Omega$, the geostrophic flow $v_{\tan}$ given by (10.297) is perpendicular to the tangential surface gradient $\nabla^*\Xi$ of the sea surface topography on Ω. This is a remarkable feature of the geostrophic velocity field. The currents flow along and not across the lines of constant sea surface topography.

As already known, the knowledge of the dynamic topography allows the determination of the geostrophic flow by taking the surface curl gradient (see Fig. 10.27). Conversely, the knowledge of the geostrophic flow implies

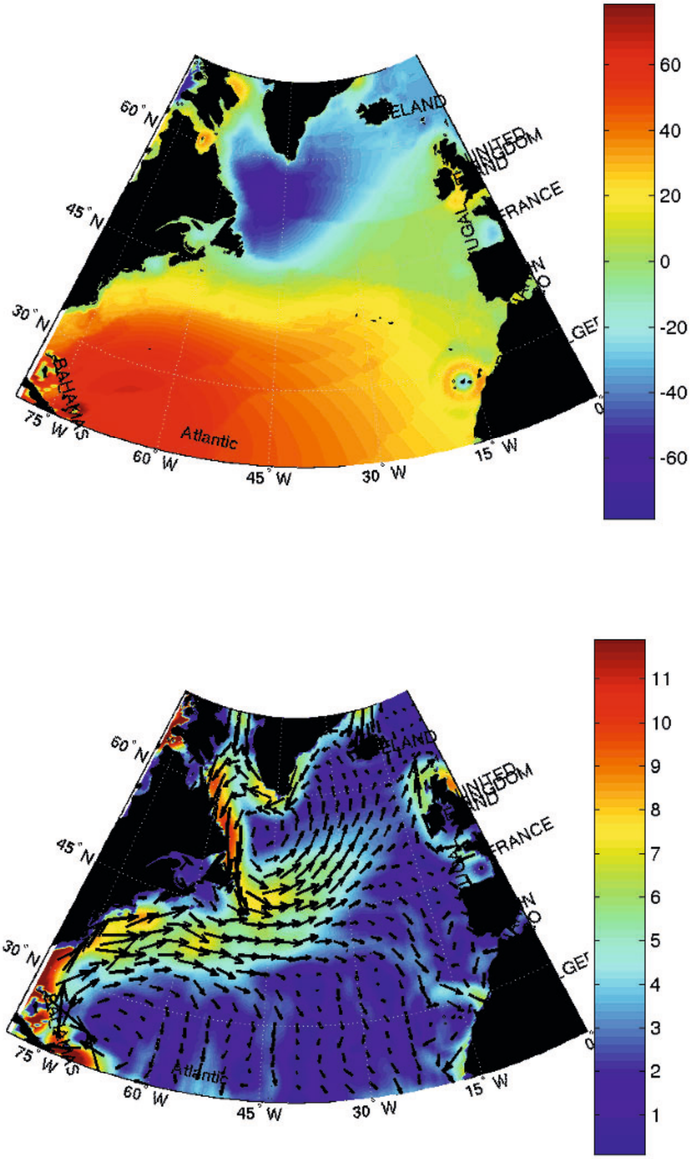


Fig. 10.27: Dynamic topography [cm] and geostrophic flow [cm/s] of the Gulf stream, Geomathematics Group, TU Kaiserslautern, D. Michel (2005) (see also W. Freeden et al. (2005)).

the dynamic topography by taking Green's surface identity with respect to L^* (cf. Section 2.6)

$$\Xi(\xi) = \frac{|\omega| R}{\pi W_0} \int_\Omega \frac{1}{1 - \xi \cdot \eta} (\eta \wedge \xi) \cdot (\eta \cdot \varepsilon^3) v_{\tan}(R\eta) \, d\omega(\eta). \tag{10.298}$$

10.7 Elastic Field

In the case of motion of material in the Earth's interior, we are concerned with solid material in the interior of the Earth such that the assumption of a perfect fluid is not valid anymore.

For small displacements $d : \Omega_R^{\text{int}} \times \mathbb{R} \to \mathbb{R}^3$, Euler's equation of motion can be linearized in the following way:

$$\rho(x,t) \frac{\partial^2 d}{\partial t^2}(x,t) = f(x,t) + \nabla_x \cdot \mathbf{f}(x,t), \tag{10.299}$$

where $\mathbf{f}(x,t) = (f_{i,j}(x,t))_{i,j=1,2,3}$ with

$$f_{i,j}(x,t) = \sum_{k=1}^{3} \sum_{l=1}^{3} \Xi_{i,j,k,l}(x,t) \frac{1}{2} \left(\frac{\partial d_k(x,t)}{\partial x_l} + \frac{\partial d_l(x,t)}{\partial x_k}, \right) \tag{10.300}$$

$i, j \in \{1,2,3\}$. The occuring tensor of rank 4, $\boldsymbol{\Xi}$, is called elasticity tensor with the following symmetries

$$\Xi_{i,j,k,l} = \Xi_{k,l,i,j} = \Xi_{i,j,l,k}. \tag{10.301}$$

An idealized case is an *isotropic medium*, where we have

$$\Xi_{i,j,k,l}(x,t) = \tilde{\lambda}(x,t) \delta_{ij} \delta_{kl} + \tilde{\mu}(x,t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{10.302}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the so-called *Lamè parameters.*

Lemma 10.19. *Under the assumption of an isotropic medium, we have for the Cauchy stress tensor*

$$\mathbf{f}(x,t) = \tilde{\lambda}(x,t)(\nabla_x \cdot d(x,t))\mathbf{i} + \tilde{\mu}(x,t) \left(\nabla_x \otimes d(x,t) + (\nabla_x \otimes d(x,t))^T \right). \tag{10.303}$$

Proof. For the components of the Cauchy stress tensor, we have

$$
\begin{aligned}
f_{i,j}(x,t) \\
&= \frac{1}{2}\sum_{k,l=1}^{3}\left(\tilde{\lambda}(x,t)\delta_{ij}\delta_{kl}+\tilde{\mu}(x,t)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\right)\left(\frac{\partial d_k(x,t)}{\partial x_l}+\frac{\partial d_l(x,t)}{\partial x_k}\right)\\
&= \frac{1}{2}\left(\tilde{\lambda}(x,t)\delta_{ij}\sum_{k=1}^{3}\left(\frac{\partial d_k(x,t)}{\partial x_k}+\frac{\partial d_k(x,t)}{\partial x_k}\right)\right.\\
&\qquad\left.+\tilde{\mu}(x,t)\left(\frac{\partial d_i(x,t)}{\partial x_j}+\frac{\partial d_j(x,t)}{\partial x_i}+\frac{\partial d_j(x,t)}{\partial x_i}+\frac{\partial d_i(x,t)}{\partial x_j}\right)\right)\\
&= \tilde{\lambda}(x,t)\delta_{ij}(\nabla_x\cdot d(x,t))+\tilde{\mu}(x,t)\left(\frac{\partial d_i(x,t)}{\partial x_j}+\frac{\partial d_j(x,t)}{\partial x_i}\right).
\end{aligned}
$$

□

Lemma 10.20. *Under the assumption of an isotropic medium, Euler's equation of motion becomes*

$$
\begin{aligned}
\rho(x)\frac{\partial^2 d(x,t)}{\partial t^2} &= f(x,t)+(\tilde{\mu}(x,t)+\tilde{\mu}(x,t))\nabla_x(\nabla_x\cdot d(x,t))\\
&\quad+(\nabla_x\cdot d(x,t))\nabla_x\tilde{\lambda}(x,t)+\tilde{\mu}(x,t)\Delta_x d(x,t)\\
&\quad+(\nabla_x\otimes d(x,t)+\ (\nabla_x\otimes d(x,t))^T)\nabla_x\mu(x,t).
\end{aligned}
$$

Proof. For the proof of this assertion, we have to calculate the divergence of the Cauchy stress tensor. We split this calculation into two parts and for the first part we obtain

$$
\begin{aligned}
&\nabla_x\cdot(\tilde{\lambda}(x,t)(\nabla_x\cdot d(x,t))\mathbf{i})\\
&=\nabla_x\cdot\begin{pmatrix}\tilde{\lambda}(x,t)(\nabla_x\cdot d(x,t)) & 0 & 0\\ 0 & \tilde{\lambda}(x,t)(\nabla_x\cdot d(x,t)) & 0\\ 0 & 0 & \tilde{\lambda}(x,t)(\nabla_x\cdot d(x,t))\end{pmatrix}\\
&=\nabla_x(\tilde{\lambda}(x,t)\nabla_x\cdot d(x,t))\\
&=(\nabla_x d(x,t))\nabla_x\tilde{\lambda}(x,t)+\tilde{\lambda}(x,t)\nabla_x(\nabla_x\cdot d(x,t)).
\end{aligned}
$$

For the second part, we first get

$$
\begin{aligned}
&\nabla_x\cdot\left(\nabla_x\otimes d(x,t)+(\nabla_x\otimes d(x,t))^T\right)\\
&\quad=\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial d_j(x,t)}{\partial x_i}+\frac{\partial d_i(x,t)}{\partial x_j}\right)\right)_{i=1,2,3}\\
&\quad=\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\frac{\partial d_j(x,t)}{\partial x_i}\right)_{i=1,2,3}+\left(\sum_{j=1}^{3}\frac{\partial^2 d_i(x,t)}{\partial x_j^2}\right)_{i=1,2,3}\\
&\quad=\nabla_x(\nabla_x\cdot d(x,t))+\Delta_x d(x,t).
\end{aligned}
$$

Thus, we obtain for the second part of $\nabla \cdot \mathbf{f}$,

$$\begin{aligned}
&\nabla_x \cdot (\tilde{\mu}(x,t)(\nabla_x \otimes d(x,t) + (\nabla_x \otimes d(x,t))^T)) \\
&= \tilde{\mu}(x,t)(\nabla_x(\nabla_x \cdot d(x,t)) + \Delta_x d(x,t)) \\
&+ \left(\nabla_x \otimes d(x,t) + (\nabla_x \otimes d(x,t))^T\right) \nabla_x \tilde{\mu}(x,t),
\end{aligned}$$

which finishes the proof. □

If the Lamé constants $\tilde{\lambda}$ and $\tilde{\mu}$ are real constants that are not dependent on the spatial variable x, then we are confronted with a *homogeneous medium*. If we, furthermore, neglect the body forces f in the Cauchy–Navier equation, we obtain a simplified version of this equation given by

$$\rho(x)\frac{\partial^2}{\partial t^2} d(x,t) = (\tilde{\lambda} + \tilde{\mu})\nabla_x(\nabla_x \cdot d(x,t)) + \tilde{\mu}\Delta_x d(x,t). \tag{10.304}$$

Moreover, when we treat equilibrium problems of an isotropic homogeneous elastic body, the field equations reduce to the *Navier equation* (also called the *Cauchy–Navier equation*)

$$\tilde{\mu}\Delta_x d(x) + \left(\tilde{\lambda} + \tilde{\mu}\right) \nabla_x(\nabla_x \cdot d(x)) = 0, \quad x \in \Omega_R^{\mathrm{int}}. \tag{10.305}$$

This equation plays in the theory of elasticity the same part as the Laplace equation in the theory of harmonic functions, and it formally reduces to it for $\tilde{\mu} = 1, \tilde{\lambda} = -1$.

The Cauchy–Navier equation admits the equivalent formulation

$$\diamondsuit_x d(x) = \Delta_x d(x) + \tilde{\tau}\nabla_x(\nabla_x \cdot d(x)) = 0, \quad x \in \Omega_R^{\mathrm{int}}, \tag{10.306}$$

where

$$\tilde{\tau} = \frac{1}{1 - 2\tilde{\delta}}, \quad \tilde{\delta} = \frac{\tilde{\lambda}}{2(\tilde{\lambda} + \tilde{\mu})} \tag{10.307}$$

($\tilde{\delta}$ is called *Poisson's ratio*). Since

$$\Delta_x d(x) = \nabla_x(\nabla_x \cdot d(x)) - \nabla_x \wedge (\nabla_x \wedge d(x)), \quad x \in \Omega_R^{\mathrm{int}}, \tag{10.308}$$

we equivalently have

$$\diamondsuit_x d(x) = \left(\tilde{\lambda} + 2\tilde{\mu}\right) \nabla_x(\nabla_x \cdot d(x)) - \tilde{\mu}\nabla_x \wedge (\nabla_x \wedge d(x)), \quad x \in \Omega_R^{\mathrm{int}}. \tag{10.309}$$

Suppose now that d is a (sufficiently often differentiable) vector field satisfying the Navier equation. Then it follows that

$$\begin{aligned}
0 = \tilde{\mu}\nabla_x \cdot (\diamondsuit_x d(x)) &= \nabla_x \cdot \left(\tilde{\mu}\Delta_x d(x) + \left(\tilde{\lambda} + \tilde{\mu}\right) \nabla_x \cdot (\nabla_x(\nabla_x \cdot d(x)))\right) \\
&= \tilde{\mu}\Delta_x(\nabla_x \cdot d(x)) + \left(\tilde{\lambda} + \tilde{\mu}\right) \Delta_x(\nabla_x \cdot d(x)) \\
&= \left(\tilde{\lambda} + 2\tilde{\mu}\right) \Delta_x(\nabla_x \cdot d(x)),
\end{aligned} \tag{10.310}$$

$$\begin{aligned} 0 = \tilde{\mu}\nabla_x \wedge (\diamondsuit_x d(x)) &= \tilde{\mu}\Delta_x(\nabla_x \wedge d(x)) + \left(\tilde{\lambda} + \tilde{\mu}\right) \nabla_x \wedge (\nabla_x(\nabla_x \cdot d(x))) \\ &= \tilde{\mu}\Delta_x(\nabla_x \wedge d(x)), \end{aligned} \tag{10.311}$$

$$\begin{aligned} 0 = \tilde{\mu}\Delta_x(\diamondsuit_x d) &= \tilde{\mu}\Delta_x\Delta_x d(x) + \left(\tilde{\lambda} + \tilde{\mu}\right) \nabla_x(\Delta_x(\nabla_x \cdot d(x))) \\ &= \tilde{\mu}\Delta_x\Delta_x d(x). \end{aligned} \tag{10.312}$$

Summarizing our results, we therefore obtain for a sufficiently often differentiable field $d : \overline{\Omega_R^{\text{int}}} \to \mathbb{R}^3$ satisfying $\diamondsuit_x d(x) = 0\ ,\ x \in \Omega_R^{\text{int}}$:

$$\Delta_x(\nabla_x \cdot d(x)) = 0, \quad x \in \Omega_R^{\text{int}}, \tag{10.313}$$

$$\Delta_x(\nabla_x \wedge d(x)) = 0, \quad x \in \Omega_R^{\text{int}}, \tag{10.314}$$

$$\Delta_x(\Delta_x d(x)) = 0, \quad x \in \Omega_R^{\text{int}}. \tag{10.315}$$

In other words, our considerations have led to the conclusions that the displacement field d is biharmonic, and its divergence and curl are harmonic.

This shows a deep relation between linear elasticity and potential theory. Moreover it should be noted that, according to the invariance of the differential operators ∇, Δ with respect to orthogonal transformations, we are able to derive that $\diamondsuit d = 0$ is equivalent to $\diamondsuit \left(\mathbf{t}^T d(\mathbf{t}\cdot)\right) = 0$ for all orthogonal transformations $\mathbf{t}$ (for $d \in \mathrm{c}^{(2)}(\Omega_R^{\text{int}})$).

Let nav_n (more explicitly: $\text{nav}_n(\mathbb{R}^3)$) be the class of homogeneous vector polynomials of degree n satisfying Navier's equations in $\mathbb{R}^3$:

$$\text{nav}_n = \left\{ u \in \text{hom}_n \,\middle|\, \diamondsuit u = \Delta u + \tau \nabla(\nabla \cdot u) = 0,\ \tau = \frac{\tilde{\lambda} + \tilde{\mu}}{\tilde{\mu}} \right\}. \tag{10.316}$$

Remark 10.21. If $\tau = 0$, (10.316) leads back to the space of $\text{harm}_n(\mathbb{R}^3)$ of vectorial harmonic polynomials (well known from W. Freeden et al. (1994)).

Every vector field $u \in \text{nav}_n$ can be written in the form

$$u(x) = \sum_{j=0}^{n} c_{n-j}(x_1, x_2)\ x_3^j,\ x \in \mathbb{R}^3, \qquad x = (x_1, x_2, x_3)^T, \tag{10.317}$$

where $c_{n-j} : \mathbb{R}^2 \to \mathbb{R}^3$ denote homogeneous vector polynomials of degree $n - j$. It readily can be seen that $\diamondsuit u$ allows the following representation:

$$\begin{aligned} \diamondsuit_x u(x) &= \Delta_x u(x) + \tau \nabla_x(\nabla_x \cdot u(x)) \qquad (10.318) \\ &= \mathbf{a}\frac{\partial^2}{\partial x_3^2} u(x) + \mathbf{b}_x \frac{\partial}{\partial x_3} u(x) + \mathbf{c}_x,\ x \in \mathbb{R}^3, \end{aligned}$$

where we have used the matrix operators $\mathbf{a}, \mathbf{b}_x, \mathbf{c}_x$ given by

$$\mathbf{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\tau \end{pmatrix},$$

$$\mathbf{b}_x = \begin{pmatrix} 0 & 0 & \tau\frac{\partial}{\partial x_1} \\ 0 & 0 & \tau\frac{\partial}{\partial x_2} \\ \tau\frac{\partial}{\partial x_1} & \tau\frac{\partial}{\partial x_2} & 0 \end{pmatrix},$$

$$\mathbf{c}_x = \begin{pmatrix} (1+\tau)\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} & \tau\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} & 0 \\ \tau\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} & \frac{\partial^2}{\partial x_1^2} + (1+\tau)\frac{\partial^2}{\partial x_2^2} & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \end{pmatrix}.$$

Observing the fact that

$$\frac{\partial u}{\partial x_3}(x_1, x_2, x_3) = \sum_{j=0}^{n-1}(j+1)c_{n-j-1}(x_1, x_2)x_3^j, \tag{10.319}$$

$$\frac{\partial^2 u}{\partial x_3^2}(x_1, x_2, x_3) = \sum_{j=0}^{n-2}(j+2)(j+1)c_{n-j-2}(x_1, x_2)x_3^j \tag{10.320}$$

we get from (10.317) the recursion relation

$$(j+2)(j+1)\mathbf{a}c_{n-j-2}(\tilde{x}) + (j+1)\mathbf{b}_x c_{n-j-1}(\tilde{x}) + \mathbf{c}_x c_{n-j}(\tilde{x}) = 0, \tag{10.321}$$

$\tilde{x} = (x_1, x_2)^T$, $j = 0, ..., n-2$. Since the matrix $\mathbf{a}$ is regular (notice that $\tau \neq -1$), all polynomials c_j are determined provided that c_n and c_{n-1} are known.

By summarizing our results, we obtain the following theorem.

Theorem 10.22. *Let c_n, $c_{n-1} : \mathbb{R}^2 \to \mathbb{R}^3$ be homogeneous polynomials of degree n, $n-1$, respectively. For $j = 0, ..., n-2$ we define recursively*

$$\mathbf{a}c_{n-j-2}(x_1, x_2) = -\frac{1}{(j+2)(j+1)}\big((j+1)\mathbf{b}_x c_{n-j-1}(x_1, x_2) + \mathbf{c}_x c_{n-j}(x_1, x_2)\big).$$

Then $u_n : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$u_n(x_1, x_2, x_3) = \sum_{j=0}^{n} c_{n-j}(x_1, x_2)x_3^j$$

is a homogeneous polynomial of degree n in $\mathbb{R}^3$ satisfying the Navier equation $\diamondsuit_x u_n(x) = 0$, $x \in \mathbb{R}^3$. Moreover, the number of linearly independent homogeneous polynomials is equal to the total number of coefficients of c_n and c_{n-1}, that is

$$\mathrm{d}(\mathrm{nav}_n) = 3(2n+1)\ . \tag{10.322}$$

Remark 10.23. We know (see, e.g., W. Freeden et al. (1994))) that homogeneous harmonic polynomials of different degree are orthogonal (in the l^2-sense). This fact, however, is not true for the spaces nav_n ($\tau \neq 0$), as the following example shows. The vector fields

$$u_0(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; u_2(x) = \begin{pmatrix} x_1 x_2 \\ -\frac{\tau}{2(\tau+3)}(x_1^2 + x_2^2 + x_3^2) \\ 0 \end{pmatrix}$$

are elements of nav_0 and nav_2, respectively. But it follows by an easy calculation that

$$\int\limits_{\Omega} u_0(\xi) \cdot u_2(\xi) \; d\omega(\xi) = -\frac{2\pi\tau}{\tau+3} \neq 0.$$

Nevertheless, we are able to prove the following result.

Theorem 10.24. *Let* $u_n \in \mathrm{nav}_n, u_m \in \mathrm{nav}_m$. *Then*

$$(u_n, u_m)_{l^2(\Omega)} = \int_{\Omega} u_n(\xi) \cdot u_m(\xi) \; d\omega(\xi) = 0$$

if $|n-m| \neq 2$ *and* $n \neq m$.

Proof. Applying Green's formula and the Gauss theorem, we see that

$$\begin{aligned}
0 &= \int_{|x|\le 1} (u_n(x) \cdot \diamondsuit_x u_m(x) - u_m(x) \cdot \diamondsuit_x u_n(x)) \; dx \\
&= \int_{|x|\le 1} (u_n(x) \cdot \Delta_x u_m(x) - u_m(x) \cdot \Delta_x u_n(x)) \; dx \qquad (10.323) \\
&\qquad + \tau \int_{|x|\le 1} (u_n(x) \cdot \nabla_x(\nabla_x \cdot u_m(x)) - u_m(x) \cdot \nabla_x(\nabla_x \cdot u_n(x))) \; dx \\
&= (m-n) \int_{|x|=1} u_n(x) \cdot u_m(x) \; d\omega(x) \\
&\qquad + \tau \int_{|x|\le 1} (\nabla_x \cdot (u_n(x) \nabla_x \cdot u_m(x)) - \nabla_x \cdot (u_m(x) \nabla_x \cdot u_n(x))) \; dx \\
&= (m-n) \int_{|x|=1} u_n(x) \cdot u_m(x) \; d\omega(x) \\
&\qquad + \tau \int_{|x|=1} ((x \cdot u_n(x)) \nabla_x \cdot u_m(x) - (x \cdot u_m(x)) \nabla_x \cdot u_n(x)) \; d\omega(x).
\end{aligned}$$

The functions $x \mapsto (\nabla_x \cdot u_m)(x)$ and $x \mapsto (\nabla_x \cdot u_n)(x), x \in \mathbb{R}^3$, are harmonic and the functions

$$x \mapsto x \cdot u_n(x) \quad \text{and} \quad x \mapsto x \cdot u_m(x), \; x \in \mathbb{R}^3$$

are biharmonic. For example,

$$\begin{aligned}\Delta_x(x\cdot u_n(x)) &= x\cdot\Delta_x u_n(x)+2\nabla_x\cdot u_n(x) \\ &= 2\nabla_x\cdot u_n(x)-\tau(x\cdot\nabla_x)\nabla_x\cdot u_n(x) \\ &= (2-\tau(n-1))\nabla_x\cdot u_n(x)\end{aligned} \tag{10.324}$$

so that we have $\Delta_x\Delta_x(x\cdot u_n(x))=0,\ x\in\mathbb{R}^3$. In an analogous way, it follows that $\Delta_x\Delta_x(x\cdot u_m(x))=0,\ x\in\mathbb{R}^3$. Therefore (see W. Freeden et al. (1994)), there exist scalar homogeneous harmonic polynomials H_{n-1}, H_{n+1}, H_{m-1}, and H_{m+1} of degree $n-1, n+1, m-1$, and $m+1$, respectively, with

$$x\cdot u_n(x)=H_{n+1}(x)+|x|^2H_{n-1}(x) \tag{10.325}$$

and

$$x\cdot u_m(x)=H_{m+1}(x)+|x|^2H_{m-1}(x). \tag{10.326}$$

According to our assumptions, we have $m-1\neq n+1$ and $m+1\neq n-1$. Thus we find

$$\int_{|x|=1}(x\cdot u_n(x))\nabla_x\cdot u_m(x)\ d\omega(x)=0, \tag{10.327}$$

$$\int_{|x|=1}(x\cdot u_m(x))\nabla_x\cdot u_n(x)\ d\omega(x)=0. \tag{10.328}$$

Hence, Equation (10.323) reduces to

$$0=\int_\Omega u_n(\xi)\cdot u_m(\xi)\ d\omega(\xi) \tag{10.329}$$

if $n\neq m$. This is the required result. □

Next, we are interested in giving explicit representations of homogeneous polynomials of degree n which solve the Navier equation in $\mathbb{R}^3$. This can be done, for example, by using the recursion formula (10.321). But we are also able to use known information about scalar homogeneous harmonic polynomials. We start with a preparatory lemma.

Lemma 10.25. *Let $H_n:\mathbb{R}^3\to\mathbb{R},\ n\geq 0$, be a scalar homogeneous harmonic polynomial of degree n. Then*

(i) $\Delta_x(H_n(x)x)=2\nabla_x H_n(x)$,

(ii) $\Delta_x(|x|^m H_n(x))=m(m+2n+1)|x|^{m-2}H_n(x),\ m\geq 2$,

(iii) $\Delta_x\left(x^2\nabla_x H_n(x)\right)=2(2n+1)\nabla_x H_n(x)$.

Proof. The formulas (i), (ii), and (iii) can be obtained by straightforward calculations. □

We are now interested in the following lemma.

Lemma 10.26. *Let $H_n : \mathbb{R}^3 \to \mathbb{R}$ be a homogeneous harmonic polynomial of degree n. Then the following identities are valid:*

(i) *For all $x \in \mathbb{R}^3$, $\diamondsuit_x(\nabla_x H_n(x)) = 0$.*

(ii) *For all $x \in \mathbb{R}^3$, $\diamondsuit_x(x \wedge \nabla_x H_n(x)) = 0$.*

(iii) *For all $x \in \mathbb{R}^3$, $\diamondsuit_x(xH_n(x) + \alpha_n |x|^2 \nabla_x H_n(x)) = 0$, where*

$$\alpha_n = -\frac{\tilde{\lambda}(3+n) + \tilde{\mu}(5+n)}{2\left(n\tilde{\lambda} + \tilde{\mu}(3n+1)\right)} \ . \tag{10.330}$$

(iv) *For all $x \in \mathbb{R}^3$, $\diamondsuit_x(H_n(x)\varepsilon^k + \beta_n |x|^2 \nabla_x \nabla_x \cdot (H_n(x)\varepsilon^k)) = 0$, where*

$$\beta_n = -\frac{\tilde{\lambda} + \tilde{\mu}}{\left(2\tilde{\lambda} + 6\tilde{\mu}\right) n - 2\tilde{\lambda} - 4\tilde{\mu}} \ . \tag{10.331}$$

(v) *For all $x \in \mathbb{R}^3$, $\diamondsuit_x\big(H_n(x)\varepsilon^k + \gamma_n(\varepsilon^k \cdot \nabla_x H_n(x))x\big) = 0$, where*

$$\gamma_n = -\frac{\tilde{\lambda} + \tilde{\mu}}{(n+2)\tilde{\lambda} + (n+4)\tilde{\mu}} \ . \tag{10.332}$$

Proof. The formulas can be obtained by elementary calculations. □

Lemma 10.26 enables us to develop three important systems of polynomial solutions of the Navier equation.

Lemma 10.27. *Let $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ be a linearly independent system of scalar homogeneous harmonic polynomials of degree n. Then the functions $w_{n,j,k} : \mathbb{R}^3 \to \mathbb{R}^3$, $k = 1, 2, 3$, defined by*

$$w_{n,j,k}(x) = H_{n,j}(x)\varepsilon^k + \beta_n |x|^2 \nabla_x \left(\nabla_x \cdot \left(H_{n,j}(x)\varepsilon^k\right)\right), \ x \in \mathbb{R}^3, \tag{10.333}$$

form a set of $3(2n+1)$ linearly independent elements of $\mathrm{nav}_n(\mathbb{R}^3)$, where β_n is given by (10.331).

Lemma 10.28. *Let $\{H_{n,j}\}_{j=1,\ldots,2n+1}$ be a linearly independent system of scalar homogeneous harmonic polynomials of degree n. Then the functions $v_{n,j,k} : \mathbb{R}^3 \to \mathbb{R}^3, k = 1, 2, 3$, defined by*

$$v_{n,j,k}(x) = H_{n,j}(x)\varepsilon^k + \gamma_n \left(\varepsilon^k \cdot \nabla_x H_{n,j}(x)\right) x, \ x \in \mathbb{R}^3, \tag{10.334}$$

form a set of $3(2n+1)$ linearly independent elements of $\mathrm{nav}_n(\mathbb{R}^3)$, where γ_n is given by (10.332).

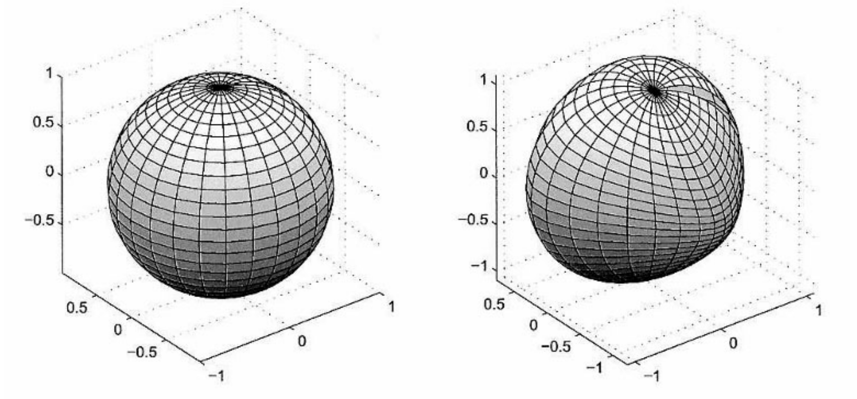


Fig. 10.28: Reference (*left*) and deformed (*right*) configurations of Ω associated to the displacement function $x \mapsto xH_1(x) + \alpha_1|x|^2\nabla_x H_1(x), \tilde{\lambda} = 2, \tilde{\mu} = 3, H_1(x) = x \cdot \varepsilon^2, x \in \mathbb{R}^3$.

Remark 10.29. The system (10.333) can be found in A. Lurje (1963), while the system (10.334) has been discussed in H. Bauch (1981), W. Freeden, R. Reuter (1989). Unfortunately, both systems are not orthogonal invariant, that is, $\mathbf{t}^T v_{n,j,k}(\mathbf{t}\cdot)$ (resp. $\mathbf{t}^T w_{n,j,k}(\mathbf{t}\cdot)$) generally is not a member of the span of the system $\{v_{n,j,k}\}$ (resp. $\{w_{n,j,k}\}$). A polynomial system showing this property will be listed now (for the case n=2 see Figs. 10.28, 10.29, and 10.30).

Lemma 10.30. *Let* $\{H_{k,j}\}_{\substack{k=n-1,n,n+1 \\ j=1,\ldots,2n+1}}$ *be a linearly independent system of scalar homogeneous harmonic polynomials. Then, the functions* $u^{(i)}_{n,j} : \mathbb{R}^3 \to \mathbb{R}^3, i = 1, 2, 3,$ *defined by*

$$u^{(1)}_{n,j}(x) = H_{n-1,j}(x)x + \alpha_{n-1}|x|^2\nabla_x H_{n-1,j}(x), \quad n = 1, 2, ..., \; j = 1, ..., 2n-1, \tag{10.335}$$

$$u^{(2)}_{n,j}(x) = \nabla_x H_{n+1,j}(x), \; n = 0, 1, ..., \; j = 1, ..., 2n+3, \tag{10.336}$$

$$u^{(3)}_{n,j}(x) = x \wedge \nabla_x H_{n,j}(x), \; n = 1, 2, ..., \; j = 1, ..., 2n+1, \tag{10.337}$$

form a set of $3(2n+1)$ *linearly independent elements of* $\mathrm{nav}_n(\mathbb{R}^3)$*, where* α_n *is given by (10.330).*

The functions $u^{(2)}_{n,j}, u^{(3)}_{n,j}$ are characterized by the properties:

$$\nabla_x \cdot u^{(2)}_{n,j}(x) = 0,\ \nabla_x \wedge u^{(2)}_{n,j}(x) = 0, \tag{10.338}$$

$$x \cdot u^{(3)}_{n,j}(x) = 0,\ \nabla_x \cdot u^{(3)}_{n,j}(x) = 0. \tag{10.339}$$

From a physical point of view, this means that $u^{(2)}_{n,j}$ is a *poloidal field* (i.e., a vector field free of dilatation and torsion), while $u^{(3)}_{n,j}$ is a *toroidal field.* Only the functions $u^{(1)}_{n,j}$ are responsible for volume change.

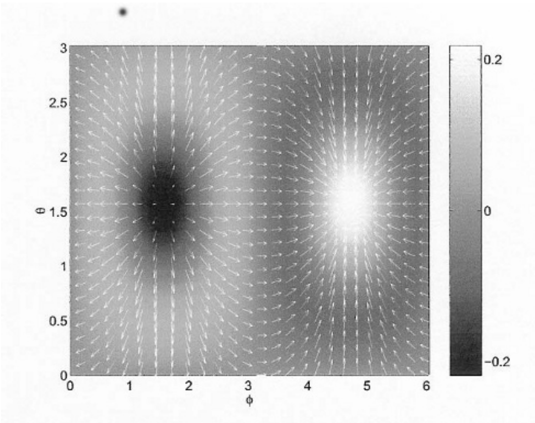


Fig. 10.29: Radial (*grey*) and tangential (*arrows*) displacements of Ω associated to the displacement function $x \mapsto xH_1(x) + \alpha_1|x|^2\nabla_x H_1(x), \tilde{\lambda} = 2, \tilde{\mu} = 3, H_1(x) = x \cdot \varepsilon^2, x \in \mathbb{R}^3$.

Remark 10.31. There is a very interesting relation between the systems $\{w_{n,j,k}\}$, $\{v_{n,j,k}\}$, $k = 1, 2, 3$, introduced above and the system $\{u^{(i)}_{n+1,j}\}, i = 1, 2, 3$. Replacing $H_{n-1,j}$ by $\sum_{k=1}^{3} \varepsilon^k \cdot \nabla H_{n,j}$ (note that $\varepsilon^k \cdot \nabla H_{n,j}$ is a homogeneous harmonic polynomial of degree $n-1$ due to a result in W. Freeden et al. (1994)) in the representation of $u^{(1)}_{n,j}$, we obtain a field $z_{n,j}$ defined as follows:

$$z_{n,j}(x) = \left(\sum_{k=1}^{3} \left(\varepsilon^k \cdot \nabla_x H_{n,j}(x) \right) \right) x + \alpha_{n-1} x^2 \nabla_x \left(\sum_{k=1}^{3} \left(\varepsilon^k \cdot \nabla_x H_{n,j}(x) \right) \right). \tag{10.340}$$

It is clear that $z_{n,j}$ satisfies the Navier equation. Moreover, it is easy to see that $\gamma_n = (-\beta_n)/\alpha_{n-1}$. But this shows that

$$v_{n,j} = w_{n,j} - \frac{\beta_n}{\alpha_{n-1}} z_{n,j}, \quad n = 0, 1, \ldots, j = 1, \ldots, 2n+1, \tag{10.341}$$

where we have used the abbreviations

$$v_{n,j} = \sum_{k=1}^{3} v_{n,j,k}, \quad w_{n,j} = \sum_{k=1}^{3} w_{n,j,k}. \tag{10.342}$$

Assuming that the scalar system $\{H_{n,j}\}_{n=0,1,\ldots,j=1,\ldots,2n+1}$ forms an orthonormal system of homogeneous harmonic polynomials with respect to $\mathrm{L}^2(\Omega)$, the following orthogonal relations can be guaranteed:

$$\int_\Omega u^{(i)}_{n,j}(\xi) \cdot u^{(k)}_{n,l}(\xi)\, d\omega(\xi) = 0 \qquad \text{if } i \neq k \text{ or } j \neq l, \tag{10.343}$$

$$\int_\Omega u^{(i)}_{n,j}(\xi) \cdot u^{(i)}_{m,k}(\xi)\, d\omega(\xi) = 0 \qquad \text{if } n \neq m \text{ or } j \neq k, i = 1,2,3, \tag{10.344}$$

$$\int_\Omega u^{(3)}_{n,j}(\xi) \cdot u^{(i)}_{m,k}(\xi)\, d\omega(\xi) = 0 \qquad \text{if } i = 1,2 \qquad . \tag{10.345}$$

This shows us the following lemma.

Lemma 10.32. *The space* $\mathrm{nav}_n, n > 0$*, defined by (10.316) can be decomposed into three subspaces* $\mathrm{nav}^{(i)}_n, i = 1,2,3$*, given by*

$$\mathrm{nav}^{(i)}_n = \operatorname*{span}_{j=1,\ldots,2n+1} \left\{ u^{(i)}_{n,j} \right\} \tag{10.346}$$

such that

$$\mathrm{nav}_n = \mathrm{nav}^{(1)}_n \oplus \mathrm{nav}^{(2)}_n \oplus \mathrm{nav}^{(3)}_n. \tag{10.347}$$

Moreover, we have the following dimensions:

$$d\left(\mathrm{nav}^{(1)}_n\right) = 2n-1,\ d\left(\mathrm{nav}^{(2)}_n\right) = 2n+3,\ d\left(\mathrm{nav}^{(3)}_n\right) = 2n+1. \tag{10.348}$$

For $n = 0$*,*

$$\mathrm{nav}_0 = \mathrm{nav}^{(2)}_0 = \operatorname*{span}_{j=1,2,3} u^{(2)}_{0,j}, \quad d\left(\mathrm{nav}_0\right) = 3. \tag{10.349}$$

As mentioned above, the spaces $\mathrm{nav}^{(i)}_n, i = 1,2,3$, are orthogonal invariant in the sense that $u \in \mathrm{nav}^{(i)}_n$ is equivalent to $\mathbf{t}^T u(\mathbf{t}\cdot) \in \mathrm{nav}^{(i)}_n$, $i = 1,2,3$, for every orthogonal transformation $\mathbf{t}$. Thus, we have found a decomposition of nav_n into three invariant subspaces.

Next, assume that $w^{(i)}_n$ is a member of $\mathrm{nav}^{(i)}_n$. Consider the space $h^{(i)}_n$ of all linear combinations of functions $w^{(i)}_n(\mathbf{t}\cdot)$, where $\mathbf{t}$ is an orthogonal transformation:

$$h^{(i)}_n = \operatorname{span}\left\{ w^{(i)}_n(\mathbf{t}\cdot) \,\middle|\, \mathbf{t} \in O(3) \right\}. \tag{10.350}$$

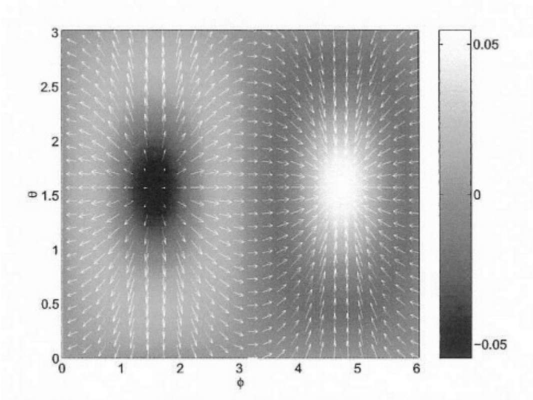


Fig. 10.30: Radial (*grey*) and tangential (*arrows*) displacements of $\Omega_{1/2}$ associated to the displacement function $x \mapsto xH_1(x) + \alpha_1 |x|^2 \nabla_x H_1(x), \tilde{\lambda} = 2, \tilde{\mu} = 3, H_1(x) = x \cdot \varepsilon^2, x \in \mathbb{R}^3$.

Then it is clear that $0 < d(h_n^{(i)}) \leq d(\mathrm{nav}_n^{(i)})$. Moreover, it can be shown that there exists no orthogonal invariant subspace in $\mathrm{nav}_n^{(i)}$. Thus, it follows immediately that $\mathrm{nav}_n^{(i)} = h_n^{(i)}$. This leads us to the following lemma.

Lemma 10.33. *Let* $w_n^{(i)}$ *be of class* $\mathrm{nav}_n^{(i)}$. *Then, there exist* $d(\mathrm{nav}_n^{(i)})$ *orthogonal transformations* $\mathbf{t}_j, j = 1, ..., d(\mathrm{nav}_n^{(i)})$, *such that any element* $u^{(i)} \in \mathrm{nav}_n^{(i)}$ *can be written in the form*

$$u^{(i)} = \sum_{j=1}^{d\left(\mathrm{nav}_n^{(i)}\right)} c_j^{(i)} \mathbf{t}_j^T w_n^{(i)}(\mathbf{t}_j \cdot), \tag{10.351}$$

where $c_j^{(i)}$ *are real numbers.*

Finally, we formulate the addition theorem for the system $\{u_{n,j}^{(k)}\}$ developed in Lemma 10.30. By separation of radial and angular tangential components, we first obtain after simple calculations

$$u_{n,j}^{(1)}(x) = \gamma_n^{(1)}(|x|) y_{n-1,j}^{(1)}(\xi) + \delta_n^{(1)}(|x|) y_{n-1,j}^{(2)}(\xi), \tag{10.352}$$

$$u_{n,j}^{(2)}(x) = \gamma_n^{(2)}(|x|) y_{n+1,j}^{(1)}(\xi) + \delta_n^{(2)}(|x|) y_{n+1,j}^{(2)}(\xi), \tag{10.353}$$

$$u_{n,j}^{(3)}(x) = \gamma_n^{(3)}(|x|) y_{n,j}^{(3)}, \tag{10.354}$$

where we have used the abbreviations

$$\gamma_n^{(1)}(|x|) = |x|^n(1+(n-1)\alpha_{n-1}), \tag{10.355}$$
$$\delta_n^{(1)}(|x|) = |x|^n\alpha_{n-1}\sqrt{(n-1)n}, \tag{10.356}$$
$$\gamma_n^{(2)}(|x|) = |x|^n(1+n), \tag{10.357}$$
$$\delta_n^{(2)}(|x|) = |x|^n\sqrt{(n+1)(n+2)}, \tag{10.358}$$
$$\gamma_n^{(3)}(|x|) = |x|^n\sqrt{n(n+1)}\ . \tag{10.359}$$

Remembering the addition theorem for vector spherical harmonics (see Theorem 5.31), we obtain the following theorem.

Theorem 10.34. *For* $x, y \in \mathbb{R}^3,\ x = r\xi, y = \rho\eta,\ r = |x|, \rho = |y|$,

$$\begin{aligned}
&\sum_{j=1}^{2n-1} u_{n,j}^{(1)}(x) \otimes u_{n,j}^{(1)}(y)\\
&\quad = \gamma_n^{(1)}(r)\gamma_n^{(1)}(\rho)\mathbf{p}_{n-1}^{(1,1)}(\xi,\eta) + \gamma_n^{(1)}(r)\delta_n^{(1)}(\rho)\mathbf{p}_{n-1}^{(1,2)}(\xi,\eta)\\
&\qquad + \delta_n^{(1)}(r)\gamma_n^{(1)}(\rho)\mathbf{p}_{n-1}^{(2,1)}(\xi,\eta) + \delta_n^{(1)}(r)\delta_n^{(1)}(\rho)\mathbf{p}_{n-1}^{(2,2)}(\xi,\eta),
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{2n+3} u_{n,j}^{(2)}(x) \otimes u_{n,j}^{(2)}(y)\\
&\quad = \gamma_n^{(2)}(r)\gamma_n^{(2)}(\rho)\mathbf{p}_{n+1}^{(1,1)}(\xi,\eta), + \gamma_n^{(2)}(r)\delta_n^{(2)}(\rho)\mathbf{p}_{n+1}^{(1,2)}(\xi,\eta)\\
&\qquad + \delta_n^{(2)}(r)\gamma_n^{(2)}(\rho)\mathbf{p}_{n+1}^{(2,1)}(\xi,\eta) + \delta_n^{(2)}(r)\delta_n^{(2)}(\rho)\mathbf{p}_{n+1}^{(2,2)}(\xi,\eta),
\end{aligned}$$

$$\sum_{j=1}^{2n+1} u_{n,j}^{(3)}(x) \otimes u_{n,j}^{(3)}(y) = \gamma_n^{(3)}(r)\gamma_n^{(3)}(\rho)\mathbf{p}_n^{(3,3)}(\xi,\eta).$$

In particular, we find the following result.

Lemma 10.35. *If* $x \in \mathbb{R}^3, r = |x|, x = r\xi$*, then*

$$\sum_{j=1}^{2n-1} \left|u_{n,j}^{(1)}(x)\right|^2 = r^{2n}\left((1+(n-1)\alpha_{n-1})^2 + \alpha_{n-1}^2 n(n-1)\right)\frac{2n-1}{4\pi},$$

$$\sum_{j=1}^{2n+3} \left|u_{n,j}^{(2)}(x)\right|^2 = r^{2n}\frac{(n+1)(2n+3)^2}{4\pi},$$

$$\sum_{j=1}^{2n+1} \left|u_{n,j}^{(3)}(x)\right|^2 = r^{2n}\frac{n(n+1)(2n+1)}{4\pi}.$$

From our considerations given above, it is clear that there are different ways of computing linearly independent systems of homogeneous polynomial solutions to the Navier equations. Of course, the recursion procedure of Theorem 10.22 can be used to derive an algorithm quite analogously to the method used for scalar homogeneous polynomials.

Next, we are interested in determining elastic potentials corresponding to vector spherical harmonics as boundary values.

Lemma 10.36. *Let* $v^{(i)}_{n,j}, \mathbb{R}^3 \to \mathbb{R}^3, i = 1,2,3,$ *be defined by*

$$\begin{aligned} v^{(1)}_{n,j}(x) &= H_{n,j}(x)x + \alpha_n \left(x^2 - 1\right) \nabla_x H_{n,j}(x), && (10.360)\\ & \quad n = 0,1,...,j = 1,...,2n+1, \\ v^{(2)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} \left(\nabla_x H_{n,j}(x) - n v^{(1)}_{n,j}(x)\right), && (10.361)\\ & \quad n = 1,2,...,j = 1,...,2n+1, \\ v^{(3)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} x \wedge \nabla_x H_{n,j}(x), && (10.362)\\ & \quad n = 1,2,...,j = 1,...,2n+1, \end{aligned}$$

where

$$\alpha_n = -\frac{n\tau + 2 + 3\tau}{2(n(\tau+2)+1)}, \tag{10.363}$$

$$H_{n,j}(x) = |x|^n Y_{n,j}(\xi), \qquad x = |x|\xi,\ \xi \in \Omega. \tag{10.364}$$

Then $v^{(i)}_{n,j}$ *satisfies the Cauchy–Navier equation* $\diamondsuit v^{(i)}_{n,j}(x) = 0$ *in* Ω^{int} *with* $v^{(i)}_{n,j}|\Omega = y^{(i)}_{n,j}$.

Proof. It is not hard to see that

$$\begin{aligned} \diamondsuit_x v^{(1)}_{n,j}(x) &= 2\nabla_x H_{n,j}(x) + \tau(3+n)\nabla_x H_{n,j}(x) && (10.365)\\ &\quad + \alpha_n((6 + 4(n-1))\nabla_x H_{n,j}(x) + 2n\tau \nabla_x H_{n,j}(x)) \\ &= 0, \\ \diamondsuit_x v^{(2)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} (\diamondsuit_x \nabla_x H_{n,j}(x)) && (10.366)\\ &\quad - n(n(n+1))^{-\frac{1}{2}} \left((\diamondsuit_x) v^{(1)}_{n,j}(x)\right) \\ &= 0, \\ \diamondsuit_x v^{(3)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} \diamondsuit_x (x \wedge \nabla_x H_{n,j}(x)) && (10.367)\\ &= -2\nabla_x \wedge \nabla_x H_{n,j}(x) \\ &= 0. \end{aligned}$$

Using the polar coordinates $x = r\xi,\ r = |x|,\ \xi \in \Omega$, we obtain after simple calculations

$$\begin{aligned} v^{(1)}_{n,j}(x) &= \sigma^{(1)}_n(r) y^{(1)}_{n,j}(\xi) + \tau^{(1)}_n(r) y^{(2)}_{n,j}(\xi), && (10.368)\\ v^{(2)}_{n,j}(x) &= \sigma^{(2)}_n(r) y^{(1)}_{n,j}(\xi) + \tau^{(2)}_n(r) y^{(2)}_{n,j}(\xi), && (10.369)\\ v^{(3)}_{n,j}(x) &= \sigma^{(3)}_n(r) y^{(3)}_{n,j}(\xi), && (10.370) \end{aligned}$$

where

$$\sigma_n^{(1)}(r) = r^{n-1}\left(r^2 + n\alpha_n\left(r^2-1\right)\right), \tag{10.371}$$
$$\sigma_n^{(2)}(r) = (n(n+1))^{-\frac{1}{2}} n(1+n\alpha_n) r^{n-1}\left(1-r^2\right), \tag{10.372}$$
$$\sigma_n^{(3)}(r) = r^n, \tag{10.373}$$
$$\tau_n^{(1)}(r) = \alpha_n (n(n+1))^{+\frac{1}{2}} r^{n-1}\left(r^2-1\right), \tag{10.374}$$
$$\tau_n^{(2)}(r) = r^{n-1}\left(1 - n\alpha_n\left(r^2-1\right)\right). \tag{10.375}$$

This shows us that $v_{n,j}^{(i)-} = v_{n,j}^{(i)}|\Omega = y_{n,j}^{(i)}$, as required. □

It should be mentioned that

$$v_{n,j}^{(1)} = u_{n+1,j}^{(1)} - \alpha_n \nabla H_{n,j},\ n = 0,1,...,j=1,...,2n+1,$$
$$v_{n,j}^{(2)} = (n(n+1))^{-\frac{1}{2}}\left(u_{n-1,j}^{(2)} - n v_{n,j}^{(1)}\right),\ n=1,2,...,j=1,...,2n+1.$$

Thus the polynomial solution $v_{n,j}^{(i)}$, $i = 1,2$ corresponding to $y_{n,j}^{(i)}$ on Ω is not homogeneous.

Remark 10.37. Observe that, under the assumption $3\tilde{\lambda} + 2\tilde{\mu} > 0,\ \tilde{\mu} > 0$, it follows that

$$\tau = \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\mu}} = \frac{1}{3} + \frac{3\tilde{\lambda}+2\tilde{\mu}}{3\tilde{\mu}} > \frac{1}{3}. \tag{10.376}$$

Therefore, it is not difficult to deduce that for all $n \geq 3$

$$|\alpha_n| = \frac{1}{2}\frac{n\tau+3\tau+2}{n\tau+2n+1} \tag{10.377}$$
$$= \frac{1}{2}\frac{1+\frac{3\tau}{n\tau}+\frac{2}{n\tau}}{1+\frac{2n}{n\tau}+\frac{1}{n\tau}} \leq \frac{1}{2}\frac{2+\frac{2}{n\tau}}{1+\frac{1}{n\tau}} \leq 1, \tag{10.378}$$

while for all $n \geq 1$

$$|\alpha_n| \leq \frac{1}{2} + \frac{\frac{3}{2n}}{1+\frac{2}{n\tau}} \leq 2. \tag{10.379}$$

The sequence (α_n) therefore is uniformly bounded with respect to τ.

Remark 10.38. Let us denote by $v_{n,j}^{(i);R} : \mathbb{R}^3 \to \mathbb{R}^3$ the vector fields

$$v_{n,j}^{(1);R}(x) = \sigma_n^{(1)}\left(\frac{|x|}{R}\right) y_{n,j}^{(1)}(\xi) + \tau_n^{(1)}\left(\frac{|x|}{R}\right) y_{n,j}^{(2)}(\xi), \tag{10.380}$$
$$v_{n,j}^{(2);R}(x) = \sigma_n^{(2)}\left(\frac{|x|}{R}\right) y_{n,j}^{(1)}(\xi) + \tau_n^{(2)}\left(\frac{|x|}{R}\right) y_{n,j}^{(2)}(\xi), \tag{10.381}$$
$$v_{n,j}^{(3);R}(x) = \sigma_n^{(3)}\left(\frac{|x|}{R}\right) y_{n,j}^{(3)}(\xi), \tag{10.382}$$

where $x = |x|\xi, |x| \le R$ and $\sigma_n^{(1)}, \tau_n^{(1)}$ are given as follows

$$\sigma_n^{(1)}\left(\frac{|x|}{R}\right) = \left(\frac{|x|}{R}\right)^{n-1}\left(\left(\frac{|x|}{R}\right)^2 + n\alpha_n\left(\frac{|x|}{R}\right)^2 - 1\right), \tag{10.383}$$

$$\tau_n^{(1)}\left(\frac{|x|}{R}\right) = \left(\frac{|x|}{R}\right)^{n-1} \alpha_n(n(n+1))^{1/2}\left(\left(\frac{|x|}{R}\right)^2 - 1\right), \tag{10.384}$$

$$\sigma_n^{(2)}\left(\frac{|x|}{R}\right) = \left(\frac{|x|}{R}\right)^{n-1} n(1+n\alpha_n)(n(n+1))^{1/2}\left(1 - \left(\frac{|x|}{R}\right)^2\right), \tag{10.385}$$

$$\tau_n^{(2)}\left(\frac{|x|}{R}\right) = \left(\frac{|x|}{R}\right)^{n-1}\left(1 - n\alpha_n\left(\left(\frac{|x|}{R}\right)^2 - 1\right)\right), \tag{10.386}$$

$$\sigma_n^{(3)}\left(\frac{|x|}{R}\right) = \left(\frac{|x|}{R}\right)^{n} \tag{10.387}$$

with

$$\tau = \frac{\tilde{\lambda} + \tilde{\mu}}{\tilde{\mu}}. \tag{10.388}$$

Then $v_{n,j}^{(i);R}$ is the unique solution of the first boundary-value problem

$$v_{n,j}^{(i);R} \in \mathrm{c}\left(\overline{\Omega_R^{\mathrm{int}}}\right) \cap \mathrm{c}^{(2)}\left(\overline{\Omega_R^{\mathrm{int}}}\right), \Diamond v_{n,j}^{(i);R} = 0 \text{ in } \overline{\Omega_R^{\mathrm{int}}}, \tag{10.389}$$

corresponding the boundary values

$$v_{n,j}^{(i);R}|\Omega_R = y_{n,j}^{(i)}. \tag{10.390}$$

We easily obtain the following theorem (see T. Gervens (1989)).

Theorem 10.39. *Suppose that f is of class $\mathrm{c}(\Omega)$. Then, the unique solution u of the Dirichlet problem $u \in \mathrm{c}^{(2)}(\Omega^{\mathrm{int}}) \cap \mathrm{c}(\overline{\Omega^{\mathrm{int}}})$, $\Diamond u = 0$ in Ω^{int} $u|\Omega = f$ is representable in the form*

$$u(x) = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\ v_{n,j}^{(i)}(x)$$

for all $x \in K$ with $K \subset \Omega_{\mathrm{int}}$ and $\mathrm{dist}(\overline{K}, \Omega) > 0$, where $(f^{(i)})^\wedge(n,j)$ are the Fourier coefficients of f with respect to the system $\{y_{n,j}^{(i)}\}$

$$(f^{(i)})^\wedge(n,j) = \left(f, y_{n,j}^{(i)}\right)_{\mathrm{l}^2(\Omega)} = \int_\Omega f(\eta) \cdot y_{n,j}^{(i)}(\eta)\ d\omega(\eta).$$

From Lemma 10.36, it is not difficult to determine the *stress vector field* $T_\nu(v^{(i)}_{n,j})(x)$ for any point $x \in \Omega_{\text{int}}$:

$$\begin{aligned} |x| T_\nu \left(v^{(1)}_{n,j}\right)(x) &= \left(\tilde{\mu}(n+2) + \tilde{\lambda}(n+3) + \alpha_n\left(\tilde{\lambda}+\tilde{\mu}\right)\right) H_{n,j}(x)x \\ &\quad + \left(\tilde{\mu} + 2\tilde{\mu} n\alpha_n\right) x^2 \nabla_x H_{n,j}(x) - 2\alpha_n \tilde{\mu}(n-1)\nabla_x H_{n,j}(x), \\ &\qquad n = 0,1,\ldots, j = 1,\ldots,2n+1, \\ |x| T_\nu \left(v^{(2)}_{n,j}\right)(x) &= (n(n+1))^{-\frac{1}{2}}(2\tilde{\mu}(n-1))\nabla_x H_{n,j}(x) - nT_\nu\left(v^{(1)}_{n,j}(x)\right), \\ &\qquad n = 1,2,\ldots, j = 1,\ldots,2n+1, \\ |x| T_\nu \left(v^{(3)}_{n,j}\right)(x) &= (n(n+1))^{-\frac{1}{2}}\tilde{\mu}(n-1) x \wedge \nabla_x H_{n,j}(x), \\ &\qquad n = 1,2,\ldots, j = 1,\ldots,2n+1. \end{aligned}$$

This leads us to the following theorem.

Theorem 10.40. *Let f be of class $\mathrm{c}(\Omega)$. Suppose that u is the solution of the inner Dirichlet problem u of the Dirichlet problem $u \in \mathrm{c}^{(2)}(\Omega^{\text{int}}) \cap \mathrm{c}(\overline{\Omega^{\text{int}}})$, $\diamondsuit u = 0$ in Ω^{int}, $u|\Omega = f$. Then*

$$|x| T_\nu(u)(x) = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) T_\nu\left(v^{(i)}_{n,j}\right)(x)$$

for each $x \in \Omega_{\text{int}}$.

Next, we note that the fields $v^{(i)}_{n,j}$ admit a decomposition into curl-free and divergence-free parts. For that purpose, we formulate the following lemma (see T. Gervens (1989)).

Lemma 10.41. *Under the assumptions of Lemma 10.36*

$$\begin{aligned} v^{(1)}_{n,j}(x) &= \delta_n \nabla_x \left(x^2 H_{n,j}(x)\right) + \varepsilon_n \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x)\right)x\right), \\ v^{(2)}_{n,j}(x) &= (n(n+1))^{-\frac{1}{2}} \nabla_x \left(H_{n,j}(x) - n\delta_n x^2 H_{n,j}(x)\right) \\ &\quad -(n(n+1))^{-\frac{1}{2}} n\varepsilon_n \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x)\right)x\right), \\ v^{(3)}_{n,j}(x) &= -(n(n+1))^{-\frac{1}{2}} \nabla_x \wedge (H_{n,j}(x)x), \end{aligned}$$

where

$$\delta_n = \frac{n+3+2n\alpha_n}{2(2n+3)}, \quad \varepsilon_n = \frac{2n\alpha_n - 1}{2(2n+3)}. \tag{10.391}$$

Proof. Elementary calculations show us that

$$\nabla_x \left(x^2 H_{n,j}(x)\right) = 2H_{n,j}(x)x + \nabla_x H_{n,j}(x), \tag{10.392}$$

and

$$\begin{aligned}
& \nabla_x \wedge \nabla_x \wedge \left(\left(x^2 H_{n,j}(x) \right) x \right) \qquad (10.393) \\
&= -\nabla_x \left(x^2 x \wedge \nabla_x H_{n,j}(x) \right) \\
&= -2x \wedge (x \wedge \nabla_x H_{n,j}(x)) + x^2 \nabla_x \wedge (\nabla_x \wedge H_{n,j}(x) x) \\
&= -2n H_{n,j}(x) x + 2x^2 \nabla_x H_{n,j}(x) + x^2 \nabla_x (\nabla_x \cdot H_{n,j}(x) x) \\
&\quad -x^2 \Delta_x (H_{n,j}(x) x) \\
&= -2n H_{n,j}(x) x + (n+3) x^2 \nabla_x H_{n,j}(x) \ .
\end{aligned}$$

This implies

$$\begin{aligned}
& H_{n,j}(x) x \qquad (10.394) \\
& \quad = (2(2n+3))^{-1} \left((n+3) \nabla_x \left(x^2 H_{n,j}(x) \right) - \nabla_x \wedge \nabla_x \wedge \left(x^2 H_{n,j}(x) x \right) \right), \\
& x^2 \nabla_x H_{n,j}(x) \\
& \quad = (2n+3)^{-1} \left(n \nabla_x \left(x^2 H_{n,j}(x) \right) + \nabla_x \wedge \nabla_x \wedge \left(x^2 H_{n,j}(x) x \right) \right) \ .
\end{aligned}$$

Therefore, the vector fields $v^{(i)}_{n,j}, i = 1,2,3$, can be written as indicated by Lemma 10.41. □

Lemma 10.41 leads us to the following result.

Theorem 10.42. *For given $f \in \mathrm{c}(\Omega)$, the uniquely determined solution u of the Dirichlet problem u of the Dirichlet problem $u \in \mathrm{c}^{(2)}(\Omega^{\mathrm{int}}) \cap \mathrm{c}(\overline{\Omega^{\mathrm{int}}})$, $\Diamond u = 0$ in Ω^{int}, $u|\Omega = f$ is given by*

$$u(x) = \nabla_x Z_1(x) + \nabla_x \wedge \nabla_x \wedge \left(x^2 Z_2(x) x \right) + \nabla_x \wedge (Z_3(x) x)$$

for all $x \in K$ with $K \subset \Omega_{\mathrm{int}}$ and $\mathrm{dist}(\overline{K}, \Omega) > 0$, where the functions $Z_i, i = 1,2,3$, can be written as follows:

$$\begin{aligned}
Z_1(x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \Bigg((f^{(1)})^\wedge(n,j) \delta_n x^2 H_{n,j}(x) \\
&\quad + \frac{(f^{(2)})^\wedge(n,j) \sigma_n}{\sqrt{n(n+1)}} \left(H_{n,j}(x) - n \delta_n x^2 H_{n,j}(x) \right) \Bigg), \\
Z_2(x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left((f^{(1)})^\wedge(n,j) - n \ (n(n+1))^{-1/2} (f^{(2)})^\wedge(n,j) \right) \right. \\
&\quad \left. \varepsilon_n H_{n,j}(x) \right), \\
Z_3(x) &= -\sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} (n(n+1))^{-1/2} (f^{(3)})^\wedge(n,j) H_{n,j}(x),
\end{aligned}$$

where $\sigma_0 = 0$ and $\sigma_n = 1$ for $n > 0$.

Obviously, the vector fields $u_i, i = 1, 2, 3$, given by

$$u_1(x) = \nabla_x Z_1(x), \quad (10.395)$$
$$u_2(x) = \nabla_x \wedge \nabla_x \wedge \left(x^2 Z_2(x) x\right), \quad (10.396)$$
$$u_3(x) = \nabla_x \wedge (Z_3(x)x) \quad (10.397)$$

satisfy

$$\nabla_x \wedge u_1(x) = 0, \quad (10.398)$$
$$\nabla_x \cdot u_2(x) = 0, \quad x \cdot \nabla_x \wedge u_2(x) = 0, \quad (10.399)$$
$$\nabla_x \cdot u_3(x) = 0, \quad x \cdot u_3(x) = 0, \quad (10.400)$$

for all $x \in K \subset \Omega_{\text{int}}$ with $\text{dist}(\overline{K}, \Omega) > 0$. The vector field u_2 is of poloidal type, while u_3 is of toroidal type.

Finally, we discuss the Neumann problem of determining polynomial solutions from given surface tractions on the unit sphere (see T. Gervens (1989)).

Lemma 10.43. *The vector fields $w_{n,j}^{(i)}, i = 1, 2, 3$, defined by*

$$w_{n,j}^{(1)}(x) = \zeta_n \left(H_{n,j}(x)x + \alpha_n x^2 \nabla_x H_{n,j}(x) - \frac{1 + 2n\alpha_n}{2(n-1)} \nabla_x H_{n,j}(x) \right),$$
$$n = 0, 2, 3, ..., j = 1, ..., 2n+1,$$
$$w_{1,j}^{(1)}(x) = 3\zeta_1 \left(H_{1,j}(x)x + \alpha_1 x^2 \nabla_x H_{1,j}(x) \right),$$
$$j = 1, 2, 3,$$
$$w_{n,j}^{(2)}(x) = (n(n+1))^{-\frac{1}{2}} (2\tilde{\mu}(n-1))^{-1/2} \nabla_x H_{n,j}(x) - (n(n+1))^{-\frac{1}{2}} n w_{n,j}^{(1)}(x),$$
$$n = 2, 3, ..., j = 1, ..., 2n+1,$$
$$w_{n,j}^{(3)}(x) = (n(n+1))^{-\frac{1}{2}} (\tilde{\mu}(n-1))^{-1} x \wedge \nabla_x H_{n,j}(x),$$
$$n = 2, 3, ..., j = 1, ..., 2n+1,$$

where

$$\alpha_n = -\frac{n\tau + 2 + 3\tau}{2(n(\tau+2)+1)}, \quad \zeta_n = \frac{1}{(\tilde{\lambda} + \tilde{\mu})(3 + n + 2n\alpha_n) - \tilde{\mu}},$$

$$H_{n,j}(x) = |x|^n Y_{n,j}(\xi), \quad x = |x|\xi, \quad \xi \in \Omega,$$

satisfy $w_{n,j}^{(i)} \in \mathrm{c}^{(2)}(\Omega^{\text{int}}) \cap \mathrm{c}(\overline{\Omega^{\text{int}}})$, $\diamondsuit u = 0$ *in* Ω^{int}, *and*

$$T_\nu \left(w_{n,j}^{(1)} \right)^- = y_{n,j}^{(1)}, \quad n = 0, 2, 3, ..., \quad j = 1, ..., 2n+1,$$
$$T_\nu \left(w_{1,j}^{(1)} \right)^- = 2y_{n,j}^{(1)} - \sqrt{2}\, y_{n,j}^{(2)}, \quad j = 1, 2, 3,$$
$$T_\nu \left(w_{n,j}^{(i)} \right)^- = y_{n,j}^{(i)}, \quad i = 2, 3; \quad n = 2, 3, ..., \quad j = 1, ..., 2n+1.$$

Note that ζ_n is well defined for all $n \geq 1$ provided that $3\tilde{\lambda} + 2\tilde{\mu} > 0$, $\tilde{\mu} > 0$.

We conclude our considerations with the following theorem.

Theorem 10.44. *Suppose that f is of class $\mathrm{c}^{(0,\gamma)}(\Omega)$, i.e., γ – Hölder continuous on Ω satisfying the conditions*

$$\int_\Omega f(\xi)\ d\omega(\xi) = 0, \quad \int_\Omega (f(\xi) \wedge \xi)\ d\omega(\xi) = 0. \tag{10.401}$$

Then the series

$$u = (f^{(1)})^\wedge(0,1) w_{0,1}^{(1)} + \sum_{j=1}^{3} (f^{(1)})^\wedge(1,j) w_{1,j}^{(1)} + \sum_{i=1}^{3} \sum_{n=2}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) w_{n,j}^{(i)}$$

solves Neumann's problem $u \in \mathrm{c}^{(2)}(\Omega^{\mathrm{int}}) \cap \mathrm{c}^{(1,\gamma)}(\overline{\Omega^{\mathrm{int}}})$, $\diamondsuit u = 0$ in Ω^{int}, $T_\nu(u) = f$ on every $K \subset \Omega_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Omega) > 0$.

The extension of our results to the sphere Ω_R around the origin with radius R is obvious (see W. Freeden et al. (1990)) and will not be worked out here.

10.8 Density Distribution

A classical problem in Earth's sciences is gravimetry, i.e., the determination of the Earth's mass density distribution from measurements of the gravitational potential or related quantities. From a mathematical point of view, the gravimetry problem amounts to the inversion of a Fredholm integral equation of first kind involving Newton's law of gravitation (see, e.g., L.L. Helms (1969), V. Michel (2002a, 2002b), W. Walter (1971) and the references therein). The gravimetry problem is ill-posed, as the inversion is not continuous (for more details see L. Ballani et al. (1993), E.W. Grafarend (1982), E. Groten (1979), W.A. Heiskanen, H. Moritz (1967), H. Moritz (1980), W. Torge (1991) and the references therein). However, this is not the only reason for the ill-posedness of the solution of the gravimetry problem. Within Hadamard's classification (existence, stability, uniqueness), we are confronted with the following situation: (Existence) It is well known that the gravitational potential is harmonic outside the Earth. Therefore, the Fredholm integral equation is unsolvable if the right hand side is non-harmonic. Moreover, there even does not exist a solution for a certain set of harmonic right hand sides. However, in our approach, it is not difficult to give a necessary and sufficient condition for the existence of a solution. Furthermore, the image of the corresponding Fredholm integral operator is dense in the space of harmonic functions with respect to the L^2-topology. Moreover, a perturbed potential outside the image can still be treated in such a way that approximations to the exact solution of the unperturbed problem can be found in an appropriate way. (Stability) The inversion of the operator, i.e., the determination of a density distribution that corresponds to a given potential, is not continuous. This means that unavoidable errors in the measurements of the potential are able to lead to a completely different density function. In consequence, regularization procedures are unavoidable. (Uniqueness) The most serious difficulty is the non-uniqueness of the solution. Essential parts of the density distribution cannot be reconstructed from the gravitational potential (for more details see V. Michel (1998), W. Freeden, V. Michel (2004), V. Michel (2005), V. Michel, K. Wolf (2008)). For every arbitrary density distribution, there exists an infinite-dimensional set of different density distributions which generate exactly the same potential. Within this context, it should be noted that a square-integrable function on a sphere, i.e., the surface of a ball, can be approximated arbitrarily well by a harmonic function. However, this is not true for square-integrable functions defined on the whole ball, i.e., including the interior. The reason is that in the second case, the anharmonic functions come into play. Therefore, a determination of a harmonic function as density distribution makes no sense if the anharmonic part of the solution is not taken into account. It should be noted that a radially symmetric density distribution, such as the standard layer model PREM (Preliminary Reference Earth Model), has a constant harmonic part, such that it will never be possible to obtain the characteristic layers of the Earth's interior if only harmonic functions are in use. The considerations of this work definitely show that methods only with harmonic functions are not able to solve the gravimetry problem. However, in general, it is necessary to include an anharmonic concept that is supposed to determine the inner composition of the Earth.

From the mathematical point of view, the gravimetry problem can be formulated by a Fredholm integral equation of the first kind,

$$V = \int_{\Omega_R^{\text{int}}} \frac{F(y)}{|y - \cdot|} \, dV(y), \tag{10.402}$$

where Ω_R^{int} is the Earth's interior, i.e., the inner space of the sphere with (mean) Earth's radius R around the origin, F is the unknown mass density function, and V is the gravitational potential, which is usually only given on a finite discrete set of points. In V. Michel (1998), V. Michel (1999), and V. Michel (2002), a multiscale approach to this problem is developed. Moreover, in W. Freeden, V. Michel (2004), V. Michel (2005), this theory is extended to the more general case of a regular (Earth's) surface Σ. It should be remarked that we are not concerned here with the determination of the potential F from the usual observables of V, such as gravity disturbances, the radial derivative, the gradient, or the Hessian of V on satellite's orbits.

We simply assume that V itself is given on the (actual) Earth's surface Ω_R (supposed here to be spherical). For more details on spaceborne gravimetry, see V. Michel (2005).

Definition 10.45. The so-called gravimetry operator $\tilde{T}$ on $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$ is given by

$$V(y) = \left(\tilde{T}F\right)(y) = \int_{\Omega_R^{\mathrm{int}}} \frac{F(x)}{|x-y|}\, dV(x), \quad y \in \overline{\Omega_R^{\mathrm{ext}}}, \tag{10.403}$$

$F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$.

Theorem 10.46. *The operator introduced by Definition 10.45 is bounded.*

Proof. By the Cauchy-Schwarz inequality, we obtain

$$(F, \frac{1}{|\cdot - y|})_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})} \leq \|F\|_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})} \left\| \frac{1}{|\cdot - y|} \right\|_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})}. \tag{10.404}$$

Then

$$\begin{aligned} |TF|^2_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})} &= \int_{\Sigma_{\mathrm{int}}} (F, \frac{1}{|\cdot - y|})^2_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})}\, dV(y) \qquad (10.405) \\ &\leq \|F\|^2_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})} \int_{\Omega_R^{\mathrm{int}}} \int_{\Omega_R^{\mathrm{int}}} \frac{1}{|x-y|}\, dV(x)\, dV(y). \end{aligned}$$

By introducing polar coordinates we find

$$\int_{\Omega_R^{\mathrm{int}}} \frac{1}{|x|^2}\, dV(x) = \int_0^R \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} r^2 \sin(\vartheta)\, d\vartheta\, d\varphi\, dr = 4\pi R. \tag{10.406}$$

Thus

$$\|T\|^2 = \sup_{\substack{F \in \mathrm{L}^2(\Omega_R^{\mathrm{int}}) \\ F \neq 0}} \frac{\|TF\|^2_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})}}{\|F\|^2_{\mathrm{L}^2(\Omega_R^{\mathrm{int}})}} \leq 4\pi R. \tag{10.407}$$

□

Common Earth models like PREM (cf. A. Dziewonski, D.L. Anderson (1981), A. Dziewonski, D.L. Anderson (1984)) consider the density to be radially symmetric. This property is inherited by the operator.

Theorem 10.47. *For radially symmetric functions, i.e., all* $F \in \mathrm{L}^2(\Omega_R^{\mathrm{int}})$ *satisfying* $F(x) = \tilde{F}(|x|)$ *with* $\tilde{F} \in \mathrm{L}^2[0,R]$*, the image under* $\tilde{T}$ *is radially symmetric*

$$(\tilde{T}F)(r\xi) = -\frac{2\pi}{r} \int_0^R s\tilde{F}(s)(|r-s| - (r+s))\, ds, r \in [0,R],\ \xi \in \Omega, \tag{10.408}$$

i.e., $\tilde{T}F$ *only depends on the radius* r.

Proof. Let $x = r\xi$ and $y = s\eta$, with $r, s \in [0, R]$ and $\xi, \eta \in \Omega$. Then, the Funk-Hecke formula shows us

$$\begin{aligned}(\tilde{T}F)(x) &= \int_{\Omega_R^{\mathrm{int}}} \frac{F(y)}{|x-y|}\, dV(y) \\ &= \int_0^R s^2 \tilde{F}(s) \int_\Omega \frac{1}{\sqrt{r^2+s^2-2(r\xi)\cdot(s\eta)}}\, d\omega(\eta)\, ds \\ &= \int_0^R s^2 \tilde{F}(s) 2\pi \int_{-1}^1 \frac{1}{\sqrt{r^2+s^2-2rst}}\, dt\, ds,\end{aligned}$$

The last equations only depends on r, i.e., the solution is radially symmetric. For the explicit calculation, we use

$$\int_{-1}^1 \frac{1}{\sqrt{a-bt}}\, dt = -\frac{2}{b}\left(\sqrt{a-b} - \sqrt{a+b}\right). \tag{10.409}$$

Observing this integral, we obtain

$$(\tilde{T}F)(r\xi) = -\int_0^R s^2 F(x) 2\pi \frac{1}{rs}\left(\sqrt{r^2+s^2-2rs} - \sqrt{r^2+s^2+2rs}\right)\, ds, \tag{10.410}$$

$\xi \in \Omega$, which is the desired result. □

The radial symmetry of the Earth's interior is obviously just a first approximation. However, deviations from this will still be small compared to the discontinuity of the radial parts.

It is a well-known fact that Newton volume integral representing the gravitational potential V as introduced by Definition 10.45 satisfies the Laplace equation in the outer space. In fact, the proof of Theorem 10.48 is an immediate consequence of the harmonicity of the integrand in $\overline{\Omega_R^{\mathrm{int}}}$.

Theorem 10.48. *Let* $F : \overline{\Omega_R^{\mathrm{int}}} \to \mathbb{R}$ *be an integrable bounded function. Then*

$$x \mapsto V(x) = \int_{\Omega_R^{\mathrm{int}}} \frac{F(y)}{|x-y|}\, dV(y) \tag{10.411}$$

satisfies

$$\Delta_x \int_{\Omega_R^{\mathrm{int}}} \frac{F(y)}{|x-y|}\, dV(y) = 0 \tag{10.412}$$

for all $x \in \Omega_R^{\mathrm{ext}}$.

Next, we are interested in showing that the Newton integral in the inner space satisfies the Poisson equation at least under some canonical conditions on the density function. Our considerations below essentially follow R. Leis (1967) and S.G. Michlin (1975).

Theorem 10.49. *Let* $F : \overline{\Omega_R^{\text{int}}} \to \mathbb{R}$ *be a continuous function. Then* V *is of class* $\mathrm{C}(\overline{\Omega_R^{\text{int}}})$. *Furthermore, we have*

$$\nabla V(x) = \int_{\overline{\Omega_R^{\text{int}}}} F(y) \nabla_x \frac{1}{|x-y|} \, dV(y). \tag{10.413}$$

Proof. We replace the fundamental solution of potential theory $S : (x, y) \mapsto S(|x-y|), x \neq y$, given by

$$S(|x-y|) = \frac{1}{|x-y|} \tag{10.414}$$

by a 'regularization' of the form

$$S^\rho(|x-y|) = \begin{cases} \dfrac{1}{2\rho}\left(3 - \dfrac{1}{\rho^2}|x-y|^2\right), & |x-y| \leq \rho \\[2ex] \dfrac{1}{|x-y|}, & |x-y| > \rho, \end{cases} \tag{10.415}$$

$\rho > 0$. In other words, by letting $r = |x-y|$, we replace

$$S(r) = \frac{1}{r}, \qquad r > 0, \tag{10.416}$$

by

$$S^\rho(r) = \begin{cases} \dfrac{1}{2\rho}\left(3 - \dfrac{1}{\rho^2} r^2\right), & r \leq \rho \\[2ex] \dfrac{1}{r}, & r > \rho. \end{cases} \tag{10.417}$$

S^ρ is continuously differentiable for all $r \geq 0$. Furthermore, $S(r) = S^\rho(r)$ for all $r > \rho$.

We set

$$V_S(x) = \int_{\overline{\Omega_R^{\text{int}}}} F(y) S(|x-y|) \, dV(y) \tag{10.418}$$

and

$$V_{S^\rho}(x) = \int_{\overline{\Omega_R^{\text{int}}}} F(y) S^\rho(|x-y|) \, dV(y). \tag{10.419}$$

The integrands of V_S and V_{S^ρ} differ only in the ball around the point x with radius ρ. Moreover, the function $F : \overline{\Omega_R^{\text{int}}} \to \mathbb{R}$ is supposed to be continuous on $\overline{\Omega_R^{\text{int}}}$. Hence, it is uniformly bounded on $\overline{\Omega_R^{\text{int}}}$. This shows us that

$$|V_S(x) - V_{S^\rho}(x)| = O\left(\int_{|x-y| \leq \rho} (S(|x-y|) - S^\rho(|x-y|) \, dV(y) \right) = O(\rho^2). \tag{10.420}$$

Therefore, V_S is of class $\mathrm{C}(\overline{\Omega_R^{\mathrm{int}}})$ as limit of a uniformly convergent sequence of continuous functions on $\overline{\Omega_R^{\mathrm{int}}}$.

Furthermore, we let

$$v_S(x) = \int_{\overline{\Omega_R^{\mathrm{int}}}} F(y) \nabla_x S(|x-y|)\, dV(y) \tag{10.421}$$

and

$$v_{S^\rho}(x) = \int_{\overline{\Omega_R^{\mathrm{int}}}} F(y) \nabla_x S^\rho(|x-y|)\, dV(y). \tag{10.422}$$

Because of $| \nabla_x S(|x-y|) |= O((S(|x-y|)^2)$, the integrals v_S and v_S^ρ exist for all $x \in \overline{\Omega_R^{\mathrm{int}}}$. It is not difficult to see that

$$\sup_{x\in\overline{\Omega_R^{\mathrm{int}}}} |v_S(x) - v_{S^\rho}(x)| = \sup_{x\in\overline{\Omega_R^{\mathrm{int}}}} |\nabla V_S(x) - \nabla V_{S^\rho}(x)| = O(\rho). \tag{10.423}$$

Consequently, v_s is a continuous vector field on $\overline{\Omega_R^{\mathrm{int}}}$. Moreover, as the relation (10.423) holds uniformly on $\overline{\Omega_R^{\mathrm{int}}}$, we obtain in connection with well-known theorems of classical analysis

$$v_S(x) = \nabla V_S(x) = \int_{\overline{\Omega_R^{\mathrm{int}}}} F(y) \nabla_x S(|x-y|)\, dV(y). \tag{10.424}$$

This is the desired result. □

Next, we come to the Poisson equation under the assumption of Hölder continuity of the function F on $\overline{\Omega_R^{\mathrm{int}}}$.

Theorem 10.50. *If F is Hölder continuous on $\overline{\Omega_R^{\mathrm{int}}}$, then the Poisson equation*

$$\Delta_x \int_{\overline{\Omega_R^{\mathrm{int}}}} F(y) \frac{1}{|x-y|}\, dV(y) = -4\pi F(x) \tag{10.425}$$

holds for all $x \in \Omega_R^{\mathrm{int}}$.

Proof. We introduce

$$H^\rho(|x-y|) = \begin{cases} \dfrac{1}{2\rho^3}\left(5 - \dfrac{3}{\rho^2}|x-y|^2\right), & |x-y| \leq \rho \\[2ex] \dfrac{1}{|x-y|^3}, & |x-y| > \rho. \end{cases} \tag{10.426}$$

With $r = |x - y|$, we have

$$H^\rho(r) = \begin{cases} \frac{1}{2\rho^3}\left(5 - \frac{3}{\rho^2} r^2\right), & r \le \rho \\ \frac{1}{r^3}, & r > \rho. \end{cases} \tag{10.427}$$

H^ρ is continuously differentiable for all $r \geq 0$. Moreover, by already known arguments, it can be shown (cf. Theorem 10.49) that the vector field

$$-\int_{\overline{\Omega_R^{\text{int}}}} F(y) H^\rho(|x-y|)(x-y)\ dV(y) \tag{10.428}$$

converges uniformly on $\overline{\Omega_R^{\text{int}}}$ to the limit field

$$\nabla V(x) = -\int_{\overline{\Omega_R^{\text{int}}}} F(y) \frac{x-y}{|x-y|^3}\ dV(y). \tag{10.429}$$

For all $x \in \mathbb{R}^3$ with $|x - y| \leq \rho$, a simple calculation yields

$$\nabla_x \cdot ((x-y) H^\rho(|x-y|)) = \frac{15}{2}\left(\frac{1}{\rho^3} - \frac{|x-y|^2}{\rho^5}\right). \tag{10.430}$$

Furthermore,

$$\int\limits_{|x-y|\le\rho} \nabla_x \cdot ((x-y) H^\rho(|x-y|))\ dV(y) = 4\pi. \tag{10.431}$$

Hence it is not hard to verify that

$$\begin{aligned} &-\nabla_x \cdot \int_{\overline{\Omega_R^{\text{int}}}} F(y) H^\rho(|x-y|)(x-y)\ dV(y) \qquad (10.432)\\ &= -\int_{|x-y|\le\rho} F(y) \nabla_x \cdot (H^\rho(|x-y|)(x-y))\ dV(y) \\ &= -4\pi F(x) \\ &\quad + \int\limits_{|x-y|\le\rho} (F(x) - F(y)) \nabla_x \cdot (H^\rho(|x-y|)(x-y))\ dV(y). \end{aligned}$$

The Hölder continuity of F assures the estimate

$$\sup_{\overline{\Omega_R^{\text{int}}}} \left| -\nabla_x \cdot \int_{\overline{\Omega_R^{\text{int}}}} F(y)(x-y) H^\rho(|x-y|)\ dV(y) + 4\pi F(x) \right| = O(\rho^\alpha) \tag{10.433}$$

uniformly as to $x \in \overline{\Omega_R^{\text{int}}}$. In an analogous way, we are able to show that the first partial derivatives of (10.428) uniformly converge to continuous limit fields. Again, well known theorems of classical analysis show us that ∇V is differentiable in $\overline{\Omega_R^{\text{int}}}$, and we have

$$\Delta_x \int_{\overline{\Omega_R^{\text{int}}}} \frac{F(y)}{|x-y|} \, dV(y) = -4\pi F(x), \quad x \in \Omega_R^{\text{int}}, \tag{10.434}$$

as required. □

Remark 10.51. Theorem 10.49 shows us that, for $x \in \overline{\Omega_R^{\text{int}}}$ and $F \in \mathrm{C}(\overline{\Omega_R^{\text{int}}})$, the improper integral

$$V(x) = \int_{\overline{\Omega_R^{\text{int}}}} F(y) \frac{1}{|x-y|} \, dV(y) \tag{10.435}$$

can be regularized by

$$\int_{\overline{\Omega_R^{\text{int}}}} F(y) S^\rho(|x-y|) \, dV(y) \tag{10.436}$$

such that

$$\lim_{\rho \to 0} \sup_{x \in \overline{\Omega_R^{\text{int}}}} \left| \int_{\overline{\Omega_R^{\text{int}}}} F(y) \frac{1}{|x-y|} \, dV(y) - \int_{\overline{\Omega_R^{\text{int}}}} F(y) S^\rho(|x-y|) \, dV(y) \right| = 0. \tag{10.437}$$

Even more, the vector field

$$\nabla V(x) = -\int_{\overline{\Omega_R^{\text{int}}}} F(y) \frac{x-y}{|x-y|^3} \, dV(y) \tag{10.438}$$

admits the regularization

$$-\int_{\overline{\Omega_R^{\text{int}}}} F(y) \nabla_x H^\rho(|x-y|) \, dV(y). \tag{10.439}$$

such that

$$\lim_{\rho \to 0} \sup_{x \in \overline{\Omega_R^{\text{int}}}} \left| \int_{\overline{\Omega_R^{\text{int}}}} F(y) \frac{x-y}{|x-y|^3} \, dV(y) - \int_{\overline{\Omega_R^{\text{int}}}} F(y) \nabla_x H^\rho(|x-y|) \, dV(y) \right| = 0. \tag{10.440}$$

Whereas boundary-value problems require tools for the approximation of functions on the boundary Ω_R (i.e., in our case, the Earth's surface), we have to deal with functions which are defined on the inner or outer space of Ω_R, i.e., on three-dimensional domains. For this purpose, the following well known theorems are important. Concerning the proofs we refer to, for example, W. Freeden (1980a) and V. Michel (1999).

Theorem 10.52. *The set of harmonic functions on a ball* $\overline{\Omega_R^{\rm int}}$,

$$\mathrm{Harm}(\overline{\Omega_R^{\rm int}}) = \left\{ F \in \mathrm{C}^{(2)}(\overline{\Omega_R^{\rm int}}) \middle| \Delta F = 0 \text{ in } \Omega_R^{\rm int} \right\}, \tag{10.441}$$

is a closed subspace of $\mathrm{L}^2(\overline{\Omega_R^{\rm int}})$. *Moreover, the inner harmonics*

$$\left\{ H_{n,j}^{\rm int}(R;\cdot) \right\}_{n=0,1,\ldots,j=1,\ldots,2n+1}, \tag{10.442}$$

given by

$$H_{n,j}^{\rm int}(R;x) = \sqrt{\frac{2n+3}{R^3}} \left(\frac{|x|}{R} \right)^n Y_{n,j}\left(\frac{x}{|x|} \right),$$

$x \in \overline{\Omega_R^{\rm int}}$, *constitute a complete orthonormal system in the Hilbert space* $\mathrm{Harm}(\overline{\Omega_R^{\rm int}})$, *with respect to the inner product* $(\cdot,\cdot)_{\mathrm{L}^2(\overline{\Omega_R^{\rm int}})}$.

Theorem 10.53. *The set of square-integrable harmonic functions on the outer space* $\overline{\Omega_R^{\rm ext}}$,

$$\mathrm{Harm}(\overline{\Omega_R^{\rm ext}}) = \left\{ F \in \mathrm{C}^{(2)}(\overline{\Omega_R^{\rm ext}}) \middle| \int_{\overline{\Omega_R^{\rm ext}}} (F(x))^2 \; dV(x) < \infty, \; \Delta F = 0 \text{ in } \Omega_R^{\rm ext} \right\}, \tag{10.443}$$

is a closed subspace of $\mathrm{L}^2(\overline{\Omega_R^{\rm ext}})$. *Moreover, the system of outer harmonics* $\{H_{-n-1,j}^{\rm ext}(R;\cdot)\}_{n=1,2,\ldots,j=1,\ldots,2n+1}$, *given by*

$$H_{-n-1,j}^{\rm ext}(R;x) = \sqrt{\frac{2n-1}{R^3}} \left(\frac{R}{|x|} \right)^{n+1} Y_{n,j}\left(\frac{x}{|x|} \right), \tag{10.444}$$

$x \in \overline{\Omega_R^{\rm ext}}$, *constitutes a complete orthonormal system in the Hilbert space* $\mathrm{Harm}(\overline{\Omega_R^{\rm ext}})$ *with respect to the inner product* $(\cdot,\cdot)_{\mathrm{L}^2(\overline{\Omega_R^{\rm ext}})}$.

Note that an outer harmonic of degree $n = 0$ possesses the form

$$H_{-1,1}^{\rm ext}(R;x) = C \frac{1}{|x|} Y_{0,1}\left(\frac{x}{|x|} \right) = \frac{C}{\sqrt{4\pi}|x|}, \quad x \in \overline{\Omega_R^{\rm ext}}, \tag{10.445}$$

$C \in \mathbb{R} \backslash \{0\}$ constant. In consequence, this function is not an element of $\mathrm{L}^2(\overline{\Omega_R^{\rm ext}})$.

The series expansion of the single pole in terms of Legendre polynomials allows us to investigate the Fredholm integral operator T in the case of a spherical surface Ω_R (see also N. Weck (1972) for a more general surface and V. Michel (1999) in the spherical case).

Theorem 10.54. *If Ω_R is a sphere with (Earth's) radius $R > 0$, then the operator* $\tilde{T} : \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}) \to \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}))$*, given by*

$$\left(\tilde{T}F\right)(y) = \int_{\overline{\Omega_R^{\mathrm{int}}}} \frac{F(x)}{|x-y|}\, dV(x), \quad y \in \overline{\Omega_R^{\mathrm{ext}}}, \tag{10.446}$$

has the null space (kernel)

$$\ker \tilde{T} = \left\{ G \in \mathrm{L}^2\left(\overline{\Omega_R^{\mathrm{int}}}\right) \,\middle|\, \int_{\overline{\Omega_R^{\mathrm{int}}}} G(x)H(x)\, dV(x) = 0,\ H \in \mathrm{Harm}\left(\overline{\Omega_R^{\mathrm{int}}}\right) \right\}, \tag{10.447}$$

i.e., $\ker \tilde{T}$ *is the* $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$*-orthogonal space of* $\mathrm{Harm}\left(\overline{\Omega_R^{\mathrm{int}}}\right)$.

Proof. Let $F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$ with $\tilde{T}F = 0$. Since $\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ is a closed subspace of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$, there exists a unique orthogonal decomposition

$$F = F_{\mathrm{harm}} + G, \tag{10.448}$$

where $F_{\mathrm{harm}} \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ and $G \perp \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$, i.e.,

$$\int_{\overline{\Omega_R^{\mathrm{int}}}} G(x)H(x)\, dV(x) = 0 \tag{10.449}$$

for all $H \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$. We have to show that $\tilde{T}F = 0$ is equivalent to $F_{\mathrm{harm}} = 0$.

For that purpose, F_{harm} allows the representation as a Fourier series in terms of inner harmonics as follows:

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} H_{n,j}^{\mathrm{int}}(R;\cdot) + G, \tag{10.450}$$

where the equality is understood in the topology of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$. Let $y \in \Omega_R^{\mathrm{ext}}$ be arbitrary but fixed. Then, the potential at y corresponding to the mass density distribution F can written in the form

$$\begin{aligned}
&\left(\tilde{T}F\right)(y) \\
&= \int_{\overline{\Omega_R^{\mathrm{int}}}} \frac{1}{|x-y|} F(x)\, dV(x) \\
&= \int_{\overline{\Omega_R^{\mathrm{ext}}}} \frac{1}{|y|} \sum_{n=0}^{\infty} \left(\frac{|\cdot|}{|y|}\right)^n \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right) Y_{n,j}\left(\frac{x}{|x|}\right).
\end{aligned}$$

Note that the sequence of functions

$$\left(\sum_{n=0}^{N} \left(\frac{|\cdot|}{|y|} \right)^n \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j} \left(\frac{y}{|y|} \right) Y_{n,j} \left(\frac{\cdot}{|\cdot|} \right) \right)_{N \in \mathbb{N}} \tag{10.451}$$

converges uniformly and, therefore, in the $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$ sense:

$$\begin{aligned} & \left\| \sum_{n=N+1}^{\infty} \left(\frac{|\cdot|}{|y|} \right)^n \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j} \left(\frac{y}{|y|} \right) Y_{n,j} \left(\frac{\cdot}{|\cdot|} \right) \right\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} \\ &= \left\| \sum_{n=N+1}^{\infty} \left(\frac{|\cdot|}{|y|} \right)^n P_n \left(\frac{y}{|y|} \cdot \frac{\cdot}{|\cdot|} \right) \right\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} \\ &\leq \sqrt{\frac{4}{3}\pi R^3} \left\| \sum_{n=N+1}^{\infty} \left(\frac{|\cdot|}{|y|} \right)^n P_n \left(\frac{y}{|y|} \cdot \frac{\cdot}{|\cdot|} \right) \right\|_{\mathrm{C}(\overline{\Omega_R^{\mathrm{int}}})} \\ &\leq \sqrt{\frac{4}{3}\pi R^3} \sum_{n=N+1}^{\infty} \left(\frac{R}{|y|} \right)^n \longrightarrow 0, \quad N \to \infty \ . \end{aligned} \tag{10.452}$$

Since the strong convergence in a Hilbert space always implies the weak convergence in the same space, we obtain

$$\begin{aligned} \left(\tilde{T} F \right)(y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{|y|^{n+1}} \frac{4\pi}{2n+1} Y_{n,j} \left(\frac{y}{|y|} \right) \\ & \quad \sqrt{\frac{R^3}{2n+3}} R^n \int_{\overline{\Omega_R^{\mathrm{int}}}} H_{n,j}^{\mathrm{int}}(R;x) F(x) \, dV(x) \\ &= R^2 \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{4\pi}{2n+1} \frac{1}{\sqrt{(2n-1)(2n+3)}} H_{-n-1,j}^{\mathrm{ext}}(R;y) \\ & \quad \int_{\overline{\Omega_R^{\mathrm{int}}}} H_{n,j}^{\mathrm{int}}(R;x) F(x) \, dV(x). \end{aligned}$$

Consequently, $\tilde{T}F = 0$ is equivalent to $(H_{n,j}^{\mathrm{int}}(\sigma;\cdot), F)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} = 0, n = 0,1,\ldots, j = 1,\ldots,2n+1$. But this means that $F_{\mathrm{harm}} = 0$. □

An appropriate coordinate transformation (see, e.g., W.A. Heiskanen, H. Moritz (1967)) allows the representation of the potential $\tilde{T}F$ in the basis $\{H^{\mathrm{R}}_{-n-1,j}\}_{\substack{n=1,2,\ldots,\\ j=1,\ldots,2n+1}}$, such that the coefficient of $H^{R}_{-1,1}$ vanish. We assume that such a coordinate transformation has already been performed such that we are able to deal with basis functions in $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$.

Theorem 10.54 implies the following corollary.

Corollary 10.55. *Let the operator* $\tilde{T}$ *be given by Theorem 10.54. If* P *is of class* $\mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$ *and if there exists a harmonic solution of the problem*

$$\tilde{T}F = P, \tag{10.453}$$

with $F \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ *unknown, then* F *is unique and given by its Fourier coefficients*

$$\begin{aligned}&\int_{\overline{\Omega_R^{\mathrm{int}}}} F(x) H_{n,j}^{\mathrm{int}}(R;x)\ dV(x)\\ &\quad = \frac{2n+1}{4\pi R^2}\sqrt{(2n-1)(2n+3)} \int_{\overline{\Omega_R^{\mathrm{int}}}} P(x) H_{-n-1,j}^{\mathrm{ext}}(R;x) dV(x),\end{aligned}$$

$n = 1, 2, \ldots, j = 1, \ldots, 2n+1$ *and*

$$\left(F, H_{0,1}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} = 0, \tag{10.454}$$

i.e.,

$$F = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{2n+1}{4\pi R^2} \sqrt{(2n-1)(2n+3)} \left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} H_{n,j}^{\mathrm{int}}(R;\cdot) \tag{10.455}$$

in the sense of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$*, where*

$$(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot))_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} = \int_{\overline{\Omega_R^{\mathrm{ext}}}} P(x) H_{-n-1}^{\mathrm{ext}}(R;x)\ dV(x). \tag{10.456}$$

The null space of the operator $\tilde{T}$, which is the $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$-orthogonal space of the space of harmonic functions on $\overline{\Omega_R^{\mathrm{int}}}$, is called the space of anharmonic functions.

$$\begin{aligned}\mathrm{Anharm}\left(\overline{\Omega_R^{\mathrm{int}}}\right) &= \left\{F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}) \,\middle|\, (F,H)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} = 0, \ \text{ if } H \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})\right\}\\ &= \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})^{\perp_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})}}\end{aligned}$$

The elements of space $\mathrm{Anharm}(\overline{\Omega_R^{\mathrm{int}}})$ are called anharmonic functions. A theoretical characterization of this space in terms of distributions and within a Sobolev space nomenclature is given in N. Weck (1972). The non -uniqueness of the solution of the gravimetry problem is a serious difficulty. Only a few publications, such as L. Ballani et al. (1993), and V. Michel (1999), have further investigated the treatment of the anharmonic functions. In our approach, we follow W. Freeden, V. Michel (2004).

Concerning the solvability of the equation $\tilde{T}F = P$, we are led to formulate another corollary of Theorem 10.54.

Corollary 10.56. *The equation $\tilde{T}F = P$ of Corollary 10.55 is solvable if and only if P is harmonic and the series*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}(2n+1)^2(2n-1)(2n+3)\left(\int_{\overline{\Omega_R^{\mathrm{int}}}} P(x)H_{-n-1,j}^{\mathrm{ext}}(R;x)\ dV(x)\right)^2 \tag{10.457}$$

is convergent.

This inequality is obtained by observing the requirement

$$F \in \mathrm{Harm}\left(\overline{\Omega_R^{\mathrm{int}}}\right) = \overline{\left\{H_{n,j}^{\mathrm{int}}(R;\cdot)\middle| n \in \mathbb{N}_0, j \in \{1,\ldots,2n+1\}\right\}}^{\|\cdot\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})}}, \tag{10.458}$$

which implies

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\int_{\overline{\Omega_R^{\mathrm{int}}}} F(x)H_{n,j}^{\mathrm{int}}(R;x)\ dV(x)\right)^2 \tag{10.459}$$

It is a well known result in functional analysis that operators of the type $\tilde{T}$ are compact, where compact operators are never continuously invertible. Note that for this purpose, the operators should be regarded as operators from $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$ to $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$, since the kernel $(x,y) \to \frac{1}{|x-y|}$ is not a member of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}} \times \overline{\Omega_R^{\mathrm{ext}}})$.

Theorem 10.57. *Let $\tilde{T}$ be given as in Theorem 10.54. Then the restricted operator*

$$\tilde{T}\big|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}) : \mathrm{Harm}\left(\overline{\Omega_R^{\mathrm{int}}}\right) \to \tilde{T}\left(\mathrm{Harm}\left(\overline{\Omega_R^{\mathrm{int}}}\right)\right) \tag{10.460}$$

is invertible. However, the inverse operator $(\tilde{T}\big|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}))^{-1}$ is discontinuous.

According to Hadamard, an inverse problem $\tilde{T}F = P$ is classified in the following way: The problem is called well-posed if the following three criteria are satisfied: (a) A solution F exists. (b) The solution F is unique. (c) The solution F is stable, i.e., $\tilde{T}^{-1}$ is continuous. Otherwise, the problem is called ill-posed. The results that we derived up to now show that the gravimetry problem (as formulated here in a spherical setup) is ill-posed. Unavoidable errors in measurements can perturb the right hand side such that a formerly solvable problem can become unsolvable. In such a case, a projection of the measured potential on the space

$$\overline{\left\{H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\middle| n \in \mathbb{N}, j \in \{1,\ldots,2n+1\}\right\}}^{\|\cdot\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}} \tag{10.461}$$

allows us to regain solvability, provided that the Fourier coefficients of the right hand side decay sufficiently fast. However, errors in measurements do not only affect the solvability. They can, in particular, seriously change the calculated solution, since the instability, i.e., the discontinuity of $(\tilde{T}|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}))^{-1}$, causes a high sensitivity of the solution to variations of the right hand side of the equation. Last but not least, the non-uniqueness of the solution has to be taken into account.

An (with respect to $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$) orthogonal basis for $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$ has been constructed in L. Ballani et al. (1993) in the spherical case. A non-orthogonal anharmonic basis has been developed in V. Michel (1999) (also in the spherical case). We omit the proofs here and only quote the results.

Theorem 10.58. *The following statements hold true.*

(a) A complete $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$*-orthogonal system in* $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$ *is given by*

$$\left\{r\xi \mapsto r^n P_{k,n}(r^2) Y_{n,j}(\xi)\right\}_{k\in\mathbb{N}, n\in\mathbb{N}_0, j\in\{1,\ldots,2n+1\}}, \tag{10.462}$$

where $\{P_{k,n}\}_{k\in\mathbb{N}; n\in\mathbb{N}_0}$ *is a system of polynomials defined by*

$$P_{k,n}(x) = \sqrt{\frac{2}{R^{2n+3}}}\, G_k\left(n+\frac{3}{2}, n+\frac{3}{2}; \frac{x}{R^2}\right) . \tag{10.463}$$

The functions $G_k, k \in \mathbb{N}_0$*, are the Jacobi polynomials, which are the only polynomials on* $[0,1]$ *that satisfy the following conditions for all* $n, m \in \mathbb{N}_0$*:*

(i) $G_n(a,b;\cdot)$ *is a polynomial of degree* n *on* $[0,1]$.
(ii) $G_n(a,b;0) = 1$.
(iii) $\int_0^1 x^{a-1}(1-x)^{b-a} G_n(a,b;x) G_m(a,b;x)\, dx = 0$ *for* $n \neq m$,

provided that $a > 0$ *and* $b > a-1$.

(b) A closed system in $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$ *is given by (see V. Michel (1999))*

$$\left\{r\xi \mapsto \left(r^{n+2k} - \frac{(2n+3)R^{2k}}{2n+2k+3}\, r^n\right) Y_{n,j}(\xi)\right\}_{k\in\mathbb{N}, n\in\mathbb{N}_0, j\in\{1,\ldots,2n+1\}} . \tag{10.464}$$

Moreover, the basis functions form polynomials of degree $\leq N \in \mathbb{N}\backslash\{1\}$ *if and only if the index triple* (k,n,j) *is within the range*

$$n \in \{0,\ldots,N-2\},\ j \in \{1,\ldots,2n+1\},\ k \in \left\{1,\ldots,\left[\frac{N-n}{2}\right]\right\},$$

where $[\cdot]$ *is the Gauss bracket, defined by* $[x] = \max\{\nu \in \mathbb{Z} | \nu \leq x\}$, $x \in \mathbb{R}$. *The set of anharmonic polynomials with degree* $\leq N$ *has the dimension* $\frac{1}{6}N^3 - \frac{1}{6}N$.

Note that the set of harmonic polynomials on $\overline{\Omega_R^{\mathrm{int}}}$ with degree $\leq N$ has the dimension $(N+1)^2$. Most surprisingly, in the case of a bandlimited reconstruction of the mass density function, the reconstructable part has a lower dimension than the null space.

The obvious advantage of the system in part (a) of Theorem 10.58 is its orthogonality. On the other hand, the system described in part (b) has a radial part, which is explicitly given, whereas the radial part of the orthogonal system has to be calculated iteratively by means of recurrence formulas.

The important role of the anharmonic functions in the theory of the gravimetry problem is also stressed if we investigate a radially symmetric density distribution which is approximately given for the mantle and the outer and inner core of the Earth. Such a structure of spherical layers leaves (almost) no information in the gravitational potential and, therefore, cannot be recovered by means of harmonic functions (see also V. Michel (1999)).

Theorem 10.59. *Let $F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$ and $G \in \mathrm{L}^2([0,R])$ be given functions such that*

$$F(x) = G(|x|) \tag{10.465}$$

for all $x \in \overline{\Omega_R^{\mathrm{int}}}$, where Ω_R is the sphere around the origin with radius $R > 0$. Then

$$\left(\tilde{T}F\right)(y) = 4\pi \int_0^R r^2 G(r)\ dr\ \frac{1}{|y|}, \quad y \in \Omega_R^{\mathrm{ext}}\ . \tag{10.466}$$

Moreover, the unique harmonic solution $H \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ of the equation $\tilde{T}H = \tilde{T}F$ is constant and given by

$$H = \frac{3}{R^3} \int_0^R r^2 G(r)\ dr\ . \tag{10.467}$$

Proof. We know that the application of the operator $\tilde{T}$ to F yields

$$\begin{aligned}
&\left(\tilde{T}F\right)(y) \\
&= \int_{\Omega_R^{\mathrm{int}}} \frac{1}{|y|} \sum_{n=0}^{\infty} \left(\frac{|x|}{|y|}\right)^n \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right) Y_{n,j}\left(\frac{x}{|x|}\right) F(x) dV(x) \\
&= \frac{1}{|y|} \sum_{n=0}^{\infty} \frac{1}{|y|^n} \frac{4\pi}{2n+1} \sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right) \int_{\Omega_R^{\mathrm{int}}} |x|^n Y_{n,j}\left(\frac{x}{|x|}\right) F(x)\ dV(x),
\end{aligned}$$

$y \in \Omega_R^{\mathrm{ext}}$, since the strong convergence in a Hilbert space implies the weak convergence in the same space. The application of the radial symmetry of

F to the inner products in the obtained series implies

$$\begin{aligned} \int_{\overline{\Omega_R^{\rm int}}} |x|^n Y_{n,j}\left(\frac{x}{|x|}\right) F(x)\, dV(x) &= \int_0^R r^{2+n} \int_\Omega Y_{n,j}(\xi) F(r\xi)\, d\omega(\xi)\, dr \\ &= \int_0^R r^{2+n} G(r)\, dr \int_\Omega Y_{n,j}(\xi)\cdot 1\, d\omega(\xi) \\ &= \sqrt{4\pi}\int_0^R r^2 G(r)\, dr\ \delta_{n0}\delta_{j1}, \end{aligned}$$

where we have observed the $L^2(\Omega)$-orthonormality of the spherical harmonics system $\{Y_{n,j}\}_{n=0,1,\ldots,j=1,\ldots,2n+1}$. Consequently, the potential of F can be written as

$$\left(\tilde{T}F\right)(y) = 4\pi \int_0^R r^2 G(r)\, dr\, \frac{1}{|y|}, \quad y \in \Omega_R^{\rm ext}, \tag{10.468}$$

since $Y_{0,1} = \frac{1}{\sqrt{4\pi}}$. If we are now looking for a harmonic function $H \in \mathrm{Harm}(\overline{\Omega_R^{\rm int}})$ with

$$H = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \int_{\overline{\Omega_R^{\rm int}}} H(y) H_{n,j}^{\rm int}(R;y)\, dV(y) H_{n,j}^{\rm int}(R;\cdot) \tag{10.469}$$

(in the sense of $L^2(\overline{\Omega_R^{\rm int}})$) that solves the equation $\tilde{T}H = \tilde{T}F$, then we obtain the identity

$$\begin{aligned} &\left(\tilde{T}H\right)(y) \\ &= \int_{\overline{\Omega_R^{\rm int}}} \frac{1}{|y|}\sum_{n=0}^{\infty}\left(\frac{|x|}{|y|}\right)^n \frac{4\pi}{2n+1}\sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right) Y_{n,j}\left(\frac{x}{|x|}\right) H(x)\, dV(x) \\ &= \int_{\overline{\Omega_R^{\rm int}}} \frac{1}{|y|}\sum_{n=0}^{\infty}\frac{1}{|y|^n}\frac{4\pi}{2n+1}\sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right)\sigma^n\sqrt{\frac{\sigma^3}{2n+3}} H_{n,j}^{\rm int}(\sigma;\cdot) H(x)\, dV(x) \\ &= \sum_{n=0}^{\infty}\frac{1}{|y|^{n+1}}\frac{4\pi}{2n+1}\sum_{j=1}^{2n+1} Y_{n,j}\left(\frac{y}{|y|}\right)\sqrt{\frac{\sigma^{2n+3}}{2n+3}}\int_{\overline{\Omega_R^{\rm int}}} H(x) H_{n,j}^{\rm int}(\sigma;x)\, dV(x) \\ &= 4\pi\int_0^\sigma r^2 G(r)\, dr\, \frac{1}{|y|}, \quad y \in \Omega_R^{\rm ext}\ . \end{aligned}$$

Obviously, the linear independence of the functions $y \mapsto \frac{1}{|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right)$ implies that all but one of the Fourier coefficients of H vanish:

$$\begin{aligned} \int_{\overline{\Omega_R^{\rm int}}} H_{n,j}^{\rm int}(R;x) H(x)\, dV(x) &= 0, \quad n \in \mathbb{N}, j = 1,\ldots,2n+1, \\ \int_{\overline{\Omega_R^{\rm int}}} H_{0,1}^{\rm int}(R;x) H(x)\, dV(x) &= \sqrt{4\pi}\int_0^R r^2 G(r)\, dr\, \sqrt{\frac{3}{R^3}}\ . \end{aligned}$$

Consequently, the unique harmonic solution H of the equation $\tilde{T}H = \tilde{T}F$ is given by

$$H = \int_0^R r^2 G(r)\ dr\ \frac{3}{R^3}\ . \qquad (10.470)$$

This is the desired result. □

This result indeed shows us that a reconstruction of the (deep) Earth's interior with a harmonic function system makes no sense. Therefore, a reliable method for the approximation of the density distribution of the Earth requires a treatment of both orthogonal projections: the harmonic part and the anharmonic part. Moreover, remember that the contribution of $H^{\mathrm{ext}}_{-1,1}$ to the (outer) gravitational potential can be neglected when applying an appropriate coordinate transformation, as we mentioned above. This operation can, therefore, physically be interpreted as filtering out the contribution of the radially symmetric density structures in the Earth's interior.

Note that the total mass of an anharmonic density function is zero.

Theorem 10.60. *Let F be a member of class* $\mathrm{Anharm}(\overline{\Omega^{\mathrm{int}}_{\mathrm{R}}})$. *Then we have*

$$\int_{\overline{\Omega^{\mathrm{int}}_R}} F(x)\ dV(x) = 0\ . \qquad (10.471)$$

Proof. Since F is $\mathrm{L}^2(\overline{\Omega^{\mathrm{int}}_R})$-orthogonal to every harmonic function on $\overline{\Omega^{\mathrm{int}}_R}$, it is in particular orthogonal to every constant function on $\overline{\Omega^{\mathrm{int}}_R}$. Thus,

$$\int_{\overline{\Omega^{\mathrm{int}}_R}} F(x) \cdot 1\ dV(x) = 0\ . \qquad (10.472)$$

□

Therefore, the constant harmonic solution, obtained in the case of a radially symmetric Earth's interior, can be interpreted as the average mass density of the Earth. In the case of PREM (the Preliminary Reference Earth Model, see, e.g., A. Dziewonski, D.L. Anderson (1981) and A. Dziewonski, D.L. Anderson (1984)), we obtain for this average density according to Theorem 10.59 the approximate value 5.5134 g/cm^3 (V. Michel (1999)).

Note that every function $H^{\mathrm{int}}_{n,j}(R;\cdot)$, $n = 1, 2, \ldots, j = 1, \ldots, 2n+1$, has the total mass zero, since $H^{\mathrm{int}}_{0,1}(R;\cdot)$ is constant and $\mathrm{L}^2(\overline{\Omega^{\mathrm{int}}_R})$-orthogonal to each $H^{\mathrm{int}}_{n,j}(R;\cdot)$, $n = 1, 2, \ldots, j = 1, \ldots, 2n+1$, such that

$$\int_{\overline{\Omega^{\mathrm{int}}_R}} F(x) H^{\mathrm{int}}_{0,1}(R;x)\ \ dV(x) \sqrt{\frac{4\pi R^3}{3}}$$

may in general be interpreted as the total mass of a mass density distribution $F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$.

Our results will now be used to investigate the inverse problem $TF = P$, where TF is a gravitational potential on $\overline{\Omega_R^{\mathrm{int}}}$ with and a mass density distribution F.

Remember the families of functions

$$\{H_{n,j}^{\mathrm{int}}(R;\cdot)\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1} \tag{10.473}$$

and

$$\{H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\}_{n\in\mathbb{N}, j=1,\ldots,2n+1}, \tag{10.474}$$

respectively, which are complete orthonormal systems in the Hilbert spaces $(\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}), (\cdot,\cdot)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})})$ and $(\mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}}), (\cdot,\cdot)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})})$. Suppose that $\{k^\wedge(n)\}_{n\in\mathbb{N}_0}$ is the symbol of $\tilde{T} : \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}) \to \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}))$, i.e.,

$$(\tilde{T}F)(y) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} k^\wedge(n) \int_{\overline{\Omega_R^{\mathrm{int}}}} F(x) H_{n,j}^{\mathrm{int}}(R;x)\ dV(x) H_{-n-1,j}^{\mathrm{ext}}(R;y),$$

$y \in \Omega_R^{\mathrm{ext}}$, $F \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})$, where $H_{-1,1}^{\mathrm{ext}}(R;\cdot)$ $(\notin \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}}))$ is a given function. We are able to formulate the following result.

Theorem 10.61. *The inverse problem*

$$\tilde{T}F = P,$$

$P \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$ *given and* $F \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ *unknown, is solvable if and only if* $P \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$ *with*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(\frac{\left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^\wedge(n)} \right)^2 < \infty\ . \tag{10.475}$$

In this case, the harmonic solution $F \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ *is unique and given by*

$$\begin{aligned}
\left(F, H_{0,1}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} &= \int_{\overline{\Omega_R^{\mathrm{int}}}} F(x) H_{0,1}^{\mathrm{int}}(R;x)\ dV(x) \\
&= 0, \\
\left(F, H_{n,j}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} &= \int_{\overline{\Omega_R^{\mathrm{int}}}} F(x) H_{n,j}^{\mathrm{int}}(x)(R;x)\ dV(x) \\
&= \int_{\overline{\Omega_R^{\mathrm{ext}}}} P(x) H_{-n-1,j}^{\mathrm{ext}}(R;x)\ dV(x) \\
&= \frac{(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot))_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^\wedge(n)},
\end{aligned}$$

$n = 1, 2, \ldots, j = 1, \ldots, 2n+1$.

As we have seen, the inverse operator $(\tilde{T}|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}))^{-1}$, defined on the image $\mathrm{im}\tilde{T} = \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}))$, is discontinuous. Due to unavoidable errors in the measurements of the gravitational field, the application of this inverse operator to the observed potential for a direct reconstruction of the mass density distribution is not reasonable. Therefore, we have to develop a method which uses the principle of regularization, i.e., we do not calculate the exact solution but determine a sequence of approximations, which continuously depend on the potential and converge to the exact solution.

Definition 10.62. Let $K : \mathcal{X} \to K(\mathcal{X}) \subset \mathcal{Y}$ be an invertible linear operator, where $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are Banach spaces. A family of linear operators $K_n : K(\mathcal{X}) \to \mathcal{X}$, $n \in \mathbb{N}_0$, is called a regularization of K^{-1} if it satisfies the following properties:

(i) K_n is continuous for every $n \in \mathbb{N}_0$.

(ii) For every $P \in K(\mathcal{X})$ the identity

$$\lim_{n\to\infty} K_n P = K^{-1} P$$

holds with respect to $\|\cdot\|_{\mathcal{X}}$. The element $K_n P$ is called an n-level regularization of the inverse problem $KF = P$.

For further details on regularizations and their application to geoscientific problems, the reader is referred to, for example, W. Freeden, F. Schneider (1998), F. Schneider (1997). In our case, the operator T is the Fredholm integral operator of the first kind given by Newton's gravitational potential. This leads us to the following statement.

Definition 10.63. Let $\tilde{T}|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}) : \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}}) \to \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}}))$ be the restriction of the linear operator to the space of harmonic functions. Then the operator $S : T(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})) \to \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$ is defined by

$$\tilde{S} = \left(\tilde{T}\middle|\mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})\right)^{-1} .$$

A regularization of the operator $\tilde{S}$ can be constructed in several ways. The 'classical' approach given by a truncated singular-value decomposition (TSVD) is discussed for example in W. Freeden, V. Michel (2004). However, a TSVD represents the regularizations $\tilde{T}_n P$ in terms of polynomials, which have no space localization. Therefore, a local determination of an approximation with high spatial resolutions requires the use of polynomials with very high degrees which can coincide with an extremely 'oscillating' behavior. For this reason, a multiscale regularization concept (based on V. Michel (1999)) is proposed in V. Michel (2005). The advantage of a multiscale regularization is the use of functions like wavelets, which are space localizing as well as frequency (momentum) localizing. Moreover, the variation of the scales allows different weightings of the two kinds of localizations.

In our approach, we restrict ourselves to the spectral reconstruction of the mass density distribution.

Harmonic part. The spectral reconstruction is based on the fact that $F = \tilde{S}P$, $P \in \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})) \cap \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$, is represented by

$$\begin{aligned}\left(F, H_{0,1}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} &= 0,\\ \left(F, H_{n,j}^{\mathrm{int}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} &= \frac{\left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^\wedge(n)},\end{aligned}$$

$n = 1, 2, \ldots, j = 1, \ldots, 2n+1$, according to Theorem 10.61. Before we are going to calculate the solution, we have to take the possible non-existence of the solution due to errors in the measurements into account. Note that we defined $\mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$ as a subset of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$.

Theorem 10.64. *Let $P \in \tilde{T}(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})) \cap \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$ be a gravitational potential. The corresponding perturbed function is given by $P + \varepsilon E$, where $E \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$ and $\varepsilon > 0$. Then the projection operator $\mathcal{P} : \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}}) \to \mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$, defined by*

$$\mathcal{P}G = \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(G, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)$$

(in the sense of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$), has the property

$$\|\mathcal{P}(P+\varepsilon E) - P\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} \leq \varepsilon \|E\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} .$$

Proof. The $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$–norm of the difference $\mathcal{P}(P+\varepsilon E) - P$ can be calculated by using Parseval's identity:

$$\begin{aligned}\|\mathcal{P}(P+\varepsilon E) - P\|^2_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} &= \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(\varepsilon E, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)^2_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}\\ &= \varepsilon^2 \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(E, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)^2_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}\\ &\leq \varepsilon^2 \|E\|^2_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} .\end{aligned}$$

□

Obviously, we are able to use the projection operator $\mathcal{P}$ to obtain the (in the sense of $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$) best possible approximation to the unperturbed potential P.

However, it does not suffice to know that $\mathcal{P}(P+\varepsilon E)$ is harmonic and square integrable on $\overline{\Omega_R^{\mathrm{ext}}}$. For the solvability of $\widetilde{TF} = \mathcal{P}(P+\varepsilon E), \quad F \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})$, we need the property $\mathcal{P}(P+\varepsilon E) \in \mathrm{im}\ \tilde{T}$. This requirement can be satisfied by an appropriate approximation, as the following theorem and its proof demonstrate.

Theorem 10.65. *The projection* $\mathcal{P}(\mathrm{im}\ \tilde{T})$ *of the image* $\mathrm{im}\ \tilde{T}$ *of the operator* $\tilde{T}$ *is dense in* $\mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$ *with respect to* $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$*:*

$$\overline{\mathcal{P}(\mathrm{im}\ \tilde{T})}^{\|\cdot\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}} = \mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}}) = \mathcal{P}\left(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})\right) \ .$$

Proof. According to the construction of the operator $\mathcal{P}$, it is clear that

$$\mathcal{P}(\mathrm{im}\ \tilde{T}) \subset \mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}}) = \mathcal{P}\left(\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})\right) \ .$$

Now let $P \in \mathrm{Harm}(\overline{\Omega_R^{\mathrm{ext}}})$ be an arbitrary function with the representation

$$P = \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)$$

in the $\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$-topology. Then the sequence $\{P_N\}_{N\in\mathbb{N}}$, defined by

$$P_N = \sum_{n=1}^{N}\sum_{j=1}^{2n+1} \left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} H_{-n-1,j}^{\mathrm{ext}}(R;\cdot),$$

obviously satisfies the properties

$$\lim_{N\to\infty} \|P - P_N\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} = 0$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \left(\frac{\left(P_N, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^{\wedge}(n)}\right)^2 \\
&\quad = \sum_{n=1}^{N}\sum_{j=1}^{2n+1} \left(\frac{\left(P, H_{-n-1,j}^{\mathrm{ext}}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^{\wedge}(n)}\right)^2 < \infty,
\end{aligned}$$

i.e., $P_N \in \mathcal{P}\left(\mathrm{im}\ \tilde{T}\right)$ for all $N \in \mathbb{N}$. □

This result does not only show that it is rather improbable to find a projected potential $\mathcal{P}(P+\varepsilon E)$ outside the image of the operator. It also tells us that such an exceptional harmonic function, for which solvability is not given, can be approximated arbitrarily well by a function that corresponds to a solvable problem. As we could see, a TSVD represents, like every bandlimited approach, a trivial way of satisfying the summability condition. W. Freeden, V. Michel (2004) show that a multiscale technique also allows the construction of non-bandlimited approximations which guarantee solvability.

Using Theorem 10.61, we can now formulate a spectral regularization technique for the gravimetry problem.

Theorem 10.66. *Let $P \in \mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})$ be an arbitrary function. Then the sequence $\{F_N\}_{N\in\mathbb{N}}$ of harmonic functions given by*

$$F_N = \sum_{n=1}^{N}\sum_{j=1}^{2n+1} \frac{(P, H^{\mathrm{ext}}_{-n-1,j}(R;\cdot))_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}}{k^\wedge(n)} H^{\mathrm{int}}_{n,j}(R;\cdot) \tag{10.476}$$

shows the property

$$\lim_{N\to\infty} \left\| \mathcal{P}P - \widetilde{TF}_N \right\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} = 0 \ .$$

Moreover, if $\mathcal{P}P \in \mathrm{im}\ \tilde{T}$, then $\{F_N\}_{N\in\mathbb{N}}$ converges to the harmonic solution F of the integral equation

$$\widetilde{TF} = \mathcal{P}P,$$

i.e.,

$$\lim_{N\to\infty} \left\| \left(T \middle| \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})\right)^{-1} (\mathcal{P}P) - F_N \right\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{int}}})} = 0 \ .$$

Proof. From the results derived above, we find

$$\widetilde{TF}_N = \sum_{n=1}^{N}\sum_{j=1}^{2n+1} \left(P, H^{\mathrm{ext}}_{-n-1,j}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})} H^{\mathrm{ext}}_{-n-1,j}(R;\cdot),$$

such that obviously $\{\widetilde{TF}_N\}_{N\in\mathbb{N}}$ converges to $\mathcal{P}P$ with respect to $\|\cdot\|_{\mathrm{L}^2(\overline{\Omega_R^{\mathrm{ext}}})}$. Moreover, according to Theorem 10.61 the harmonic solution

$$F = \left(T \middle| \mathrm{Harm}(\overline{\Omega_R^{\mathrm{int}}})\right)^{-1} (\mathcal{P}P)$$

is given by

$$F = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{\left(P, H^{\text{ext}}_{-n-1,j}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega^{\text{ext}}_R})}}{k^\wedge(n)} H^{\text{int}}_{n,j}(R;\cdot),$$

(with respect to $\mathrm{L}^2(\overline{\Omega^{\text{int}}_R})$), provided that $\mathcal{P}P \in \text{im}\ \tilde{T}$. Thus,

$$\begin{aligned} \lim_{N\to\infty} \|F - F_N\|^2_{\mathrm{L}^2(\overline{\Omega^{\text{int}}_R})} &= \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \sum_{j=1}^{2n+1} \left(\frac{\left(P, H^{\text{ext}}_{-n-1,j}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega^{\text{ext}}_R})}}{k^\wedge(n)} \right)^2 \\ &= 0\ . \end{aligned}$$

This is the wanted result. □

The approximations as defined by (10.476) require the calculation of the Fourier coefficients

$$\left(P, H^{\text{ext}}_{-n-1,j}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega^{\text{ext}}_R})} = \int_{\overline{\Omega^{\text{ext}}_R}} P(x) H^{\text{ext}}_{-n-1,j}(R;x)\ dV(x)\ .$$

In reality, we only know P on a discrete set of points, such that the above integral has to be determined numerically with an appropriate integration formula.

More explicitly, the Fourier coefficients of the potential P are given by

$$\begin{aligned} &\left(P, H^{\text{ext}}_{-n-1,j}(R;\cdot)\right)_{\mathrm{L}^2(\overline{\Omega^{\text{ext}}_R})} \\ &= \int_{\overline{\Omega^{\text{ext}}_R}} P(x) H^{\text{ext}}_{-n-1,j}(R;x)\ dV(x) \\ &= \sqrt{\frac{2n-1}{R^3}} \int_R^{\infty} r^2 \left(\frac{R}{r}\right)^{n+1} \int_{\Omega} P(r\xi) Y_{n,j}(\xi)\ d\omega(\xi)\ dr, \end{aligned}$$

$n = 1, 2, \ldots, j = 1, \ldots, 2n+1$.

Let α be the radius of a sphere A around the origin with $\alpha < R$. If we assume that the potential P can be represented by

$$P(x) = \sum_{m=0}^{\infty} \sum_{k=1}^{2m+1} \left(P, \frac{1}{\alpha} Y_{m,k}\left(\frac{\cdot}{|\cdot|}\right)\right)_{\mathrm{L}^2(A)} \frac{1}{\alpha} \left(\frac{\alpha}{|x|}\right)^{m+1} Y_{m,k}\left(\frac{x}{|x|}\right),$$

$x \in \overline{\Omega_R^{\rm ext}}$, then we find that

$$
\begin{aligned}
&\left(P, H^{\rm ext}_{-n-1,j}(R;\cdot)\right)_{{\rm L}^2(\overline{\Omega_R^{\rm ext}})} \\
&= \sqrt{\frac{2n-1}{R^3}} \\
&\quad \times \int_R^\infty \frac{R^{n+1}}{r^{n-1}} \left(\sum_{m=0}^{\infty} \sum_{k=1}^{2m+1} \left(P, Y_{m,k}\left(\frac{\cdot}{|\cdot|}\right)\right)_{{\rm L}^2(A)} \frac{\alpha^{m-1}}{r^{m+1}} Y_{m,k}, Y_{n,j} \right)_{{\rm L}^2(\Omega)} dr \\
&= \sqrt{\frac{2n-1}{R^3}} \\
&\quad \times \int_R^\infty \frac{R^{n+1}}{r^{n-1}} \sum_{m=0}^{\infty} \sum_{k=1}^{2m+1} \left(P, Y_{m,k}\left(\frac{\cdot}{|\cdot|}\right)\right)_{{\rm L}^2(A)} \frac{\alpha^{m-1}}{r^{m+1}} \left(Y_{m,k}, Y_{n,j}\right)_{{\rm L}^2(\Omega)} \, dr \\
&= \sqrt{\frac{2n-1}{R^3}} \int_R^\infty \frac{R^{n+1}}{r^{n-1}} \frac{\alpha^{n-1}}{r^{n+1}} \, dr \left(P, Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right)\right)_{{\rm L}^2(A)} \\
&= \frac{1}{\sqrt{R^3(2n-1)}} \frac{\alpha^{n-1}}{R^{n-2}} \left(P, Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right)\right)_{{\rm L}^2(A)} \\
&= \frac{1}{\sqrt{R(2n-1)}} \left(\frac{\alpha}{R}\right)^{n-1} \int_A P(x) Y_{n,j}\left(\frac{x}{|x|}\right) \, d\omega(x),
\end{aligned}
$$

$n = 1, 2, \ldots,\ j = 1, \ldots, 2n+1$. This means that, in this simplified case, the numerical integration can be reduced to the calculation of a spherical integral over a sphere A, with radius α around the origin.

Note that the integrals that have to be determined approximately are L^2-scalar products with polynomials. Due to the non-space localizing character of polynomials, the grid for the numerical integration should be equidistributed. However, this requirement does not fit the real data situation. In North America, western Europe, and Australia, gravitational data are available on a comparatively dense grid, whereas one of the lowest densities of available data points is, for example, given in the polar regions. Such datasets can be better handled by the multiscale approach as described by W. Freeden, V. Michel (2004), V. Michel (2005).

We have seen that it is only possible to recover the harmonic part of the Earth's density distribution from the gravitational potential (for graphical illustration see Figs. 10.31 and 10.32). Therefore, we need a strategy to determine an approximation to the anharmonic part of the unknown mass density function from non-gravitational data. We will represent such a priori information by linear functionals $\mathcal{F}^n : \mathrm{L}^2(\overline{\Omega_R^{\rm int}}) \to \mathbb{R}$. The idea is that, a given set of measurements related to the true mass density function F can be represented by $\mathcal{F}^n F = b_n, \quad n \in \mathcal{I} \subset \mathbb{N}$. The application of those functionals to the already calculated (approximation to the) harmonic part $F_{\rm harm}$ of F allows us to formulate an equation system for the anharmonic

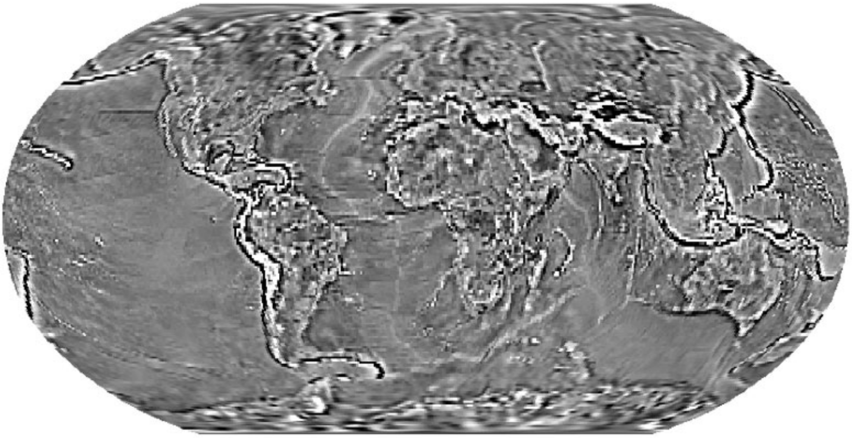


Fig. 10.31: (Harmonic) Density $\left[\frac{kg}{m^3}\right]$ from EGM96, Geomathematics Group, TU Kaiserslautern, K. Wolf (2006) (see also V. Michel, K. Wolf (2008)).

part F_{anharm} of F:

$$\mathcal{F}^n F_{\mathrm{anharm}} = b_n - \mathcal{F}^n F_{\mathrm{harm}}, \quad n \in \mathcal{I} \ . \tag{10.477}$$

Consequently, it suffices to assume that the linear functionals $\mathcal{F}^n$ are defined on $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$: $\mathcal{F}^n : \mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}}) \to \mathbb{R}$. For the determination of an anharmonic function satisfying (10.477), several methods exist. In the following, we explain a "classical" spectral approximation. For a wavelet approach, see V. Michel (2002). A spline approximation method is developed in V. Michel (1999). For alternative methods we refer to, for example, L. Ballani et al. (1993) and the references therein.

Since the available information for the mass density function of the Earth and related quantities is always finite, it suffices to assume that the index set $\mathcal{I}$ has the form $\mathcal{I} = \{1, \ldots, N\}$. In other words, our a priori information is represented by

$$\mathcal{F}^n G = b_n, \quad n = 1, \ldots, N, \tag{10.478}$$

where $G \in \mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$ is unknown.

Motivated by Theorem 10.58, we assume that we have a countable basis of $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$. In Theorem 10.58, we listed for the spherical Earth two closed systems A_i in $\mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$, namely

$$\left\{ r\xi \mapsto r^n P_{k,n}(r^2) Y_{n,j}(\xi) \right\}_{k\in\mathbb{N}, n\in\mathbb{N}_0, j\in\{1,\ldots,2n+1\}}, \tag{10.479}$$

where $P_{k,n}(x)$ is a polynomial of degree k, and

$$\left\{ r\xi \mapsto \left(r^{n+2k} - \frac{(2n+3)R^{2k}}{2n+2k+3} r^n \right) Y_{n,j}(\xi) \right\}_{k\in\mathbb{N}, n\in\mathbb{N}_0, j\in\{1,\ldots,2n+1\}} . \tag{10.480}$$

In both cases, the degree of the polynomial corresponding to an index triple (k,n,j) is $n+2k$. It appears to be reasonable to choose the enumeration $(k,n,j) \mapsto i$ in the last definition in a way such that

$$\deg \mathcal{A}_{i_1} \leq \deg \mathcal{A}_{i_2}, \text{ if } i_1 \leq i_2. \tag{10.481}$$

A fixed degree $N = n + 2k$ corresponds to pairs

$$\left\{ (n,k) \middle| k = 1, \ldots, \left[\frac{N}{2} \right], n = N - 2k \right\}, \tag{10.482}$$

where $[\cdot]$ is the Gauss bracket. Hence, the number of polynomials of degree N in one of the above mentioned systems is

$$\begin{aligned} \sum_{k=1}^{[\frac{N}{2}]} (2(N-2k)+1) &= (2N+1)\left[\frac{N}{2}\right] - 4\frac{\left[\frac{N}{2}\right]\left(\left[\frac{N}{2}\right]+1\right)}{2} \\ &= \left(2N - 2\left[\frac{N}{2}\right] - 1\right)\left[\frac{N}{2}\right], \ N \geq 2 \ . \end{aligned}$$

Note that every polynomial of degree 0 or 1 is harmonic.

Thus, the indices of $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ can be divided (in the spherical case) into consecutive sections of length $\left(2N - 2\left[\frac{N}{2}\right] - 1\right)\left[\frac{N}{2}\right]$, such that, in each section, the degree of $\mathcal{A}_i$ is constant. The arrangement of the polynomials within each index section is, therefore, not influenced by the condition (10.481) and is, consequently, from this point of view, arbitrary. We are interested in an harmonic function G satisfying (10.478). To obtain a unique solution, the most simple approach would be to take a function of the kind

$$G = \sum_{i=1}^{N} a_i \mathcal{A}_i \ . \tag{10.483}$$

The set of coefficients $\{a_i\}_{i=1,\ldots,N}$ has to be determined from the a priori conditions

$$\sum_{i=1}^{N} a_i \mathcal{F}^n \mathcal{A}_i = b_n, \quad n = 1, \ldots, N \ . \tag{10.484}$$

If the rank of the matrix

$$\left(\mathcal{F}^n \mathcal{A}_i\right)_{\substack{n=1,\ldots,N \\ i=1,\ldots,N}} \tag{10.485}$$

is less than N, we can add additional a priori information (or, alternatively, decrease the number of coefficients) to obtain a matrix of maximal rank. The solution of the obtained linear equation system can be calculated with the common algorithms such as the Householder method.

An example of such a system of functionals is given by the point functionals. Let $\{x_n\}_{n=1,\ldots,N}$ be a system of pairwise distinct points in $\overline{\Omega_R^{\mathrm{int}}}$. Then the functionals $\mathcal{F}^n : \mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}}) \to \mathbb{R}$, given by $\mathcal{F}^n G = G(x_n)$, $G \in \mathrm{Anharm}(\overline{\Omega_{\mathrm{R}}^{\mathrm{int}}})$, $n \in \{1, \ldots, N\}$, are linear. The corresponding linear equation system has the matrix $(\mathcal{A}_i(x_n))_{\substack{n=1,\ldots,N \\ i=1,\ldots,N}}$ However, in reality direct measurements of the mass density are only available at some points in the upper crust of the Earth. It is obvious that an anharmonic function calculated from such datasets cannot represent the situation in the deep Earth. This fact is stressed by the discovery that the (almost) radially symmetric layer structure of the mantle and the core is nearly fully described by the anharmonic part of the density distribution (see Theorem 10.59 and V. Michel (1999)). Therefore, additional datasets which are also influenced by deep structures have to be included in the calculations, such as data from seismology and geomagnetism. A priori information represented by point functionals $\mathcal{F}^n G = G(x_n)$ are, therefore, usually pointwise solutions of further inverse problems.

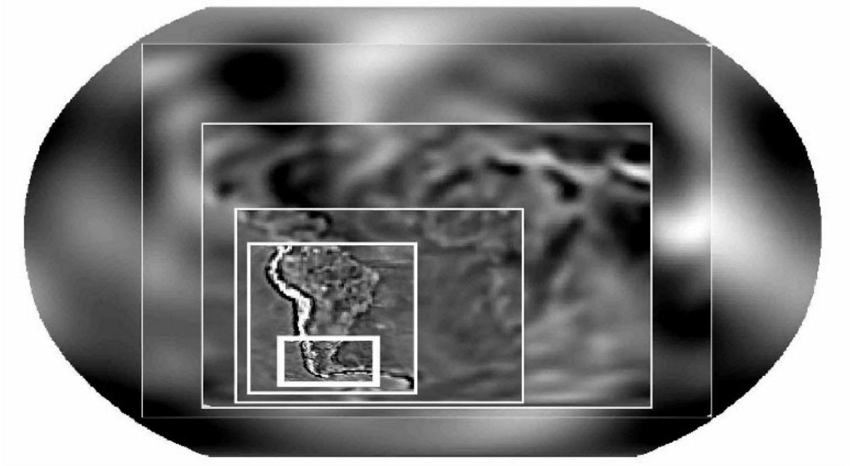


Fig. 10.32: Multiresolution of density $\left[\frac{kg}{m^3}\right]$ from EGM96, Geomathematics Group, TU Kaiserslautern, K. Wolf (2006).

10.9 Vector Outer Harmonics and the Gravitational Gradient

In what follows, we extend the results obtained for scalar outer harmonics to the vectorial case (cf. H. Nutz (2002)). It should be noted that the system of vector spherical harmonics $\{\tilde{y}^{(i)}_{n,m}\}$ and *not* the system $\{y^{(i)}_{n,m}\}$ is used to generate the set of vector outer harmonics $\{h^{(i);R}_{-n-1,m}\}$.

To be more concrete, the *vectorial outer (solid spherical) harmonics (briefly called vector outer harmonics)* $h^{(i);R}_{n,m}$ of degree n, order m, and kind i are given by

$$h^{(1);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+2} \tilde{y}^{(1)}_{n,m}\left(\frac{x}{|x|}\right), \quad n = 0, 1, \ldots, m = 1, \ldots 2n+1, \tag{10.486}$$

$$h^{(2);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n} \tilde{y}^{(2)}_{n,m}\left(\frac{x}{|x|}\right), \quad n = 1, 2, \ldots, m = 1, \ldots 2n+1, \tag{10.487}$$

$$h^{(3);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+1} \tilde{y}^{(3)}_{n,m}\left(\frac{x}{|x|}\right), \quad n = 1, 2, \ldots, m = 1, \ldots 2n+1, \tag{10.488}$$

$x \in \mathbb{R}^3 \backslash \{0\}$.

From our results about the systems $\{\tilde{y}^{(i)}_{n,m}\}$ (see Chapter 5), we are immediately able to deduce the following properties:

- $h^{(i);R}_{-n-1,m}$ is of class $\mathrm{c}^{(\infty)}(\mathbb{R}^3 \backslash \{0\})$, $i \in \{1, 2, 3\}$,
- $\Delta_x h^{(i);R}_{-n-1,m} = 0$ for all $x \in \mathbb{R}^3 \backslash \{0\}$ and $i \in \{1, 2, 3\}$, i.e., every component function $h^{(i)}_{-n-1,m} \cdot \varepsilon^k$ satisfies the Laplace equation,
- $h^{(i);R}_{-n-1,m}|\Omega_R = \frac{1}{R}\tilde{y}^{(i)}_{n,m}$, $i \in \{1, 2, 3\}$,
- $|h^{(i);R}_{-n-1,m}(x)| = O(|x|^{-1}), \quad |x| \to \infty, \ i \in \{1, 2, 3\}$,
- $$(h^{(i);R}_{-n-1,m}(x), h^{(j);R}_{-l-1,s})_{\mathrm{l}^2(\Omega_R)} = \int_{\Omega_R} h^{(i);R}_{-n-1,m}(x) \cdot h^{(j);R}_{-l-1,s}(x)\, d\omega(x) = \delta_{ij}\delta_{nl}\delta_{ms}, \quad i, j \in \{1, 2, 3\}.$$

As in the scalar case, we introduce the $\widetilde{\text{harm}}_n^{(i)}$–spaces to be

$$\widetilde{\text{harm}}_n^{(i)}(\overline{\Omega_R^{\text{ext}}}) = \underset{m=1,\ldots,2n+1}{\text{span}} \left\{ h_{-n-1,m}^{(i);R} \big| \overline{\Omega_R^{\text{ext}}} \right\}. \tag{10.489}$$

Furthermore,

$$\widetilde{\text{harm}}_0(\overline{\Omega_R^{\text{ext}}}) = \widetilde{\text{harm}}_0^{(1)}(\overline{\Omega_R^{\text{ext}}}), \tag{10.490}$$

$$\widetilde{\text{harm}}_n(\overline{\Omega_R^{\text{ext}}}) = \bigoplus_{i=1}^{3} \widetilde{\text{harm}}_n^{(i)}(\overline{\Omega_R^{\text{ext}}}). \tag{10.491}$$

As usual, $\widetilde{\text{harm}}_{p,\ldots,q}^{(i)}(\overline{\Omega_R^{\text{ext}}})$, $0_i \le p \le q$, denotes the space

$$\widetilde{\text{harm}}_{p,\ldots,q}^{(i)}(\overline{\Omega_R^{\text{ext}}}) = \bigoplus_{n=p}^{q} \widetilde{\text{harm}}_n^{(i)}(\overline{\Omega_R^{\text{ext}}}). \tag{10.492}$$

We are now able to formulate the addition theorems for the vector outer harmonics. Again, we have two choices involving Legendre tensors or Legendre vectors, respectively.

Theorem 10.67. *Let $\{h_{-n-1,m}^{(i);R}\}_{m=1,\ldots,2n+1}$ be a system of vector outer harmonics of degree n, order m, and kind i. Then, for $(x,y) \in \overline{\Omega_R^{\text{ext}}} \times \overline{\Omega_R^{\text{ext}}}$, the addition theorem for vector outer harmonics reads as follows:*

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(1);R}(x) \otimes h_{-n-1,m}^{(1);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n+2} \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(1,1)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.493}$$

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(1);R}(x) \otimes h_{-n-1,m}^{(2);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n} \left(\frac{R}{|x|} \right)^{2} \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(1,2)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.494}$$

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(1);R}(x) \otimes h_{-n-1,m}^{(3);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n+1} \left(\frac{R}{|x|} \right) \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(1,3)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.495}$$

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(2);R}(x) \otimes h_{-n-1,m}^{(1);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n} \left(\frac{R}{|y|} \right)^{2} \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(2,1)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.496}$$

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(2);R}(x) \otimes h_{-n-1,m}^{(2);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n} \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(2,2)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.497}$$

$$\sum_{m=1}^{2n+1} h_{-n-1,m}^{(2);R}(x) \otimes h_{-n-1,m}^{(3);R}(y) = \frac{1}{R^2} \left(\frac{R^2}{|x||y|} \right)^{n} \left(\frac{R}{|y|} \right) \frac{2n+1}{4\pi} \tilde{\mathbf{p}}_n^{(2,3)} \left(\frac{x}{|x|}, \frac{y}{|y|} \right), \tag{10.498}$$

$$\sum_{m=1}^{2n+1} h^{(3);R}_{-n-1,m}(x) \otimes h^{(1);R}_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n+1}\left(\frac{R}{|y|}\right)\frac{2n+1}{4\pi}\tilde{\mathbf{p}}^{(3,1)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right), \tag{10.499}$$

$$\sum_{m=1}^{2n+1} h^{(3);R}_{-n-1,m}(x) \otimes h^{(2);R}_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n}\left(\frac{R}{|x|}\right)\frac{2n+1}{4\pi}\tilde{\mathbf{p}}^{(3,2)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right), \tag{10.500}$$

$$\sum_{m=1}^{2n+1} h^{(3);R}_{-n-1,m}(x) \otimes h^{(3);R}_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n+1}\frac{2n+1}{4\pi}\tilde{\mathbf{p}}^{(3,3)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right). \tag{10.501}$$

Combining scalar and vector outer harmonics and observing the concept of Legendre vectors, we are led to the following addition theorem:

Theorem 10.68. *Let* $\{H^R_{-n-1,m}\}_{m=1,\ldots,2n+1}$ *be a system of scalar outer harmonics of degree* n *and order* m*. Suppose that* $\{h^{(i);R}_{-n-1,m}\}_{m=1,\ldots,2n+1}$ *forms the associated system of vector outer harmonics of degree* n*, order* m*, and kind* i*. Then, for* $(x,y) \in \overline{\Omega^{\mathrm{ext}}_R} \times \overline{\Omega^{\mathrm{ext}}_R}$*, the addition theorem for scalar and vector outer harmonics reads as follows*

$$\sum_{m=1}^{2n+1} h^{(1);R}_{-n-1,m}(x) H^R_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n+1}\left(\frac{R}{|y|}\right)\frac{2n+1}{4\pi}\tilde{p}^{(1)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right), \tag{10.502}$$

$$\sum_{m=1}^{2n+1} h^{(2);R}_{-n-1,m}(x) H^R_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n}\left(\frac{R}{|x|}\right)\frac{2n+1}{4\pi}\tilde{p}^{(2)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right), \tag{10.503}$$

$$\sum_{m=1}^{2n+1} h^{(3);R}_{-n-1,m}(x) H^R_{-n-1,m}(y) = \frac{1}{R^2}\left(\frac{R^2}{|x||y|}\right)^{n+1}\frac{2n+1}{4\pi}\tilde{p}^{(3)}_n\left(\frac{x}{|x|},\frac{y}{|y|}\right). \tag{10.504}$$

Our purpose now is to mention some important properties involving vector outer harmonics.

Lemma 10.69. *(Linear Independence of Vector Outer Harmonics) Assume that* $\{h^{(i);R}_{-n-1,m}\}_{\substack{i=1,2,3,n=0_i,\ldots,\\ m=1,\ldots,2n+1}}$ *is a system of vector outer harmonics as defined by (10.486), (10.487) and (10.488). Then, for all* $r > 0$*, the system*

$$\{h^{(i);R}_{-n-1,m}|\Omega_r\}_{\substack{i=1,2,3;n=0_i,\ldots,\\ m=1,\ldots,2n+1}}$$

is linearly independent.

Next we are interested in the completeness for vector outer harmonics on Ω_R. Our results can be based on the corresponding theorems of the scalar theory.

Lemma 10.70. *Let* $\{H^R_{-n-1,m}\}_{n=0,1,\ldots,m=1,\ldots,2n+1}$ *be a system of scalar outer harmonics. The n*

$$\overline{\mathrm{span}\{H^R_{-n-1,m}\varepsilon^i|\Omega_R\}}^{\|\cdot\|_{\mathrm{l}^2(\Omega_R)}} = \mathrm{l}^2(\Omega_R),$$

and

$$\overline{\mathrm{span}\{H^R_{-n-1,m})\varepsilon^i|\Omega_R\}}^{\|\cdot\|_{\mathrm{c}(\Omega_R)}} = \mathrm{c}(\Omega_R).$$

Lemma 10.70 enables us to formulate the following theorem.

Theorem 10.71. *Let* $\{h^{(i);R}_{-n-1,m}\}_{\substack{i=1,2,3;n=0_i,\ldots;\\ m=1,\ldots,2n+1}}$ *be a system of vector outer harmonics as defined by (10.486), (10.487) and (10.488). Then, for all* $r > 0$ *the following statements hold true:*

$$\mathrm{l}^2(\Omega_r) = \overline{\mathrm{span}_{\substack{i=1,2,3;n=0_i,\ldots,\\ m=1,\ldots,2n+1}}\{h^{(i);R}_{-n-1,m}|\Omega_r\}}^{\|\cdot\|_{\mathrm{l}^2(\Omega_r)}},$$

and

$$\mathrm{c}(\Omega_r) = \overline{\mathrm{span}_{\substack{i=1,2,3;n=0_i,\ldots,\\ m=1,\ldots,2n+1}}\{h^{(i);R}_{-n-1,m}|\Omega_R\}}^{\|\cdot\|_{\mathrm{c}(\Omega_r)}}.$$

The purpose of *high-low satellite-to-satellite tracking* (hi–lo SST) by use of GPS (as realized, e.g., by the German satellite CHAMP of the GFZ) is to develop the geopotential from measured ranges (geometrical distances) between a low earth orbiter (LEO) and the high flying GPS satellites. In what follows, hi-lo SST is discussed from a mathematical point of view as the problem of determining the external gravitational field of the Earth from the gradient vector at the altitude of the LEO.

In order to translate hi-lo SST into a mathematical formulation (see W. Freeden (1999), W. Freeden et al. (1999), W. Freeden et al. (2002), and, for alternative approaches, ESA (1996), ESA (1998), ESA (1999) and the references therein), we start from the following spherically oriented situation: Let Ω_R denote the Earth's surface, while Ω_S denotes the orbital surface. The arrangement of the GPS satellites is such that at least four satellites are simultaneously visible above the horizon anywhere on the Earth's surface Ω_R and the orbit Ω_S of the LEO satellite as well, all the time. Moreover, the GPS satellites (see Fig. 10.33) are supposed to be placed in circular

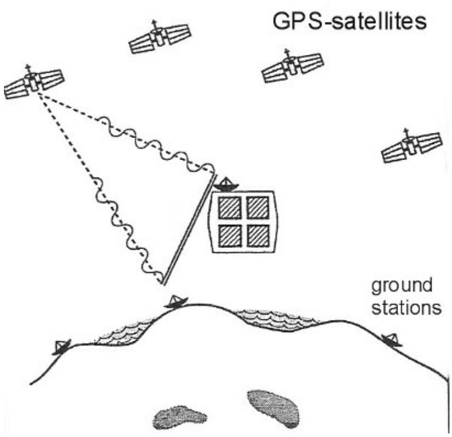


Fig. 10.33: Principle of spaceborne GPS.

orbits Ω_{γ_i} of radii around the origin with $\gamma_i \gg S$; and n is the total number of GPS satellites. To every LEO position $x \in \Omega_S$, therefore, there exist at least $m(\geq 4)$ visible GPS satellites located at $y_{l_1}, \ldots, y_{l_m}$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, such that the geometrical distances (ranges) $d_{l_i} = |x - y_{l_i}|$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, are measurable (see Fig. 10.33). Since the orbits of the GPS satellites are assumed to be known, the coordinates of the LEO satellite located at $x \in \Omega_S$ can be derived from simultaneous range measurements to the satellites. From this, the relative positions of the satellites at x and y_{l_i}, i.e., $p_{l_i} = x - y_{l_i}$, $\quad l_i \in \{1, \ldots, n\}$, $\quad i = 1, \ldots, m$, become available at time t. The relative velocities v_{l_i} and accelerations a_{l_i} are obtainable by differentiating the relative positions with respect to t. We may assume that the measurements are produced at a sufficiently dense rate so that (numerical) differentiation can be performed without any difficulty. The interesting expressions now are the relative accelerations a_{l_i}, $i = 1, \ldots, m$, all of which are determined for inertial motion (in accordance with the Newton–Euler equation) by the gravitational field only and may be equated by the difference of the gradient field of the geopotential, V, here evaluated at the locations of x and y_{l_i}, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$. To be more specific,

$$a_{l_i}(x) = (\nabla V)(x) - (\nabla V)(y_{l_i}), \quad x \in \Omega_S, \tag{10.505}$$

$i = 1, \ldots, m$. (Note that the gravitational force is considered now to be independent of time t at a certain position. In other words, we assume here that the time-like variations of the field are so slow as to be negligible.) From (10.505), it follows that

$$(\nabla V)(x) = \sum_{i=1}^{m} \alpha_i \left(a_{l_i}(x) + (\nabla V)(y_{l_i}) \right), \quad x \in \Omega_S, \tag{10.506}$$

for all selections $(\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m \alpha_i = 1$. The influence of the GPS on the choice of the coefficients $\alpha_1, \ldots, \alpha_m$ will not be investigated here. (Usually, in practice, $(\nabla V)(y_{l_i})$ are supposed to be so small as to be negligible.)

The hi-low SST problem can be described as follows: Let Ω_R be the Earth's surface, and let Ω_S denote the orbital surface of the LEO under consideration. Let there be known the gradient vectors

$$v(x) = (\nabla V)(x), \quad x \in \Omega_S, \tag{10.507}$$

at the flight positions of the LEO. Find the geopotential V on $\overline{\Omega_R^{\text{ext}}}$, i.e., on and outside the Earth's surface Ω_R.

Low–low satellite-to-satellite tracking (lo–lo SST) (as used, for example, by the GFZ/NASA two satellite configuration GRACE) is a tandem mode procedure. By lo–lo SST (see the explanations in ESA (1996, 1998, 1999), the vectors a_{l_i}, $i = 1, ..., m$, are measurable at two different positions x and x^* with $x^* = x + h(x)$, $x \in \Omega_S$, where $h : \Omega_S \to \mathbb{R}^3$ is the difference vector field between the two satellite positions (i.e., $|h(x)| \geq \iota > 0$ with ι denoting the intersatellite range).

Consequently, the mathematical scenario of the lo–lo SST problem is characterized as follows: Let there be known the vectors $v(x) = (\nabla V)(x)$ and $v(x + h(x)) = (\nabla V)(x + h(x))$, $x \in \Omega_S$. Find V on $\overline{\Omega_R^{\text{ext}}}$ from the values $v(x) - v(x + h(x))$.

We begin our considerations with the uniqueness of the SST problem from given vector values (in spherical geometry).

Theorem 10.72. *Suppose that $\mathcal{X} \subset \Omega_S$ (i.e., the subset of observational points on the satellite orbit Ω_S, $S > R$) is a dense system on Ω_S. If v satisfies $\nabla \cdot v = 0$, $\mathrm{L} \cdot v = 0$ in Ω_R^{ext} such that*

$$v(x) = 0, \quad x \in \mathcal{X},$$

then $v = 0$ in $\overline{\Omega_R^{\text{ext}}}$.

Proof. Any field v satisfying $\nabla \cdot v = 0$, $\mathrm{L} \cdot v = 0$ in Ω_R^{ext} can be expressed in the form ∇V, hence, the coordinate functions $v \cdot \varepsilon^i$, $i = 1, 2, 3$, satisfy

$$\Delta \left(v \cdot \varepsilon^i\right) = \Delta \left(\left(\varepsilon^i \cdot \nabla\right) V\right) = \left(\varepsilon^i \cdot \nabla\right) \Delta V = 0 \tag{10.508}$$

in Ω_R^{ext}. Note that the harmonic function V is arbitrarily often differentiable in Ω_R^{ext}. Moreover, according to our assumption, $(\varepsilon^i \cdot \nabla)V(x) = 0$ for all points x of the dense system $\mathcal{X}$ on Ω_S, hence, $\mathcal{X}$ is a fundamental system in Ω_R^{ext} in the sense of Definition 10.2. This implies $v \cdot \varepsilon^i = 0$ in Ω_R^{ext}, $i = 1, 2, 3$, as required. □

Furthermore, we are able to verify the following result.

Theorem 10.73. *Suppose that $\mathcal{X} \subset \Omega_S$, $S > R$, is a dense system of points on the satellite orbit Ω_S, $S > R$. If v satisfies $\nabla \cdot v =$, $\mathrm{L} \cdot v = 0$ in Ω_R^{ext} with*

$$(-x) \cdot v(x) = 0, \quad x \in \mathcal{X}, \tag{10.509}$$

then $v = 0$ in $\overline{\Omega_R^{\mathrm{ext}}}$.

Proof. We again base our arguments on the identity $v = \nabla V$. The potential $V|\overline{\Omega_S^{\mathrm{ext}}}$, $S > R$, may be expanded in terms of outer harmonics

$$V(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k) H_{-n-1,k}^{S}(x), \quad x \in \overline{\Omega_S^{\mathrm{ext}}}, \tag{10.510}$$

where $V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k)$, $n = 0, 1, \ldots$, $k = 1, \ldots, 2n+1$, are the expansion coefficients

$$V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k) = \int_{\Omega_S} V(x) H_{-n-1,k}^{S}(x) \, d\omega(x), \tag{10.511}$$

and the series expansion in (10.510) is absolutely and uniformly convergent in $\overline{\Omega_S^{\mathrm{ext}}}$. It is not hard to see that

$$-\frac{x}{|x|} \cdot (\nabla V)(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{n+1}{|x|} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k) H_{-n-1,k}^{S}(x), \quad x \in \overline{\Omega_S^{\mathrm{ext}}} \ . \tag{10.512}$$

Hence,

$$x \mapsto (-x) \cdot (\nabla V)(x), \quad x \in \overline{\Omega_S^{\mathrm{ext}}}, \tag{10.513}$$

is continuous in $\overline{\Omega_S^{\mathrm{ext}}}$, twice continuously differentiable in Ω_S^{ext}, harmonic in Ω_S^{ext}, and regular at infinity. By virtue of $(-x) \cdot (\nabla V)(x) = 0$ for all $x \in \mathcal{X}$, we therefore obtain

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k)(n+1) H_{-n-1,k}^{S}(x) = 0, \quad x \in \mathcal{X} \ . \tag{10.514}$$

Since $\mathcal{X}$ is assumed to be a dense system on Ω_S, $S > R$, the identity (10.514) holds true for all $x \in \overline{\Omega_S^{\mathrm{ext}}}$. The completeness property of the theory of spherical harmonics then tells us that

$$V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k)(n+1) = 0, \tag{10.515}$$

hence,

$$V^{\wedge_{L^2(\Omega_S)}}(n,k) = 0 \tag{10.516}$$

for all $n = 0, 1, \ldots,$ $k = 1, \ldots, 2n+1$. This yields $V = 0$ in $\overline{\Omega_S^{\text{ext}}}$. By analytical continuation, we get $V = 0$ in Ω_R^{ext}, hence, $v = 0$ in Ω_R^{ext}. This is the desired result. □

Theorem 10.73 means that the Earth's external gravitational field is uniquely recoverable from (negative) radial derivatives corresponding to a fundamental system $\mathcal{X}$ on the satellite orbit. In other words, the Earth's external gravitational field is uniquely detectable on and outside Ω_R from GPS-SST data corresponding gradient vectors given on a dense system $\mathcal{X}$ on the satellite orbit Ω_S.

In conclusion, the results concerning satellite-to-satellite tracking (SST) show that the problem of developing the gravitational potential outside the Earth from given gradients in point systems on spherical orbits is overdetermined; it suffices to prescribe, e.g., the normal (i.e. radial) component (cf. Theorem 10.73). In fact, the Earth's gravitational potential can be detected alternatively from a dense system of surface gradients (i.e., vertical deflections) on the satellite orbit. Both aspects of gravitational field determination from SST-data will be described now in more detail:

Let the gravitational potential $V \in \mathrm{C}(\overline{\Omega_R^{\text{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\text{ext}})$ satisfying $\Delta V = 0$ in $\overline{\Omega_R^{\text{ext}}}$, $|V(x)| = O(\frac{1}{|x|})$, $|\nabla V(x)| = O(\frac{1}{|x|^2})$, $|x| \to \infty$, be given in the form

$$V = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{L^2(\Omega_R)}}(n,k) H^R_{-n-1,k}. \tag{10.517}$$

We have to derive V from its gravitational gradient ∇V on the external sphere Ω_S. For that purpose, we remember the decomposition of ∇V into its horizontal and tangential parts:

$$(\nabla V)(S\xi) = \frac{\partial V(r\xi)}{\partial r}\xi \bigg|_{r=S} + \frac{1}{r}\nabla^*_\xi V(r\xi)\bigg|_{r=S}, \quad S > R. \tag{10.518}$$

First, we show that V (as defined by (10.517)) is uniquely determined by its normal derivative on Ω_S. Observing the identity

$$\begin{aligned} \partial_r V(r\xi)\,|_{r=S} &= \frac{\partial V(r\xi)}{\partial r}\bigg|_{r=S} \qquad (10.519)\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge_{L^2(\Omega_R)}}(n,k)\Lambda_{\partial_{\frac{R}{S}}}(n) H^S_{-n-1,k}(S\xi) \end{aligned}$$

with

$$\Lambda_{\partial_{\frac{R}{S}}}(n) = -\frac{n+1}{S}\left(\frac{R}{S}\right)^{n} = -\frac{n+1}{R}\left(\frac{R}{S}\right)^{n+1} \tag{10.520}$$

we find $\partial_r V|_{r=S} = 0$ if and only if $V = 0$. This means that V is uniquely determined by its radial derivative $\partial_r V$ on Ω_S.

Second, it is not difficult to verify that

$$\begin{aligned} &\frac{1}{r}\nabla^*_\xi V(r\xi)\bigg|_{r=S} \\ &= \sum_{i=1}^{2}\sum_{k=0_i}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)\lambda^{(i)}_{\nabla^*;S}(n)h^{(1)}_{-n-1,k}(S\xi), \end{aligned} \tag{10.521}$$

where

$$\lambda^{(i)}_{\nabla^*;S}(n) = \begin{cases} -\frac{n}{S}\left(\frac{R}{S}\right)^{n}\sqrt{\frac{n+1}{2n+1}}; & i=1, \\ \frac{n+1}{S}\left(\frac{R}{S}\right)^{n}\sqrt{\frac{n}{2n+1}}; & i=2. \end{cases} \tag{10.522}$$

This leads us to the conclusion that

$$\frac{1}{r}\nabla^*_\xi V(r\xi)|_{r=S} = 0 \tag{10.523}$$

if and only if

$$V(x) = \frac{C_{0,1}}{|x|}, \quad C_{0,1} \in \mathbb{R}. \tag{10.524}$$

Turning over from the gravitational potential V to the disturbing potential T as indicated by (10.139), we get the following results:

$$\begin{aligned} \partial_r T|_{r=S} = 0 &\qquad \text{if and only if} \quad T = 0, \\ \nabla^{*;S} T = 0 &\qquad \text{if and only if} \quad T = 0. \end{aligned}$$

In other words, the anomalous potential is uniquely determined by its radial derivative or its surface gradient on the (orbital) surface Ω_S, respectively.

Finally, we remember

$$\nabla H^R_{-n-1,k} = -\sqrt{\tilde{\mu}^{(1)}_n}\, h^{(1);R}_{-n-1,k} = -\sqrt{(n+1)(2n+1)}\, h^{(1);R}_{-n-1,k}. \tag{10.525}$$

This enables us to write the gravitational field ∇V as follows:

$$\nabla V = -\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)\sqrt{\tilde{\mu}^{(1)}_n}\, h^{(1);R}_{-n-1,k}. \tag{10.526}$$

Moreover, it is easily seen that

$$\int_{\Omega_S} \nabla V(y)\cdot h^{(1);S}_{-p-1,q}(y)\, d\omega(y) = -\left(\frac{R}{S}\right)^{p+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(p,q)\sqrt{\tilde{\mu}^{(1)}_n}. \tag{10.527}$$

Therefore, we finally obtain the following reformulation of V from the series representation (10.517)

$$V = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \left(\nabla V, h^{(1);S}_{-n-1,k} \right)_{\mathrm{l}^2(\Omega_S)} (\tilde{\mu}_n^{(1)})^{-1/2} \left(\frac{S}{R} \right)^{n+1} H^R_{-n-1,k}. \quad (10.528)$$

The last formula expresses the gravitational potential on $\overline{\Omega_R^{\mathrm{ext}}}$ in terms of the gravitational gradient on the satellite orbit Ω_S. Essential tools are the vector outer harmonics. The same result holds true for the anomalous potential. Clearly, in (10.528), the equality on Ω_R is understood in $\mathrm{L}^2(\Omega_R)$-sense, while the convergence on each Ω_r, $r > R$ is understood in $\mathrm{C}(\Omega_r)$-sense.

10.10 Tensor Outer Harmonics and the Gravitational Tensor

The extension of the vectorial theory of outer harmonics to the tensorial case is straightforward. By use of the systems $\{\tilde{\mathbf{y}}^{(i,k);R}_{n,m}\}$ of tensor spherical harmonics, we are able to write down a set of tensor outer harmonics $\{\mathbf{h}^{(i,k);R}_{-n-1,m}\}$, where the arguments are quite similar to the vectorial case.

The system

$$\{\mathbf{h}^{(i,k);R}_{-n-1,m}(\cdot)\}_{i,k\in\{1,2,3\}}, \; n = \tilde{0}_{ik}, \tilde{0}_{i,k}+1, \ldots, m = 1, \ldots, 2n+1 \quad (10.529)$$

of *tensor outer harmonics of degree n, order m, and kind (i,k)* is given by

$$\mathbf{h}^{(1,1);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n+3} \tilde{\mathbf{y}}^{(1,1)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.530)$$

$$\mathbf{h}^{(1,2);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n+1} \tilde{\mathbf{y}}^{(1,2)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.531)$$

$$\mathbf{h}^{(2,1);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n+1} \tilde{\mathbf{y}}^{(2,1)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.532)$$

$$\mathbf{h}^{(2,2);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n-1} \tilde{\mathbf{y}}^{(2,2)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.533)$$

$$\mathbf{h}^{(3,3);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n+1} \tilde{\mathbf{y}}^{(3,3)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.534)$$

$$\mathbf{h}^{(1,3);R}_{-n-1,m}(x) = \frac{1}{R} \left(\frac{R}{|x|} \right)^{n+2} \tilde{\mathbf{y}}^{(1,3)}_{n,m} \left(\frac{x}{|x|} \right), \quad (10.535)$$

$$\mathbf{h}^{(2,3);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n} \tilde{\mathbf{y}}^{(2,3)}_{un,m}\left(\frac{x}{|x|}\right), \tag{10.536}$$

$$\mathbf{h}^{(3,1);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n+2} \tilde{\mathbf{y}}^{(3,1)}_{n,m}\left(\frac{x}{|x|}\right), \tag{10.537}$$

$$\mathbf{h}^{(3,2);R}_{-n-1,m}(x) = \frac{1}{R}\left(\frac{R}{|x|}\right)^{n} \tilde{\mathbf{y}}^{(3,2)}_{n,m}\left(\frac{x}{|x|}\right), \tag{10.538}$$

$x \in \mathbb{R}^3\backslash\{0\}$, $n = \tilde{0}_{ik}, \tilde{0}_{ik}+1, \ldots$, $m = 1, \ldots, 2n+1$ (for the definition of the $\tilde{\mathbf{y}}^{(i,j)}_{n,m}$, see Chapter 6).

The following properties are valid:

- $\mathbf{h}^{(i,k);R}_{-n-1,m}$ is of class $\mathbf{c}^{(\infty)}(\mathbb{R}^3\backslash\{0\})$,
- $\Delta_x \mathbf{h}^{(i,k);R}_{-n-1,m}(x) = 0$ for $x \in \mathbb{R}^3\backslash\{0\}$, i.e., the component functions of $\mathbf{h}^{(i,k);R}_{n,m}$ fulfill the Laplace equation,
- $\mathbf{h}^{(i,k);R}_{-n-1,m}|\Omega_R = \frac{1}{R}\tilde{\mathbf{y}}^{(i,k)}_{n,m}$,
- $|\mathbf{h}^{(i,k);R}_{-n-1,m}(x)| = O(|x|^{-1}), \quad |x| \to \infty$,
- $(\mathbf{h}^{(i,k);R}_{-n-1,m}, \mathbf{h}^{(p,q);R}_{-l-1,s})_{\mathbf{l}^2(\Omega_R)} = \delta_{ip}\delta_{kq}\delta_{nl}\delta_{ms}$.

In analogy to the vector case, we set

$$\widetilde{\mathbf{harm}}^{(i,k)}_{n}(\overline{\Omega^{\mathrm{ext}}_R}) = \operatorname{span}\left\{\mathbf{h}^{(i,k);R}_{-n-1,m}(\cdot)|\overline{\Omega^{\mathrm{ext}}_R}\,\middle|\, m = 1, \ldots, 2n+1\right\}, \tag{10.539}$$

and

$$\widetilde{\mathbf{harm}}^{(i,k)}_{p,\ldots,q}(\overline{\Omega^{\mathrm{ext}}_R}) = \bigoplus_{n=p}^{q} \widetilde{\mathbf{harm}}^{(i,k)}_{n}(\overline{\Omega^{\mathrm{ext}}_R}), \quad \tilde{0}_{ik} \le p \le q. \tag{10.540}$$

As in the case of spherical harmonics, we define

$$\widetilde{\mathbf{harm}}_0(\overline{\Omega^{\mathrm{ext}}_R}) = \bigoplus_{i=1}^{3} \widetilde{\mathbf{harm}}^{(i,1)}_{0}(\overline{\Omega^{\mathrm{ext}}_R}) \tag{10.541}$$

$$\widetilde{\mathbf{harm}}_1(\overline{\Omega^{\mathrm{ext}}_R}) = \bigoplus_{\substack{i,k=1 \\ (i,k)\neq(2,2),(3,2)}}^{3} \widetilde{\mathbf{harm}}^{(i,k)}_{1}(\overline{\Omega^{\mathrm{ext}}_R}), \tag{10.542}$$

$$\widetilde{\mathbf{harm}}_n(\overline{\Omega^{\mathrm{ext}}_R}) = \bigoplus_{i,k=1}^{3} \widetilde{\mathbf{harm}}^{(i,k)}_{n}(\overline{\Omega^{\mathrm{ext}}_R}), \qquad n \ge 2. \tag{10.543}$$

The addition theorems can be formulated in analogy to the vectorial case both for the tensor product of two tensor outer harmonics and for the product of a scalar and a tensor outer harmonic. They are very easy to derive, but lengthy to write down; so, we will omit them.

In parallel to the vectorial case, we find the following lemma.

Lemma 10.74. *(Linear Independence of Tensor Outer Harmonics) Let* $\{\mathbf{h}^{(i,k);}_{-n-1,m}\}_{\substack{i,k=1,2,3;n=\tilde{0}_{ik},\ldots,\\ m=1,\ldots,2n+1}}$ *be a system of tensor outer harmonics as defined in (10.530)–(10.538). Then, for all* $r > 0$ *the system*

$$\left\{\mathbf{h}^{(i,k)}_{n,m}|\Omega_r\right\}_{\substack{i,k=1,2,3;n=\tilde{0}_{ik},\ldots;\\ m=1,\ldots,2n+1}} \tag{10.544}$$

is linearly independent for all Ω_r.

It is obvious from the corresponding results of scalar outer harmonics that the following lemma holds true.

Lemma 10.75. *Let* $\{H^R_{-n-1,m}\}_{n=0,1,\ldots,m=1,\ldots,2n+1}$ *be a system of scalar outer harmonics. Then*

$$\overline{\mathrm{span}\{H^R_{-n-1,m}\varepsilon^i \otimes \varepsilon^k|\Omega_r\}}^{\|\cdot\|_{\mathbf{l}^2(\Omega_r)}} = \mathbf{l}^2(\Omega_r), \tag{10.545}$$

$$\overline{\mathrm{span}\{H^R_{-n-1,m}\varepsilon^i \otimes \varepsilon^k|\Omega_r)\}}^{\|\cdot\|_{\mathbf{c}(\Omega_r)}} = \mathbf{c}(\Omega_r). \tag{10.546}$$

Finally, we mention the following theorem.

Theorem 10.76. *Let* $\{\mathbf{h}^{(i,k);R}_{-n-1,m}\}_{\substack{i,k=1,2,3,n=\tilde{0}_{ik},\ldots,\\ m=1,\ldots,2n+1}}$ *be a system of tensor outer harmonics. Then, for all* $r > 0$*, the following statements hold true:*

$$\mathbf{l}^2(\Omega_r) = \overline{\mathop{\mathrm{span}}_{\substack{i,k=1,2,3;n=\tilde{0}_{ik},\ldots,\\ m=1,\ldots,2n+1}} (\mathbf{h}^{(i,k);R}_{-n-1,m})|\Omega_r}^{\|\cdot\|_{\mathbf{l}^2(\Omega_r)}}, \tag{10.547}$$

$$\mathbf{c}(\Omega_r) = \overline{\mathop{\mathrm{span}}_{\substack{i,k=1,2,3;n=\tilde{0}_{ik},\ldots,\\ m=1,\ldots,2n+1}} (\mathbf{h}^{(i,k);R}_{-n-1,m})|\Omega_r}^{\|\cdot\|_{\mathbf{c}(\Omega_r)}}. \tag{10.548}$$

As already mentioned, current knowledge of the Earth's gravity field, as derived from various observing techniques, is incomplete. We can only expect substantial improvements by exploiting new approaches based on satellite gravitational observation methods. Our intent now is to provide an overview at the sattelite-gravity-gradiometry (SGG) techniques to be realized by the ESA satellite GOCE. The concept considered for the GOCE mission (see ESA (1999)) is satellite-gravity-gradiometry (SGG), i.e., the measurement of the relative acceleration of test masses at different locations inside *one* satellite.

In an idealized situation, free of non-gravitational influences, the acceleration vector of a proof mass in free fall at the center x of mass of a space vehicle is, according to Newton's law, equal to the gradient of the gravitational potential: $v = \nabla V$. Considering now the motion of a second proof mass at y close to x relative to the first one, its acceleration is in the linearized sense

$$v(y) \approx v(x) + \mathbf{v}(x)(y - x) \ . \tag{10.549}$$

The matrix $\mathbf{v}(x) = (\nabla v)(x)$ is the Hesse matrix

$$\mathbf{v}(x) = \nabla \otimes \nabla V(x) \tag{10.550}$$

consisting of all second order derivatives of the Earth's gravitational potential V. Because of its tensor properties, $\mathbf{v}$ is called the gravitational tensor. In other words, measurements of the relative accelerations between two test masses provide information about the second order partial derivatives of the gravitational potential V. In an ideal observational situation, the full Hesse matrix is available by an array of test masses.

An illustrative view on satellite gradiometry based on Newton's theory of gravitation is as follows: Newton, when working on his law of gravitation, is said to have been inspired by a falling apple. Referring to the theory of gravitation as the tale of the falling apple, it would be appropriate to view gradiometry as the story of two falling apples. In C.W. Misner et al. (1973) this point is made clear. In one of their examples, it is shown that by measuring the relative distance between the shortest paths taken by two ants walking on the skin of an apple, from two adjacent beginning to two adjacent end points, the geometry of its curved surface can be derived. Translated to our case, shortest path means geodesic or free fall of two test particles (apples), from the relative motion of which the geometry of the curved space can be inferred, curved by the gravitational field of the Earth. Interpreting gravity in terms of geometry in the sense of Einstein, when all nine observable gradient components are measured at a point, gradiometry shows the complete local geometry of the relative motion of adjacent proof masses in free fall. However, it is more practical to constrain their relative motion by highly sensitive springs and measure instead the tension and compression of the springs. This is equivalent to saying that a gradiometer is realized by a coupled system of highly sensitive micro accelerometers. (A gradiometer of this kind is envisaged for the GOCE mission (see ESA (1999)) planned by ESA to produce a coverage of the entire Earth with measurements.) The one-dimensional principle is shown in Fig. 10.34. The gravitational force acting on the test masses results in an elongation of the springs, where (assuming linear stiffness) the elongation is proportional to the forces. Measuring the *differences* of the two elongations gives information on the differences of the forces, which is an approximation of the space derivative of the force field, and thus an approximation of the order derivative of the potential.

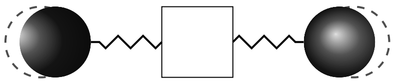


Fig. 10.34: Principle of gradiometry. Two masses are connected with springs at the satellite. Measured are differences of the elongation of the two springs.

In conclusion, the mathematical formulation of the SGG problem (after separating all non-gravitational influences) reads as follows:

Let there be known from the gravitational field v on $\overline{\Omega_R^{\text{ext}}}$ the gradients

$$\mathbf{v} = (\nabla v) = \nabla \otimes \nabla V, \tag{10.551}$$

on the orbital surface Ω_S of the LEO satellite. Find the geopotential V from the knowledge of the gravitational tensor $\nabla \otimes \nabla V$ on Ω_S.

We first deal with the problem of uniqueness corresponding to the model situation of a system $\mathcal{X} \subset \Omega_S$ of known GPS–SST data.

Theorem 10.77. *Suppose that $\mathcal{X} \subset \Omega_S$ (i.e., the subset of observational points on the satellite orbit $\Omega_S, S > R$) is a dense system on Ω_S. If $\mathbf{v}$ satisfies $\nabla \cdot \mathbf{v} = 0$, $\mathrm{L} \cdot \mathbf{v} = 0$ in Ω_R^{ext} such that*

$$\mathbf{v}(x) = 0, \quad x \in \mathcal{X},$$

then the associated field v satisfying $\nabla \cdot v = 0$, $\mathrm{L} \cdot v = 0$ in Ω_R^{ext} with $\mathbf{v} = \nabla v = \nabla^{(2)} V$ satisfies $v = 0$ in $\overline{\Omega_R^{\text{ext}}}$.

Proof. Any field $\mathbf{v}$ with $\nabla \cdot v = 0$, $\mathrm{L} \cdot v = 0$, in $\overline{\Omega_R^{\text{ext}}}$ can be expressed in the form $\nabla^{(2)} V = (\nabla \otimes \nabla) V$. Furthermore, the coordinate functions $\mathbf{v}_{ij} = \varepsilon^i \cdot \mathbf{v} \varepsilon^j$, $i, j \in \{1, 2, 3\}$, satisfy $\Delta \mathbf{v}_{ij} = 0$ in Ω_R^{ext}. This implies $\mathbf{v}_{ij} = 0$ in Ω_R^{ext}, $i, j \in \{1, 2, 3\}$, since $\mathcal{X}$ is a fundamental system in Ω_R^{ext}. From $\mathbf{v} = \nabla^{(2)} V = (\nabla \otimes \nabla) V = 0$, we finally get $V = 0$ in Ω_R^{ext} and, thus, $v = \nabla V = 0$, as required. □

In other words, the Earth's external gravitational field v is uniquely detectable on and outside the Earth's surface Ω_R if SGG data (i.e., second order derivatives of the Earth's gravitational potential V) are given on a dense system $\mathcal{X}$ (on the satellite orbit Ω_S).

Furthermore, we are able to verify the following result.

Theorem 10.78. *Suppose that $\mathcal{X}$ is a dense system of points on Ω_S. If $\mathbf{v}$ satisfies $\nabla \cdot \mathbf{v} = 0$, $\mathrm{L} \cdot \mathbf{v} = 0$ in Ω_R^{ext} with*

$$x \cdot (\mathbf{v}(x)x) = 0, \qquad x \in \mathcal{X},$$

then $v = 0$ in $\overline{\Omega_R^{\mathrm{ext}}}$ (with $\mathbf{v} = \nabla v = \nabla^{(2)} V$).

Proof. We base our arguments on the identity

$$\mathbf{v}(x) = \nabla^{(2)} V(x) = (\nabla \otimes \nabla) V(x), \quad x \in \Omega_R^{\mathrm{ext}} \ . \tag{10.552}$$

From our assumptions, it is clear that Ω_S^{ext}, $S > R$, is a strict subset of Ω_R^{ext}. Clearly, the potential $V|\Omega_S^{\mathrm{ext}}$ may be expanded in terms of outer harmonics

$$V(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k) H^S_{-n-1,k}(x), \quad x \in \Omega_S^{\mathrm{ext}}, \tag{10.553}$$

where $V^{\wedge_{\mathrm{L}^2(\Omega_s)}}(n,k)$ are the orthogonal coefficients (10.511). By elementary calculations, we get

$$\begin{aligned} \frac{x}{|x|} \cdot \left(\left(\nabla_x^{(2)} V \right)(x) \frac{x}{|x|} \right) &= \left(\frac{x}{|x|} \cdot \nabla_x \right) \left(\frac{x}{|x|} \cdot \nabla_x \right) V(x) \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{(n+1)(n+2)}{|x|^2} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k) H^S_{-n-1,k}(x), \end{aligned} \tag{10.554}$$

$x \in \overline{\Omega_S^{\mathrm{ext}}}$. Hence,

$$x \mapsto x \cdot \left(\nabla^{(2)} V \right)(x) x, \quad x \in \Omega_S^{\mathrm{ext}},$$

is a harmonic function in Ω_S^{ext}. In accordance with $x \cdot \big((\nabla^{(2)} V)(x) x \big) = 0$, $x \in \mathcal{X}$, we thus obtain

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_S)}}(n,k)(n+1)(n+2) H^S_{-n-1,k}(x) = 0, \quad x \in \mathcal{X}. \tag{10.555}$$

Since $\mathcal{X}$ is a fundamental system in Ω_R^{ext}, the identity (10.555) holds true in $\overline{\Omega_R^{\mathrm{ext}}}$. The theory of spherical harmonics then tells us that

$$V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)(n+1)(n+2) = 0, \tag{10.556}$$

hence, $V^{\wedge_{L^2(\Omega_S)}}(n,k) = 0$ for $n = 0, 1, \ldots,$ $k = 1, \ldots, 2n+1$. This yields $V = 0$ in Ω_S^{ext}. By analytical continuation, we have $V = 0$ in Ω_S^{ext}, and hence $v = \nabla V = 0$ in $\overline{\Omega_R^{\text{ext}}}$. □

Theorem 10.78 means that the Earth's external gravitational field is uniquely recoverable from 'second radial derivatives' corresponding to a fundamental system $\mathcal{X} \subset \Omega_S$.

Next, we remember the decomposition of $\nabla \otimes \nabla$ into its well-known radial and tangential parts:

$$\begin{aligned} \nabla_x \otimes \nabla_x &= \left(\xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla_\xi^*\right) \otimes \left(\xi \frac{\partial}{\partial r} + \frac{1}{r}\nabla_\xi^*\right) \\ &= \frac{\partial}{\partial r}\xi \otimes \frac{\partial}{\partial r}\xi + \frac{\partial}{\partial r}\xi \otimes \frac{1}{r}\nabla^*\xi \\ &\quad + \frac{1}{r}\nabla_\xi^* \otimes \frac{\partial}{\partial r}\xi + \frac{1}{r^2}\nabla_\xi^* \otimes \nabla_\xi^*. \end{aligned} \tag{10.557}$$

It will be seen that the operators $\partial/\partial r$, $(\partial/\partial r)^2$, $\nabla^*, \partial/\partial r\ \nabla^*$, and $\nabla^* \otimes \nabla^*$ form the constituting ingredients of the gravitational tensor $\nabla \otimes \nabla V$. It should be noted that operators $\partial/\partial r$ and ∇^* have already been described within the SST-context for specifying the gravitational gradient. Therefore, it remains to discuss the derivatives $(\partial/\partial r)^2$, $\partial/\partial r\ \nabla^*$, and $\nabla^* \otimes \nabla^*$.

First, an easy calculation gives

$$\begin{aligned} (\partial_r)^2\, V(r\xi)|_{r=S} &= \left.\frac{\partial^2 V(r\xi)}{\partial r^2}\right|_{r=S} \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge_{L^2(\Omega_R)}}(n,k)\Lambda(n) H^S_{-n-1,k}, \end{aligned} \tag{10.558}$$

where

$$\Lambda(n) = \left(-\frac{n+1}{S}\right)\left(-\frac{n+2}{S}\right)\left(\frac{R}{S}\right)^n = \frac{(n+1)(n+2)}{R^2}\left(\frac{R}{S}\right)^{n+2}. \tag{10.559}$$

Thus it follows that $(\partial_{\frac{R}{S}})^2 V = 0$ if and only if $V = 0$.

Second, we are able to write

$$\frac{1}{r}\partial_{\frac{R}{r}}\nabla^*_\xi V(r,\xi)\bigg|_{r=S} = \frac{1}{r}\frac{\partial}{\partial r}\nabla^*_\xi V(r\xi)\bigg|_{r=S} = \sum_{i=1}^{2}\sum_{n=0_i}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k)\lambda^{(i)}_{\partial_r}\nabla^{*;S}(n)h^{(i);S}_{-n-1,k}(S\xi), \tag{10.560}$$

where

$$\lambda^{(i)}_{\partial_r\nabla^{*;S}}(n) = \begin{cases} \frac{n(n+1)}{S^2}\sqrt{\frac{n+2}{2n+1}}(\frac{R}{S})^n, & i=1, \\ -\frac{(n+1)^2}{S^2}\sqrt{\frac{n}{2n+1}}(\frac{R}{S})^n, & i=2. \end{cases} \tag{10.561}$$

This shows us that

$$\left(\frac{1}{r}\partial_{\frac{R}{r}}\nabla^*_\xi V(r\xi)\right)|_{r=S} = 0 \tag{10.562}$$

if and only if

$$V(x) = \frac{C}{|x|}, \quad C\in\mathbb{R}. \tag{10.563}$$

Third, it can be shown that

$$\frac{1}{r^2}\nabla^*_\xi\otimes\nabla^*_\xi V(r\xi)|_{r=S} = \sum_{i,k=1}^{2}\sum_{n=\tilde{0}_{i,k}}^{\infty}\sum_{m=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,m)\lambda^{(i,k)}_{\nabla^{*;S}\nabla^{*;S}}(n)\mathbf{h}^{(i,k);S}_{-n-1,m}(S\xi), \tag{10.564}$$

where

$$\lambda^{(i,k)}_{\nabla^{*;S}\otimes\nabla^{*;S}}(n) = \begin{cases} \sqrt{\mu_n^{(1,1)}}\frac{n(n+1)}{S^2(2n+1)(2n+3)}(\frac{R}{S})^n &, \quad (i,k)=(1,1), \\ -\sqrt{\mu_n^{(1,2)}}\frac{(n+1)(n-1)}{S^2(2n-1)(2n+1)}(\frac{R}{S})^n &, \quad (i,k)=(1,2), \\ -\sqrt{\mu_n^{(2,1)}}\frac{n(n+2)}{S^2(2n+3)(2n+1)}(\frac{R}{S})^n &, \quad (i,k)=(2,1), \\ \sqrt{\mu_n^{(2,2)}}\frac{n(n+1)(n+2)}{S^2(2n-1)(2n+1)}(\frac{R}{S})^n &, \quad (i,k)=(2,2). \end{cases} \tag{10.565}$$

This implies

$$\frac{1}{r^2}\nabla^*_\xi\otimes\nabla^*_\xi V(r\xi)\bigg|_{r=S} = 0 \tag{10.566}$$

if and only if

$$V = \sum_{n=0}^{1}\sum_{m=1}^{2n+1} C_{n,m}H^R_{-n-1,m}, \quad C_{n,m}\in\mathbb{R}. \tag{10.567}$$

Turning over from the gravitational potential V to the disturbing potential T (where the Fourier coefficients of order 0 and 1 are zero), we get the following result:

Corollary 10.79. *For the disturbing potential T, each of the following statements is equivalent to $T = 0$:*

(i) $\partial_r T = 0$ *on a sphere at satellite's height,*

(ii) $(\partial_r)^2 T = 0$ *on a sphere at satellite's height,*

(iii) $\nabla^{*;S} T = 0$ *on a sphere at satellite's height,*

(iv) $\partial_r \nabla^{*;S} T = 0$ *on a sphere at satellite's height,*

(v) $\nabla^{*;S} \otimes \nabla^{*;S} T = 0$ *on a sphere at satellite's height.*

Finally, we mention that

$$\begin{aligned} \nabla \otimes \nabla H^R_{-n-1,k} &= \sqrt{\tilde{\mu}_n^{(1,1)}} \mathbf{h}^{(1,1);R}_{-n-1,k} \\ &= \sqrt{(n+2)(n+2)(2n-3)(2n-1)} \mathbf{h}^{(1,1);R}_{-n-1,k}. \end{aligned} \tag{10.568}$$

Consequently, we find

$$\nabla \otimes \nabla V = \sum_{k=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) \sqrt{\tilde{\mu}_n^{(1,1)}} \mathbf{h}^{(1,1);R}_{-n-1,m}. \tag{10.569}$$

This enables us to verify that

$$\int_{\Omega_S} \nabla \otimes \nabla V(y) \cdot \mathbf{h}^{(1,1);S}_{-p-1,q}(y)\, d\omega(y) = \left(\frac{R}{S}\right)^{p+2} V^{\wedge_{\mathrm{L}^2(\Omega_R)}}(p,q) \sqrt{\tilde{\mu}_n^{(1,1)}}. \tag{10.570}$$

Therefore, from (10.517), we finally obtain the following reformulation of V,

$$V = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} (\nabla \otimes \nabla V; \mathbf{h}^{(1,1);S}_{-n-1,k})_{\mathbf{l}^2(\Omega_S)} (\tilde{\mu}_n^{(1,1)})^{-1/2} (\frac{S}{R})^{n+2} H^R_{-n-1,k}. \tag{10.571}$$

This formula expresses the gravitational potential V in $\overline{\Omega_R^{\mathrm{ext}}}$ in terms of the gravitational tensor $\nabla \otimes \nabla V$ on the satellite orbit Ω_S. Essential tools are the tensor outer harmonics. The equality in (10.571) is understood in the standard sense. The result is also true for the anomalous potential.

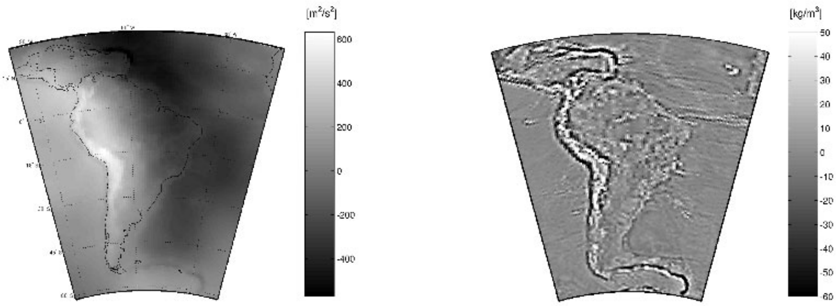


Fig. 10.35: Terrestrial EGM96-potential (*left*) and terrestrial harmonic density of the EGM96-potential (*right*) for South America (Geomathematics Group, TU Kaiserslautern (2008)).

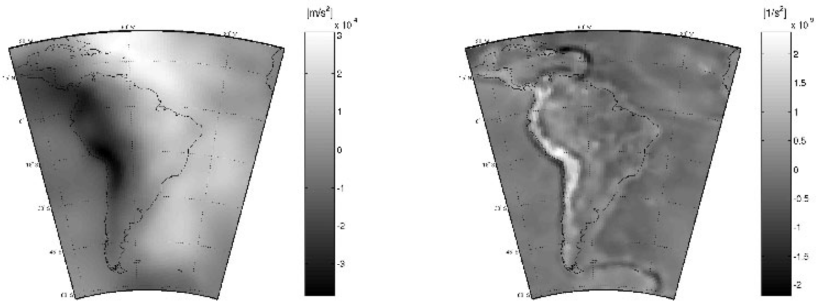


Fig. 10.36: First radial derivative at 400 km (*left*) and second radial derivative at 200 km (*right*) of the EGM96-potential for South America (Geomathematics Group, TU Kaiserslautern (2008)).

10.11 Gravity Quantities in Spherical Nomenclature

Finally, we list all essential material of our spherically oriented approach to gravity quantities (GQ) of the anomalous potential in a formal setup (see Figs. 10.35, 10.36 for getting a graphical impression):

Assume that the anomalous potential T satisfies the properties

- $T \in \mathrm{C}(\overline{\Omega_R^{\mathrm{ext}}}) \cap \mathrm{C}^{(2)}(\Omega_R^{\mathrm{ext}})$,
- $\Delta T = 0$ in $\overline{\Omega_R^{\mathrm{ext}}}$,
- T is regular at infinity,

$$|T(x))| = O\left(\frac{1}{|x|}\right), \quad |\nabla T(x)| = O\left(\frac{1}{|x|^2}\right), |x| \to \infty,$$

- $T^{\wedge_{L^2(\Omega_R)}}(n,k) = 0, \quad n = 0, 1, \text{and } k = 1, \ldots, 2n+1.$

Then T is given in form

$$T = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} T^{\wedge_{L^2(\Omega_R)}}(n,k) H^R_{-n-1,k}, \tag{10.572}$$

where the equality is understood in the sense

$$\lim_{N\to\infty} \left(\int_{\Omega_R} \left| T(y) - \sum_{n=2}^{N} \sum_{k=1}^{2n+1} T^{\wedge_{L^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(y) \right|^2 d\omega(y) \right)^{1/2} = 0 \tag{10.573}$$

and

$$\lim_{N\to\infty} \sup_{x\in\Omega_r} \left| T(x) - \sum_{n=2}^{N} \sum_{k=1}^{2n+1} T^{\wedge_{L^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(x) \right| = 0, \tag{10.574}$$

$r > R$. Relevant scalar gravity quantities (GQ) are characterized by the operator Λ^L_{GQ} given by

$$\begin{aligned} \Lambda^L_{GQ} T(x) &= \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \Lambda^L_{GQ}(n) T^{\wedge_{L^2(\Omega_R)}}(n,k) H^R_{-n-1,k}(x) \qquad (10.575) \\ &= \int_{\Omega_L} K_{\Lambda^L_{GQ}}(x,y) T(y) \, d\omega(y) \end{aligned}$$

with

$$K_{\Lambda^L_{GQ}}(x,y) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \Lambda^L_{GQ}(n) H^R_{-n-1,k}(x) H^L_{-n-1,k}(y), \tag{10.576}$$

$L \in \{R, S\}$, where R is the ground level and S is the satellite level.

By virtue of the addition theorem, we are allowed to write (10.576) in the form

$$K_{\Lambda^L_{GQ}}(x,y) = \frac{1}{RL} \sum_{n=2}^{\infty} \Lambda^L_{GQ}(n) \left(\frac{RL}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \tag{10.577}$$

where P_n is the one-dimensional Legendre polynomial of degree n. Obviously, from (10.577), we see that $K_{\Lambda^L_{GQ}}(\cdot,\cdot)$ is a zonal function provided that $x \in \Omega_R, y \in \Omega_L$. We summarize the results in Table 10.9.

Table 10.9: 'L-upward continued' scalar gravity quantities.

Symbol	**Operator**	**Quantity**
$\left(\frac{R}{L}\right)^n$	T	anomalous potential
$-\frac{n-1}{r}\left(\frac{R}{L}\right)^n$	A	gravity anomaly
$-\frac{n+1}{R}\left(\frac{R}{L}\right)^{n+1}$	$(\partial_r)\vert_{r=L}$	first radial derivative
$\left(-\frac{n+2}{R}\right)\left(-\frac{n+1}{R}\right)\left(\frac{R}{L}\right)^{n+2}$	$(\partial_r)^2\Big\vert_{r=L}$	second radial derivative

If the anomalous potential T is assumed to be bandlimited, i.e.,

$$T^{\wedge_{\mathrm{L}^2(\Omega_R)}}(n,k) = 0, \; n \geq N+1, k = 1, \ldots, 2n+1, \tag{10.578}$$

then T can be recovered from the 'L-upward continued' gravity quantities as follows

$$T(x) = \int_{\Omega_L} K^N_{(\Lambda^L_{GQ})^{-1}}(x,y)\Lambda^L_{GQ}(y) \; d\omega(y), \tag{10.579}$$

where

$$K^N_{(\Lambda^L_{GQ})^{-1}}(x,y) = \sum_{n=2}^{N}\sum_{k=1}^{2n+1} \left(\Lambda^L_{GQ}(n)\right)^{-1} H^R_{-n-1,k}(x) H^L_{-n-1,k}(y), \tag{10.580}$$

$(x,y) \in \overline{\Omega^{\mathrm{ext}}_R} \times \overline{\Omega^{\mathrm{ext}}_L}$.

The (relevant) vectorial gravity quantity in SST is the 'L-upward continued' gradient ∇T given by

$$\lambda^L_{GQ} T(x) = \int_{\Omega_L} k_{\lambda^L_{GQ}}(x,y) T(y) \; d\omega(y), \tag{10.581}$$

where

$$k_{\lambda^L_{GQ}}(x,y) = \sum_{n=2}^{\infty}\sum_{k=1}^{2n+1} \lambda^L_{GQ}(n) h^{(1);R}_{-n-1,k}(x) H^L_{-n-1,k}(y), \tag{10.582}$$

$(x,y) \in \overline{\Omega^{\mathrm{ext}}_R} \times \overline{\Omega^{\mathrm{ext}}_L}$ (see Table 10.10).

Table 10.10: 'L-upward continued' anomalous gradient.

Symbol	**Operator**	**Quantity**
$\sqrt{\tilde{\mu}_n^{(1)}}\left(\frac{R}{L}\right)^{n+1}$	∇T	anomalous field

A bandlimited potential T can be recovered from the 'L-upward continued' gradient ∇T as follows

$$T(x) = \int_{\Omega_L} k^N_{(\lambda^L_{GQ})^{-1}}(x,y) \cdot \lambda^L_{GQ} T(y)\, d\omega(y), \tag{10.583}$$

where

$$k^N_{(\lambda^L_{GQ})^{-1}}(x,y) = \sum_{n=2}^{N} \sum_{k=1}^{2n+1} \left(\lambda^L_{GQ}(n)\right)^{-1} H^R_{-n-1,k}(x) h^{(1);L}_{-n-1,k}(y), \tag{10.584}$$

$(x,y) \in \overline{\Omega^{\text{ext}}_R} \times \overline{\Omega^{\text{ext}}_L}$.

The addition theorem enables us to rewrite $k_{\lambda^L_{GQ}}(\cdot,\cdot)$ in the form

$$k_{(\lambda^L_{GQ})^{-1}}(x,y) = \frac{1}{RL} \sum_{n=2}^{N} \frac{2n+1}{4\pi} \left(\frac{L}{|y|}\right)^{n+2} \left(\frac{R}{|x|}\right)^{n+1} \tilde{p}_n^{(1)} \left(\frac{y}{|y|} \cdot \frac{x}{|x|}\right). \tag{10.585}$$

As (relevant) tensorial gravity quantity for satellite-gravity-gradiometry (SGG), we finally mention the 'L-upward continued' anomalous tensor $\nabla \otimes \nabla T$ given by

$$\boldsymbol{\lambda}^L_{GQ} T = \int_{\Omega_R} \mathbf{k}_{\boldsymbol{\lambda}^L_{GQ}}(x,y) T(y)\, d\omega(y), \tag{10.586}$$

where

$$\mathbf{k}_{\boldsymbol{\lambda}^L_{GQ}}(x,y) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \boldsymbol{\lambda}^L_{GQ}(n) \mathbf{h}^{(1,1);R}_{-n-1,k}(x) H^L_{-n-1,k}(y), \tag{10.587}$$

$(x,y) \in \overline{\Omega^{\text{ext}}_R} \times \overline{\Omega^{\text{ext}}_L}$ (see Table 10.11).

Table 10.11: 'L-upward continued' anomalous tensor.

symbol	Operator	Quantity
$\sqrt{\tilde{\mu}_n^{(1,1)}} \left(\frac{R}{L}\right)^{n+2}$	$\nabla \otimes \nabla T$	anomalous tensor

10.12 Pseudodifferential Operators and Geomathematics

In the preceding chapters of this book, we put particular emphasis on the usefulness and the beauty of orthogonality and rotational invariance for sphere oriented applications. It has been shown how scalar, vectorial and tensorial problems can be dealt with in a unified concept. All these things led quite naturally to zonal kernels of different types. In this section, we like to focus our attention on (invariant) pseudodifferential operators and their usefulness in the described setting.

An invariant pseudodifferential operator (for brevity, we will leave out the word invariant in the following) can be seen as a linear operator such that its effect on the orthogonal system of spherical harmonics is only on the degree, but not on the order, i.e.,

$$\Lambda Y_{n,m} = \Lambda^\wedge(n) Y_{n,m}, \quad m = 1, \ldots, 2n+1. \tag{10.588}$$

The spherical symbol $\Lambda^\wedge(n)$ has many appealing properties. For example, it is easily seen that

$$(\Lambda' + \Lambda'')^\wedge(n) = (\Lambda')^\wedge(n) + (\Lambda'')^\wedge(n) \tag{10.589}$$

and

$$(\Lambda'\Lambda'')^\wedge(n) = (\Lambda')^\wedge(n)(\Lambda'')^\wedge(n), \tag{10.590}$$

for all $n = 0, 1, \ldots$

Since pseudodifferential operators on the sphere belong to the traditional equipment, we do not give a general description (within a Sobolev space framework). Instead, we restrict ourselves to certain geophysically relevant examples of pseudodifferential operators. A more detailed discussion (e.g., from the point of physical geodesy) can be found in S.L. Svensson (1983) and W. Freeden et al. (1998).

The perhaps best known scalar example is the Laplace Beltrami operator on the unit sphere Ω with the symbol $(\Delta^*)^\wedge(n) = -n(n+1), n = 0, 1, ...$

(1) *Beltrami operator* Δ^*. It is not invertible since $(\Delta^*)^\wedge(0) = 0$, but $-\Delta^* + \frac{1}{4}$ has the symbol $\{(-\Delta^* + \frac{1}{4})^\wedge(n)\}$ with $(-\Delta^* + \frac{1}{4})^\wedge(n) = (n + \frac{1}{2})^2$, $n = 0, 1, ...$ and, hence, has an inverse $(-\Delta^* + \frac{1}{4})^{-1}$ which is a rational pseudodifferential operator of order -2, i.e. $n \mapsto ((-\Delta^* + \frac{1}{4})^{-1})^\wedge(n)$, $n \in \mathbb{N}_0$, is a rational function of order -2. More generally, $(-\Delta^* + \frac{1}{4})^s$ is a rational pseudodifferential operator of order $2s$ and has the spherical symbol $\{((-\Delta^* + \frac{1}{4})^s)^\wedge(n)\}$ with

$$\left(\left(-\Delta^* + \frac{1}{4}\right)^s\right)^\wedge(n) = \left(n + \frac{1}{2}\right)^{2s}, \quad n = 0, 1, ... \tag{10.591}$$

(2) *Green's integral operator*. The operator Λ given by

$$\Lambda(U)(\xi) = \frac{1}{4\pi} \int_\Omega G(\Delta^*; \xi \cdot \eta) U(\eta)\, d\omega(\eta), \quad \xi \in \Omega \tag{10.592}$$

has the spherical symbol $\{(\Lambda)^\wedge(n)\}$, where

$$(\Lambda)^\wedge(n) = \begin{cases} 0 & \text{for} \quad n = 0 \\ -1/(n(n+1)) & \text{for} \quad n = 1, 2, \ldots \end{cases} \tag{10.593}$$

Note that the operator Λ given by

$$\Lambda(U)(\xi) = \frac{1}{4\pi} \int_\Omega G(\Delta^* + \frac{1}{4}; \xi \cdot \eta) U(\eta)\, d\omega(\eta), \quad \xi \in \Omega \tag{10.594}$$

has the spherical symbol $\{(\Lambda)^\wedge(n)\}$, where

$$(\Lambda)^\wedge(n) = \begin{cases} -4 & \text{for} \quad n = 0 \\ -1/(n(n+1) + \frac{1}{4}) = -1/(n + \frac{1}{2})^2 & \text{for} \quad n = 1, 2, \ldots \end{cases} \tag{10.595}$$

A pseudodifferential operator Λ satisfying $\Lambda^\wedge(n) \to 0$ is called a *smoothing operator*, because $\Lambda^\wedge(n) \to 0$ means that the higher order harmonics are subdued by the operator. An example is the Green integral operator. The Beltrami operator has an opposite effect, because the higher order harmonics are amplified.

This concept can be generalized in natural way to the vectorial, tensorial and mixed cases. A pseudodifferential operator mapping scalar functions to vector fields on Ω is given by a symbol

$$\lambda^\wedge(n) = (\lambda^\wedge_{(1)}(n), \lambda^\wedge_{(2)}(n), \lambda^\wedge_{(3)}(n)) \tag{10.596}$$

such that

$$\lambda Y_{n,m} = \lambda^\wedge_{(1)}(n) y^{(1)}_{n,m} + \lambda^\wedge_{(2)}(n) y^{(2)}_{n,m} + \lambda^\wedge_{(3)}(n) y^{(3)}_{n,m}. \tag{10.597}$$

Similarly, we can use such an operator representation for the mapping of vector fields to scalar ones:

$$\lambda y^{(i)}_{n,m} = \lambda^\wedge_{(i)}(n) Y_{n,m}. \tag{10.598}$$

Obvious generalizations lead to pseudodifferential operators mapping scalar fields to tensor fields, vector fields to tensor fields, and so on.

In this context, the surface gradient $o^{(2)} = \nabla^*$, e.g., can be interpreted as a pseudodifferential operator with symbol

$$(\nabla^*)^\wedge(n) = (0, \sqrt{n(n+1)}, 0), \tag{10.599}$$

since $\nabla^* Y_{nm} = \sqrt{n(n+1)} y^{(2)}_{n,m}$. The surface curl gradient $o^{(3)} = \mathrm{L}^*$ defined by $\mathrm{L}^*_\xi = \xi \wedge \nabla^*_\xi, \xi \in \Omega$, can be seen to have the symbol

$$(\mathrm{L}^*)^\wedge(n) = (0, 0, \sqrt{n(n+1)}). \tag{10.600}$$

Using this terminology the operators $O^{(i)}$ are found to be pseudodifferential operators of order 0 if $i = 1$ and order 1 if $i = 2, 3$. More explicitly, we have

$$\begin{aligned} (O^{(1)})^\wedge(n) &= 1, \\ (O^{(i)})^\wedge(n) &= \sqrt{n(n+1)}, \quad i = 2, 3. \end{aligned}$$

Similarly, the tensorial case can be attacked.

$$\begin{aligned} \boldsymbol{\lambda} Y_{n,m} &= \boldsymbol{\lambda}^\wedge_{(1,1)}(n) \mathbf{y}^{(1,1)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(1,2)}(n) \mathbf{y}^{(1,2)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(1,3)}(n) \mathbf{y}^{(1,3)}_{n,m} \\ &+ \boldsymbol{\lambda}^\wedge_{(2,1)}(n) \mathbf{y}^{(2,1)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(2,2)}(n) \mathbf{y}^{(2,2)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(2,3)}(n) \mathbf{y}^{(2,3)}_{n,m} \\ &+ \boldsymbol{\lambda}^\wedge_{(3,1)}(n) \mathbf{y}^{(3,1)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(3,2)}(n) \mathbf{y}^{(3,2)}_{n,m} + \boldsymbol{\lambda}^\wedge_{(3,3)}(n) \mathbf{y}^{(3,3)}_{n,m}. \end{aligned} \tag{10.601}$$

For example, the symbol of the operator $\mathbf{o}^{(1,2)}$ defined by $\mathbf{o}^{(1,2)}_\xi = \xi \otimes \nabla^*_\xi$, $\xi \in \Omega$, can be characterized by

$$(\mathbf{o}^{(1,2)})^\wedge(n) \begin{pmatrix} 0 & \sqrt{\mu_n^{(1,2)}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{n(n+1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10.602}$$

In particular, the operators $O^{(i,i)}$, $i = 1, 2, 3$, can be interpreted to be pseudodifferential operators of order 0, while the operators $O^{(i,k)}$ are of order 1 for $(i,k) \in \{(1,2),(1,3),(2,1),(3,1)\}$, or of order 2 if $(i,k) \in \{(2,3),(3,2)\}$, respectively. Their symbols are given by

$$(O^{(i,k)})^\wedge(n) = \begin{cases} 1 & \text{if } (i,k)=(1,1) \\ \sqrt{2} & \text{if } (i,k) \in \{(2,2),(3,3)\} \\ \sqrt{n(n+1)} & \text{if } (i,k) \in \{(1,2),(1,3),(2,1),(3,1)\} \\ \sqrt{n(n+1)(n(n+1)-2)} & \text{if } (i,k) \in \{(2,3),(3,2)\}. \end{cases}$$

The strong connection of orthogonal invariance and zonal kernels gets obvious, when applying these pseudodifferential operators to Legendre kernels. To be more concrete, the addition theorem of spherical harmonics is on the one hand

$$P_n(\xi \cdot \eta) = \frac{4\pi}{2n+1} \sum_{m=1}^{2n+1} Y_{n,m}(\xi) Y_{n,m}(\eta). \tag{10.603}$$

On the other hand, we already know that the vectorial Legendre vector kernel satisfies

$$^v p_n^{(i)}(\xi,\eta) = (\mu_n^{(i)})^{1/2} o_\xi^{(i)} P_n(\xi \cdot \eta) = \frac{4\pi}{2n+1} \sum_{m=1}^{2n+1} y_{n,m}^{(i)}(\xi) Y_{n,m}(\eta). \tag{10.604}$$

That means that the transition from $P_n(\xi \cdot \eta)$ to $^v p_n^{(i)}(\xi,\eta)$ can be described with the application of the pseudodifferential operator $\delta_{(i)}$ with

$$\delta_{(i)}^\wedge(n) = (\delta_{i1}, \delta_{i2}, \delta_{i3}). \tag{10.605}$$

Similarly, the transition from $P_n(\xi \cdot \eta)$ to $\mathbf{p}_n^{(i,k)}(\xi,\eta)$ can be seen as the application of a (tensorial) pseudodifferential operator with symbol

$$\boldsymbol{\delta}_{(i,j)}^\wedge(n) = \begin{pmatrix} \delta_{1i}\delta_{1j} & \delta_{1i}\delta_{2j} & \delta_{1i}\delta_{3j} \\ \delta_{2i}\delta_{1j} & \delta_{2i}\delta_{2j} & \delta_{3i}\delta_{3j} \\ \delta_{2i}\delta_{1j} & \delta_{2i}\delta_{2j} & \delta_{3i}\delta_{3j} \end{pmatrix} \tag{10.606}$$

Thus, we have developed an efficient and clear calculus for the solution of many problems in sphere oriented geomathematics. We can switch from orthogonal systems with scalar, vectorial or tensorial spherical harmonics, to zonal kernels or pseudodifferential operators to get the best representation for the underlying problems. Furthermore, as illustrated in many examples in this book, efficient numerical schemes can be implemented with the help of these techniques.

10.13 Bibliographical Notes

This chapter is essentially led by the contents of the German priority research programme 'Mass Transport and Mass Distribution in the Earth System' (DFG-SPP 1257) (see K.H. Ilk et al. (2004) and the references therein). The particular role of gravitation as a key observable for mass distribution even from space was already pointed out in ESA-reports of the last century (cf. ESA (1996, 1998, 1999)). Our background of classical physical geodesy concerned with Earth's gravitational field determination is taken from standard monographs (E. Groten (1979), W.A. Heiskanen, H. Moritz (1967), W. Torge (1991)). The locally oriented 'zooming-in' techniques for determining disturbance potential, gravity anomalies, deflections of the vertical etc are based on different approximation methods leading to a palette of new spherical base systems, for example, Dirac sequences (see, e.g., W. Freeden, M. Schreiner (1995, 2006, 2007). W. Freeden, U. Windheuser (1997), W. Freeden (1999), M.J. Fengler et al. (2004)), regularizations of Green's functions (W. Freeden, M. Schreiner (2006), W. Freeden, C. Mayer (2003, 2007)), T. Fehlinger et al. (2007, 2008), W. Freeden, K. Wolf (2008), space- and frequency limited zonal kernel functions (see, e.g., S.L. Svensson (1983), W. Freeden (1990b), W. Freeden et al. (1996, 1998a), F. Bauer et al. (2006), W. Freeden, K. Hesse (2002), M. Schreiner (2003)). The (geostrophic) modeling of ocean circulation is in close orientation to the work of J. Pedlovsky (1979), R.S. Nerem et al. (1990) and many others, multiscale approximation of geostrophic flow is applied numerically by W. Freeden et al. (2005), D. Michel (2006), T. Fehlinger et al. (2007). The vectorial systems in Earth's deformation based on the Cauchy-Navier equations go back to V.D. Kupradze (1965), E.W. Grafarend (1982, 1986), T. Gervens (1989), W. Freeden et al. (1990), M.K. Abeyratne, W. Freeden and C. Mayer (2003), W. Freeden, V. Michel (2004b). Starting from important investigations by N. Weck (1972), L. Ballani et al. (1993a,b), the theses of V. Michel (1999, 2002) and V. Michel (2002, 2005), W. Freeden, V. Michel (2004b), V. Michel, K. Wolf (2008) significantly contribute to the specification of function systems involved in the inversion of Newton's potential, i.e., the so-called gravimetry problem. Finally, it should be mentioned that the concepts of satellite technology (SST, SGG) are essentially influenced by the ideas of R. Rummel and his former Delft and recent Munich working groups (see, e.g., R. Rummel (1986, 1997)). Their major interest in spaceborne data is globally reflected orthogonal expansions in terms of (frequency localizing) vector and tensor spherical harmonics (cf. R. Rummel, M. van Gelderen (1992), R. Rummel et al. (1993)). The research of the Geomathematics Group, Kaiserslautern, is much more concerned with locally oriented modeling by means of space-limited (i.e., locally supported) vector and tensor zonal functions (see M. Schreiner 1994, 1997),

W. Freeden et al. (1994, 1998a)). In this respect the results obtained for inverse problems by multiresolution are important keystones (see W. Freeden (1990b), F. Schneider (1996, 1997), W. Freeden et al. (1997, 2002), W. Freeden, F. Schneider (1998c), W. Freeden (1999), W. Freeden, V. Michel (2004b)).

This book is dedicated to the memory of Prof. Dr. Claus Müller, RWTH Aachen, who died on February 6, 2008.

Concluding Remarks

Today's geosciences profit so much from the possibilities that result from highly advanced electronic measurement concepts, modern computer technology and, most of all, artificial satellites. In fact, the exceptional situation of getting simultaneous and complementary observations from a multiple of low-orbiting satellites opens new opportunities to contribute significantly to the understanding of our planet, its climate, its environment and about an expected shortage of natural resources. All of a sudden, key parameters for the study of the dynamics of our planet and the interaction of its solid part with ice, oceans, atmosphere etc become accessible. In this context, new types of vector and tensor data measured on (almost) spherical reference surfaces such as the (spherical) Earth or (near-)circular orbits are very likely the greatest challenge. These data help geodesists to determine the Earth's gravitational field from spaceborne gravity sensors, the oceanographers to see the oceans flow, people from geomagnetics to get insight in the spatio-time variation of the magnetic field, solid Earth physicists to better understand the dynamics of the Earth's interior, meteorologists to simulate wind fields, a.s.o. However, (non-standard) observations and data having a different type, location, and distribution cannot be handled by traditional modeling and simulation techniques. This is the reason why adequate components of mathematical thinking, adapted formulations of theories and models, and economical and efficient numerical developments are indispensable. Up to now, the modeling of vector and tensor data is done on global scale by orthogonal expansions by means of polynomial structures such as (certain types of) vector and tensor spherical harmonics. But so far, they can not keep pace with the prospects and the expectations of the 'Earth system sciences'. Moreover, there is an increasing need for high-precision modeling on local areas. In this respect, zonal kernel functions, i.e., in the jargon of constructive approximation, radial basis functions, become more and more important because of their space localizing properties even in the vectorial and tensorial context. The current book shows that the addition theorem of the theory of spherical harmonics enables us to express all types of zonal kernel functions in terms of a one-dimensional function, viz. the Legendre polynomial. In other words, additive clustering of spherical harmonics generates specific classes of space localizing zonal kernel functions, i.e., Legendre series expansions, ready for approximation within scalar, vectorial, and tensorial framework. Furthermore, our investigations demonstrate that the closer the Legendre series expansion is to the Dirac kernel, the more localized is the zonal kernel in space, and the more economical is its role in (spatial) local computation. In addition, the Funke–Hecke formula provides the natural tool for establishing convolutions of spherical fields against zonal kernels. In consequence, by specifying scaling functions, i.e., sequences of zonal functions tending to the Dirac kernel, (space-localized) filtered versions of (square-integrable) spherical fields are obtainable by convolution leading to 'zooming-in' approximation within a multiscale procedure. Altogether, the vectorial and tensorial counterparts of the Legendre polynomial are the essential keystones in our work. They enable the transition from spherical harmonics via zonal kernels up to the Dirac kernel. In addition, the Funk–Hecke formula and its consequences in spherical convolutions opens new methological perspectives for global as well as local approximation in vectorial and tensorial physically motivated application.

It should be remarked that only the joint use of mathematical technology and highly accurate sensors combining (globally available) spaceborne data with local airborne and/or terrestrial observations will contribute to a deeper knowledge of the Earth system and, in turn, to the development of sustainable strategies to safeguard the human habitat for future generations. In this respect, the spherically oriented structures, methods and procedures presented in this book form an essential step for handling terrestrial, airborne, and spaceborne data under relevant physical as well as numerical assumptions.

Finally, the authors want to point out that much of the material of this book can be readily formulated for non-spherical reference surfaces. Nevertheless, it remains to work with more realistic geometries such as (actual) Earth's surface, real satellite orbits, etc. This is the great challenge for future research.

List of Symbols

In the following definitions, the first digit refers to the the chapter in which the notation occurs and the second to the section within the chapter.

Basic Nomenclature

$\mathbb{N}_0$	set of non–negative integers:	2.1
$\mathbb{N}$	set of positive integers:	2.1
$\mathbb{Z}$	set of integers:	2.1
$\mathbb{R}$	set of real numbers:	2.1
$\mathbb{C}$	set of complex numbers:	2.1
$\mathbb{R}^3$	three-dimensional Euclidean space:	2.1
x, y, z	elements of $\mathbb{R}^3$:	2.1
$x \cdot y$	scalar product of vectors:	2.1
$x \wedge y$	vector product of vectors:	2.1
$x \otimes y$	dyadic product of vectors:	2.1
$\lvert x \rvert$	Euclidean norm of x:	2.1
$\varepsilon^i, \ i = 1, 2, 3$	canonical orthonormal basis in $\mathbb{R}^3$:	2.1
δ_{ij}	Kronecker symbol:	2.1
$\mathbf{i}$	identity tensor:	2.1
$\mathbf{t}$	orthogonal matrix:	2.1
$\mathbf{t}^T$	transpose of the matrix $\mathbf{t}$:	2.1
$\det \mathbf{t}$	determinant of the matrix $\mathbf{t}$:	2.1
$\operatorname{trace} \mathbf{t}$	trace of tensor $\mathbf{t}$:	2.1
$\mathbf{s} \otimes \mathbf{t}$	tensor product of tensors:	2.1
$\mathbf{s} \cdot \mathbf{t}$	scalar product of tensors:	2.1
Γ	point set in $\mathbb{R}^3$:	2.2
$\partial\Gamma$	boundary of Γ:	2.2
$\overline{\Gamma}$	closure of Γ:	2.2
$F \lvert M$	restriction of F to M:	2.2
∇	gradient:	2.2
L	curl gradient:	2.2
$\nabla\cdot, \mathrm{div}$	divergence:	2.2
$\mathrm{L}\cdot, \mathrm{curl}$	curl:	2.2
Δ	Laplace operator:	2.2

Spherical Nomenclature

Spherical Harmonics (with respect to $o^{(i)}, O^{(i)}; \mathbf{o}^{(i,k)}, O^{(i,k)}$-operators)

Spherical Harmonics (with respect to $\tilde{o}^{(i)}, \tilde{O}^{(i)}; \tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$-Operators)

$H_{n,j}$ .... scalar homogeneous harmonics polynomials of degree n and order j: 3.3
$Y_{n,j}$ .... spherical harmonic of degree n

and order j: 3.4
P_n ....Legendre polynomial of degree n: 3.4
$P_{n,m}$ .... associated Legendre function of degree n and order m:3.12

$\tilde{o}_n^{(i)}, \tilde{O}_n^{(i)}$ ....adjoint operators (vectorial context):5.14
$\widetilde{y}_{n,j}^{(i)}$ ....vector spherical harmonics of degree n, order j, and type i(with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$): 5.3
$\widetilde{p}_n^{(i)}(\cdot,\cdot)$ ....Legendre vector kernel of degree n and type i (with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$):5.14
$\widetilde{p}_n(\cdot,\cdot)$ ....Legendre vector kernel of degree n (with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$):5.14
${}^v\widetilde{\mathbf{p}}_n^{(i,k)}(\cdot,\cdot)$ ....(vectorial) Legendre rank–2 tensor kernel of degree n and type (i,k) (with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$):5.14
${}^v\widetilde{\mathbf{p}}_n(\cdot,\cdot)$ ......(vectorial) Legendre rank–2 tensor kernel of degree n and type (i,k) (with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$):5.14

$\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$ ....adjoint operators (tensorial context):5.14
$\widetilde{\mathbf{y}}_{n,j}^{(i,k)}$ .... tensor spherical harmonics of degree n, order j, and type (i,k)(with respect to $\tilde{\mathbf{o}}^{(i,k)}, K^{(i,k)}$):6.13
${}^t\widetilde{\mathbf{p}}_n^{(i,k)}(\cdot,\cdot)$ ....(tensorial) Legendre rank–2 tensor kernel of degree n and type (i,k) (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$):6.11
$\widetilde{\mathbf{p}}_n(\cdot,\cdot)$ ....(tensorial) Legendre rank–2 tensor kernel of degree n (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$):6.11
$\widetilde{\mathbf{P}}_n^{(i,k,l,m)}(\cdot,\cdot)$. (tensorial) Legendre rank–4 tensor kernel of degree n and type (i,k,l,m) (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$): 6.9
$\widetilde{\mathbf{P}}_n(\cdot,\cdot)$ ....(tensorial) Legendre rank–4 tensor kernel of degree n (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$): 6.9

Spherical Harmonic Spaces

Harm_n .... space of scalar spherical harmonics of degree n: 3.1

$\mathrm{harm}_n^{(i)}$ .... space of vector spherical harmonics of degree n and type (i) (with respect to $o^{(i)}, O^{(i)}$): 5.3

harm_n .... space of vector spherical harmonics of degree n (with respect to $o^{(i)}, O^{(i)}$): 5.3

$\mathbf{harm}_n^{(i,k)}$ .... space of tensor spherical harmonics of degree n and type (i,k) (with respect to $o^{(i,k)}, O^{(i,k)}$):6.13

$\mathbf{harm}_n$ .... space of tensor spherical harmonics of degree n (with respect to $o^{(i,k)}, O^{(i,k)}$):6.13

$\widetilde{\mathrm{harm}}_n^{(i)}$ .... space of vector spherical harmonics of degree n and type (i) (with respect to $\tilde{o}^{(i)}, \tilde{O}^{(i)}$):5.14

$\widetilde{\mathrm{harm}}_n$ .... space of vector spherical harmonics of degree n (with respect to $\tilde{\mathbf{o}}^{(i)}, \tilde{O}^{(i)}$):5.14

$\widetilde{\mathbf{harm}}_n^{(i,k)}$ .... space of tensor spherical harmonics of degree n and type (i,k) (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$):6.13

$\widetilde{\mathbf{harm}}_n$ .... space of tensor spherical harmonics of degree n (with respect to $\tilde{\mathbf{o}}^{(i,k)}, \tilde{O}^{(i,k)}$):6.13

Zonal Kernel Functions

$K(\xi \cdot \eta)$ ....scalar zonal kernel function:7.1

$k^{(i)}(\cdot,\cdot)$ .... (vectorial) zonal kernel function of type i (with respect to the Legendre $\{p_n^{(i)}(\cdot,\cdot)\}$ – system):8.3
$k(\cdot,\cdot)$ ....(vectorial) zonal kernel function (with respect to the Legendre $\{p_n(\cdot,\cdot)\}$– system):8.3
${}^v\mathbf{k}^{(i)}(\cdot,\cdot)$ .... (vectorial) rank-2 tensor zonal kernel function of type i (with respect to the Legendre $\{{}^v\mathbf{p}_n^{(i,i)}(\cdot,\cdot)\}$ –system) :8.3
${}^v\mathbf{k}(\cdot,\cdot)$ .... (vectorial) rank-2 tensor zonal kernel function (with respect to the Legendre $\{{}^v\mathbf{p}_n(\cdot,\cdot)\}$ –system) :8.3
${}^t\mathbf{k}^{(i,k)}(\cdot,\cdot)$ .... (tensorial) rank-2 tensor zonal kernel function of type (i,k) (with respect to the Legendre $\{{}^t\mathbf{p}_n^{(i,k)}(\cdot,\cdot)\}$–system) :9.2
${}^t\mathbf{k}(\cdot,\cdot)$ ....(tensorial) rank-2 tensor zonal kernel function (with respect to the Legendre $\{{}^t\mathbf{p}_n(\cdot,\cdot)\}$ –system) :9.2
$\mathbf{K}^{(i,k)}(\cdot,\cdot)$ .... (tensorial) rank-4 tensor zonal kernel function of type (i,k) (with respect to the Legendre $\{\mathbf{P}_n^{(i,k,i,k)}(\cdot,\cdot)\}$ –system):9.2
$\mathbf{K}(\cdot,\cdot)$ ....(tensorial) rank-4 tensor zonal kernel function (with respect to the Legendre $\{\mathbf{P}_n(\cdot,\cdot)\}$ –system):9.2

$\widetilde{k}^{(i)}(\cdot,\cdot)$ ....(vectorial) zonal kernel function of type i (with respect to the Legendre $\{\widetilde{p}_n^{(i)}(\cdot,\cdot)\}$ –system):8.3
$\widetilde{k}(\cdot,\cdot)$ ....(vectorial) zonal kernel function (with respect to the Legendre $\{\widetilde{p}_n(\cdot,\cdot)\}$ –system):8.3
${}^v\widetilde{\mathbf{k}}^{(i)}(\cdot,\cdot)$ .... (vectorial) rank-2 tensor zonal kernel function of type i (with respect to the Legendre $\{{}^v\widetilde{\mathbf{p}}_n^{(i,i)}(\cdot,\cdot)\}$ –system):8.3
${}^v\widetilde{\mathbf{k}}(\cdot,\cdot)$ .... (vectorial) rank-2 tensor zonal kernel function (with respect to the Legendre $\{{}^v\widetilde{\mathbf{p}}_n(\cdot,\cdot)\}$ –system) :8.3
${}^t\widetilde{\mathbf{k}}^{(i,k)}(\cdot,\cdot)$ .... (tensorial) rank-2 tensor zonal kernel function of type (i,k) (with respect to the Legendre $\{{}^t\widetilde{\mathbf{p}}_n^{(i,k)}(\cdot,\cdot)\}$ –system):9.2
${}^t\widetilde{\mathbf{k}}(\cdot,\cdot)$ ....(tensorial) rank-2 tensor zonal kernel function (with respect to the Legendre $\{\widetilde{\mathbf{p}}_n(\cdot,\cdot)\}$ –system) :9.2
$\widetilde{\mathbf{K}}^{(i,k)}(\cdot,\cdot)$ .... (tensorial) rank-4 tensor zonal kernel function of type (i,k) (with respect to the Legendre $\{\widetilde{\mathbf{P}}_n^{(i,k,i,k)}(\cdot,\cdot)\}$ –system):9.2
$\widetilde{\mathbf{K}}(\cdot,\cdot)$ ....(tensorial) rank-4 tensor zonal kernel function (with respect to the Legendre $\{\widetilde{\mathbf{P}}_n(\cdot,\cdot)\}$ –system):9.2

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